INDIAN STATISTICAL INSTITUTE
M.STAT Second Year
2016-17 Semester II

Computational Finance
Midterm Examination

Points for each question is in brackets. Total Points 100.
Students are allowed to bring 2 pages of hand-written notes
Duration: 3 hours

1. Consider the Black Scholes PDE
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$
   (a) Use a mesh of equal space steps of size δS and equal time steps of size δt, central differences for space derivatives, backward differences of time derivatives, to obtain the explicit finite difference equations
$$V_n^m = a_n V_{n-1}^{m+1} + b_n V_n^{m+1} + c_n V_n^{m+1}$$
   where V_n^m is the finite difference approximation to $V(n\delta S, m\delta t)$ and
$$\begin{aligned} a_n &= \frac{1}{2}(\sigma^2 n^2 - rn)\delta t \\ b_n &= 1 - (\sigma^2 n^2 + r)\delta t \\ c_n &= \frac{1}{2}(\sigma^2 n^2 + rn)\delta t \end{aligned}$$
   (b) Why is this an explicit method? What boundary and initial/final conditions are appropriate?
   (c) Derive the corresponding implicit finite difference equations.

2. (a) Assume that you have a random number generator from Unif(0,1). Briefly indicate an algorithm for generating a random variable from distribution with pdf $\frac{1}{16}\sqrt{x} + 1/6, 0 < x < 4$ using the following procedure:
   - Generate X from Uniform (0,4)
   - Generate Y from a distribution with pdf $\frac{3}{16}\sqrt{x}, 0 < x < 4$.
   - Combine X and Y suitably. Calculate actual realizations (one each) to demonstrate your method using the following values for the uniform random numbers: 0.6443 0.2077 0.3111

   (b) Suppose you want to evaluate the integral of the function $f(x) = e^x$ over $[0, 1]$ using Monte Carlo simulation from Unif(0,1). Use the control variable method with $g(x) = x$ and show that the variance is reduced by a factor of 60. Is there much additional improvement if you use a general quadratic function of x?

3. (a) Use the Ito formula to show that, for any integer $k \geq 2$,
$$\mathrm{E}W(t)^k = \frac{1}{2}k(k-1)\int_0^t \mathrm{E}W(s)^{k-2}ds,$$
   and use this to evaluate the sixth moment of the standard normal distribution.

(b) A Perpetual Option: Assume that the share prices of Stock follows a geometric Brownian motion with zero drift and volatility 1. Consider an option with no date of expiration that pays the owner $\exp \beta\tau_\alpha$ at the first time τ_α that the share price of Stock reaches α (if ever). Here β and α are positive real numbers, and $S_0 < \alpha$. Calculate the arbitrage price at time 0 of this option.

[Hint: Use $\mathrm{P}(\tau_\alpha \leq t)$=$2\mathrm{P}(S_t > \alpha)$, by the reflection principle.]

4. (a) Show that the Gaussian quadrature formula with n nodes is of order $2n - 1$. That is, if we choose as nodes the n roots of a polynomial of order n, within a family of orthogonal polynomials, then for any polynomial f of order $2n - 1$, we have $\int_a^b f(x)w(x)dx = \sum_{i=1}^n w_i f(x_i)$.

   (b) When approximating an integral by the trapezoid method, show that the error is bounded by $\frac{M}{12}(b - a)^3$ where $M = \sup f''(x)$.