INDIAN STATISTICAL INSTITUTE

M.STAT Second Year

2017-18 Semester II

Computational Finance

Final Examination

Points for each question is in brackets. Total Points 100.
Students are allowed to bring 4 pages (one-sided) of hand-written notes
Duration: 3 hours

1. (10+5) Show that the Gaussian quadrature formula with n nodes is of order $2n-1$. That is, if we choose as nodes the n roots of a polynomial of order n, within a family of orthogonal polynomials, then for any polynomial f of order $2n-1$, we have $\int_a^b f(x)w(x)dx = \sum_{i=1}^n w_i f(x_i)$, where given a non-negative weight function $w(x)$, the nodes x_i and weights w_i have to be chosen properly. What are the proper weights when $w(x)$ is the Gaussian weight function?

2. (8+5+7) We are interested in generating the path of a Brownian motion at time points $0 < t_1 < t_2 \cdots < t_n$

   (a) Describe the Cholesky construction by explicitly deriving the lower triangular matrix A such that $AA^T = C$, where C is the covariance matrix.

   (b) How does the random walk construction use the form of A above to reduce the number of computations from $O(n^2)$ to $O(n)$?

   (c) Write one advantage and one disadvantage of using the principal component construction over the random walk construction.

3. (10+10) Let C denote the copula of the two random variables X and Y. Assume that the marginal cdfs are continuous and strictly increasing.

   (a) Show that $P[\max(X,Y) \le t] = C(F_X(t), F_Y(t))$

   (b) Prove that the Spearman correlation coefficient $\rho(X,Y)$ is given by the formula

$$\rho(X,Y) = 12 \int_0^1 \int_0^1 uvC(u,v)dudv - 3$$

4. (8+7+10) Consider the generalized Vasicek family below to model the term structure of interest rates:

$$Y(x) = \theta_1 - \theta_2\theta_4 \frac{1 - e^{-x/\theta_4}}{x} + \theta_3\theta_4 \frac{(1 - e^{-x/\theta_4})^2}{4x}$$

   (a) Derive an analytical formula for the instantaneous forward rate.

   (b) Derive an analytical formula for the price of zero coupon bonds.

   (c) Given data on the price of bonds of different maturities for a series of days, how would you forecast the prices for the next day using this model and an AR(1) structure on the evolution of the parameters.

5. (7+8+5) Suppose ϵ_t is stationary strong ARCH(1) process; that is $\mathrm{E}(\epsilon_t|\mathcal{F}_{t-1}) = 0$, $\mathrm{Var}(\epsilon_t|\mathcal{F}_{t-1}) = \sigma_t^2$ and $\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2$.

   (a) Show that ϵ_t is white noise.

   (b) In addition, assume that $\mathrm{E}(\epsilon_t^4) = c < \infty$ and $Z \sim \mathcal{N}(0,1)$. Show that ϵ_t^2 follows an AR(1) process.

   (c) What is the main difference between GARCH-type models and Stochastic Volatility models?