INDIAN STATISTICAL INSTITUTE
M.STAT Second Year
2016-17 Semester II

Computational Finance
Back Paper Examination

Points for each question is in brackets. Total Points 100.
Students are allowed to bring 4 pages (one-sided) of hand-written notes
Duration: 3 hours

1. (12+3)Consider the Black Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

   (a) Use a mesh of equal space steps of size δS and equal time steps of size δt, central differences for space derivatives, backward differences of time derivatives, to obtain the explicit finite difference equations

$$V_n^m = a_n V_{n-1}^{m+1} + b_n V_n^{m+1} + c_n V_{n+1}^{m+1}$$

   where V_n^m is the finite difference approximation to $V(n\delta S, m\delta t)$ and

$$\begin{aligned} a_n &= \frac{1}{2}(\sigma^2 n^2 - rn)\delta t \\ b_n &= 1 - (\sigma^2 n^2 + r)\delta t \\ c_n &= \frac{1}{2}(\sigma^2 n^2 + rn)\delta t \end{aligned}$$

   (b) Why is this an explicit method? What boundary and initial/final conditions are appropriate?

2. (5+15) The objective of this problem is to show that Binomial Option pricing formula with

$$u = e^{\sigma\sqrt{\Delta t}}, d = 1/u \quad \text{and} \quad p = \frac{e^{r\Delta t} - d}{u - d}$$

   approximately solves the BS equation as $\Delta t \to 0$.

   (a) Write down the relationship induced by no arbitrage, between prices at time t and $t + \Delta t$, in the single period Binomial model.

   (b) Use Taylor expansion and take limit at $\Delta t \to 0$ to show that this leads to the Black Scholes PDE.

3. (10) Show that for estimating $E[f(Z)]$ based on an antithetic pair $(Z, -Z)$, where $Z \sim \mathcal{N}(0, I)$, antithetic sampling eliminates all variance if f is antisymmetric.

4. (10+7+8)

   (a) For a given level α compute the Value at Risk and Expected shortfall under the following distributional assumption in the loss distribution of the portfolio:

i. Exponential with rate parameter λ.

ii. Standard Pareto distribution with shape parameter ξ, location parameter 0 and scale parameter 1, that is, the pdf is

$$f_\xi(x) = (1+\xi x)^{-(1+1/\xi)} \quad \text{if} \quad x > 0$$

(b) For each $\alpha \in (0,1)$, derive an equation that the rate parameter λ and the shape parameter ξ must satisfy in order for the values of VaR(α) for the two distributions to be the same.

(c) Assuming that the parameters λ and ξ satisfy the relationship in (b) above, compare the corresponding values of the expected short fall in the two models and comment on the differences.

5. (10+10) Let C denotes the copula of the two random variables X and Y. Assume that the marginal cdfs are continuous and strictly increasing.

   (a) Show that $P[\max(X,Y) \le t] = C(F_X(t), F_Y(t))$

   (b) Prove that the Spearman correlation coefficient $\rho(X,Y)$ is given by the formula

$$\rho(X,Y) = 12 \int_0^1 \int_0^1 uvC(u,v)dudv - 3$$

6. (10) Explain the problem of cointegration of two time series.