Quiz 2, ASM1
Time allowed 30 mins

Suppose $X_1, \cdots, X_n$ are iid from a distribution with pdf f. Define

$$\hat{f}(t) = \frac{1}{n} \sum_{i=1}^{n} K_h(X_i - t)$$

where $K_h(x) = \frac{1}{h} K(\frac{x}{h}), K \geq 0, \int K(x)dx = 1, \int xK(x)dx = 0, \int x^2 K(x)dx = 1.$ Prove the following

1. $\mathrm{E}\left(\hat{f}(t)\right) = f(t) + \frac{h^2}{2} f''(t) + o(h^2)$
2. $\mathrm{Var}\left(\hat{f}(t)\right) = \frac{1}{nh^2}\mathrm{E}\left(K^2\left(\frac{X_1 - t}{h}\right)\right) - \frac{1}{n} f^2(t) + o(\frac{h^2}{n})$
3. $\mathrm{E}\left(K^2\left(\frac{X_1 - t}{h}\right)\right) = hf(t) \int K^2(y)dy + o(h)$
4. $\mathrm{Var}\left(\hat{f}(t)\right) = \frac{1}{nh} f(t) \int K^2(y)dy + o(\frac{1}{nh})$

For proving any part, you may use the results of the previous parts even if you could not prove them.