MODULI TOPOLOGY

Abstract. Notes from a seminar based on the section 3 of the paper: Picard groups of moduli problems (by Mumford).

1. Grothendieck Topology

We can define a topology on any set $S$ provided the subsets designated as *open sets* satisfy certain topological axioms. But if the fundamental notions are categories and sheaves, not sets, then we need a different definition of *topology* which should be in a sense the ”minimal” structure on a category, so that one can define a sheaf and give a suitable cohomology theory. This structure is known as Grothendieck topology.

**Definition 1.1.** Let $C$ be a category. A Grothendieck topology on $C$ is an assignment to each object $U$ of $C$, a collection of morphisms in $C$, $\{U_i \to U\}$, called ”coverings of $U$” such that,

1. *Fibred products exist.*
2. If $V \to U$ is an isomorphism, then the set $\{V \to U\}$ is a covering of $U$. If $\{U_i \to U\}$ is a covering of $U$ and $\{U_{ij} \to U_i\}$ is a covering of $U_i$ for each index $i$, then the collection of composites $\{U_{ij} \to U\}$ is a covering of $U$.
3. If $V \to U$ is a morphism and $\{U_i \to U\}$ is a covering, then the set $\{U_i \times_U V \to V\}$ is a covering of $V$.

The axioms do not describe the ”open sets”, but the coverings of a space. Instead of intersections, we have fibered products. The union operation of standard topology plays no role in Grothendieck topology.

A category with a Grothendieck topology is called a *site*.

2. Moduli Topology

Assumptions and conventions:

1. $k$ is an algebraically closed field.
2. All schemes are separated and of finite type over $k$.
3. Throughout this note, we have fixed a non-negative integer $g$.

**Definition 2.1.** A curve (of genus $g$) is a connected, reduced, one-dimensional scheme $C$, such that $H^1(C, O_C) = g$. 

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Definition 2.2. A "family of curves" over a scheme $S$, is a flat projective morphism $\pi : X \to S$ such that fibres over all closed points are curves.

Definition 2.3. A "morphism" $F$ of one family $\pi_1 : X_1 \to S_1$ to another $\pi_2 : X_2 \to S_2$ is a fibered diagram of schemes:

$$
\begin{array}{ccc}
X_1 & \longrightarrow & X_2 \\
\pi_1 \downarrow & & \pi_2 \\
S_1 & \longrightarrow & S_2
\end{array}
$$

We say $F$ is flat/smooth/etale if the morphism from $S_1$ to $S_2$ is flat/smooth/etale.

Definition 2.4. Given a family of curves $\pi : X \to S$ and a morphism $g : T \to S$, the "induced family of curves" over $T$ is the natural projection $p_2 : X \times_SS T \to T$.

We will denote the category of families of curves by $\mathcal{M}$. We try to define a Grothendieck topology (in the most natural way) on $\mathcal{M}$.

Definition 2.5 (Provisional form). The moduli topologies $\mathcal{M}^*_{\text{et}}, \mathcal{M}^*_{\text{smooth}}$ and $\mathcal{M}^*_{\text{flat}}$ are as follows:

1. Open sets are families of curves.
2. Morphism between open sets are morphism between families i.e fibered diagrams.
3. A collection of such morphisms

$$
\begin{array}{ccc}
X_\alpha & \longrightarrow & X \\
\downarrow & & \downarrow \\
S_\alpha & \xrightarrow{g_\alpha} & S
\end{array}
$$

is called a covering if $S = \bigcup_\alpha g_\alpha S_\alpha$ and if each $g_\alpha$ is etale, smooth or flat respectively.

To get a feel of this extra structure on $\mathcal{M}$, we will show that this is a topology. We first have a fact which will be used several times in these notes.

Lemma 2.6. Let $\mathcal{C}$ be any category (with fibred products). Consider the following diagram of morphisms in $\mathcal{C}$ (where solid squares are fibred products),

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\exists!} & X_2 \\
\downarrow & & \downarrow \\
X_3 & \xleftarrow{} & \\
\downarrow & & \downarrow \\
S_1 & \xrightarrow{} & S_2
\end{array}
$$

$X_1$ and $X_2$ are connected by a unique morphism.

The only unique morphism $X_3 \to X_2$ is $X_3 \to X_2$. A similar situation is true for $X_2 \to X_3$ and $S_2 \to S_3$.
If we have a map $S_1 \rightarrow S_2$ factoring $S_1 \rightarrow S_3$ via $S_2$ then there exists a unique map $X_1 \rightarrow X_2$ such that the dotted square is also a fibred product.

Proof. Exercise. \qed

We now continue with the proof that $\mathcal{M}$ with the extra structure is a topology.

1) Fibre product exists: We claim that fibre product of families $\pi_i : X_i \rightarrow S_i$ ($i = 1, 2$) over $\pi_0 : X_0 \rightarrow S_0$ is given by $\pi : X_1 \times_{X_0} X_2 \rightarrow S_1 \times_{S_0} S_2$.

To see this, let $\pi' : X' \rightarrow S'$ be another family such that there exists a diagram of morphism of families,

Thus $\pi'$ is pullback of the family $\pi_0$ under the composition map $S' \rightarrow S_i \rightarrow S_0$ ($i = 1$ or 2). By universal property, this factors as $S' \rightarrow S_1 \times_{S_0} S_2 \rightarrow S_0$. It is easy to check that the pullback of $\pi_0$ under the natural map $S_1 \times_{S_0} S_2 \rightarrow S_0$ is $\pi$, thus we get the morphism $\pi' \rightarrow \pi$ (by lemma 2.6) which is unique as the base maps are.

We will denote the fibre product by $\pi_1 \times_{\pi_0} \pi_2$.

2) Second axiom of topology is easy to verify (coverings of a covering is a covering). Isomorphism between families are ofcourse coverings.

3) Third axiom: Let $\pi_\alpha : X_\alpha \rightarrow S_\alpha$ be a covering of $\pi : X \rightarrow S$. Let $\pi' : X' \rightarrow S'$ be another family with a morphism $\pi' \rightarrow \pi$. We claim that the projection $\pi_\alpha \times_{\pi} \pi' \rightarrow \pi'$ is a covering of $\pi'$. Thus we want to show that $\bigcup_\alpha p_\alpha(S_\alpha \times_{S} S') = S'$, where $p_\alpha$ is the natural projection.

Let $S' \ni s' \mapsto s \in S$ under the given morphism. Since $\pi_\alpha : X_\alpha \rightarrow S_\alpha$ is a covering, there exists some $\beta$ such that $S_\beta$ has a preimage of $s$. Call that preimage $s_\beta$, then the pair $(s_\beta, s') \in S_\beta \times_{S} S'$. This proves the claim.

2.1. Absolute product exists. Let $\pi_i : X_i \rightarrow S_i$ ($i = 1, 2$) be families of curves. We claim that their absolute product exists.

First note that the obvious candidate for the absolute product $\pi_1 \times \pi_2 : X_1 \times X_2 \rightarrow S_1 \times S_2$ is not the correct one, as (e.g) the projection $\pi_1 \times \pi_2 \rightarrow \pi_i$ is not a morphism of families.

We also note that the projection $S_1 \times S_2 \rightarrow S_i$ induces families of curves,

\[
\begin{array}{ccc}
X_1 \times S_2 & & X_2 \times S_1 \\
\phi_1 & \downarrow & \phi_2 \\
S_1 \times S_2 & & \\
\end{array}
\]
Now, we assume the data that there exists a third family $\pi' : X' \to S'$ with morphism of families $\pi' \to \pi_1$ and $\pi' \to \pi_2$. We claim that this data is equivalent to having maps $\pi' \to \phi_1$ and $\pi' \to \phi_2$. One side is easy, that is, if we are given maps $\pi' \to \phi_i$ it will immediately give maps $\pi' \to \phi_i \to \pi_i$, $(i = 1, 2)$ by composition of morphisms between families,

$$
\begin{array}{c}
X' \to X_1 \times S_2 \to X_1 \\
\downarrow \quad \downarrow \\
S' \to S_1 \times S_2 \to S_1
\end{array}
$$

So assume that we are given morphism of families $\pi' \to \pi_1$ and $\pi' \to \pi_2$. By universal property of $X_1 \times S_2$, there exists a diagram where the two squares are fibred product,

$$
\begin{array}{c}
X' \quad X_1 \times S_2 \quad X_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
S' \quad S_1 \times S_2 \quad S_1
\end{array}
$$

By lemma 2.6, it follows that there exists maps $\pi' \to \phi_i$ $(i = 1, 2)$. This proves the claim.

The upshot is that we now have maps from $\pi'$ to families having same base. In particular, to have compatibility, we must have isomorphisms over $S'$ of the families of curves,

$$
(X_1 \times S_2) \times_{S_1 \times S_2} S' \quad X' \quad (X_2 \times S_1) \times_{S_1 \times S_2} S'
$$

and we are looking for a universal object in the category of families, which have maps to $\phi_1$ and $\phi_2$. Such a family (if it exists) must have a unique map from $\pi'$. In particular, denoting such a family by $\pi : X \to S$, we must have a unique map $S' \to S$ and a map $S \to S_1 \times S_2$ (coming via projections to $\phi_i$).

To preserve the compatibility mentioned above, we must also have the following commutative diagram,

$$
\begin{array}{c}
(X_1 \times S_2) \times_{S_1 \times S_2} S' \quad (X_2 \times S_1) \times_{S_1 \times S_2} S' \\
\downarrow \quad \downarrow \\
(X_1 \times S_2) \times_{S_1 \times S_2} S \quad (X_2 \times S_1) \times_{S_1 \times S_2} S
\end{array}
$$

This diagram and the uniqueness of the morphism $S' \to S$, will also give uniqueness (upto a unique isomorphism) of the family $\pi : X \to S$.

Such an $S$ exists and a proof of existence can be found in [3]. Thus the absolute product exists.
**Definition 2.7.** The absolute product of the families $\pi_i : X_i \to S_i$, $(i = 1, 2)$, will be denoted by

$$\pi : (X_1, X_2) \to Isom(\pi_1, \pi_2)$$

Since absolute products exist in moduli topology, we will formally add a final object $M$. Fibre products over $M$ are nothing but the absolute products.

But we must define the coverings of this final object. The problem is this that if $\pi : X \to S$ is part of a covering of $M$ then, let $\pi' : X' \to S'$ be any other family, it must have a map to $M$ by definition of final object and hence (by last axiom of Grothendieck topology) the morphism from the product family $(X, Y) \to Isom(\pi, \pi')$ to $\pi'$ must be part of a covering of $\pi'$.

In particular, depending upon the topology we are working in i.e. $M_{et}, M_{smooth}$ or $M_{flat}$, the projection $Isom(\pi, \pi') \to S'$ must be etale, smooth or flat respectively.

**Definition 2.8.** A family of curves $\pi : X \to S$ is etale, smooth or flat over $M$ if, for all other families $\pi' : X' \to S'$, the projection from $Isom(\pi, \pi') \to S'$ is etale, smooth or flat.

2.2. Smooth/Flat families over $M$. We will prove a criteria for smoothness of families of curves over $M$. For the rest of this section, $A$ will denote a local artin $k$-algebra and $I \subset A$ is an ideal.

Let $\pi : X \to S$ be a smooth family over $M$. We claim that whenever we are given a diagram of solid arrows,

![Diagram](image)

where $Y \to Spec(A)$ and $Y_0 \to Spec(A/I)$ are families of curves, and where the two solid squares are morphisms of families. Then, there exist maps along dotted lines, making the third square also a fibred product.

By universal property of product, the first square factors as,

$$Y_0 \quad Y \quad Spec(A/I) \quad Spec(A)$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$Y_0 \rightarrow (X, Y) \rightarrow Y \rightarrow Spec(A/I) \rightarrow Isom(\pi, \phi) \rightarrow Spec(A)$$
By the assumption $\pi : X \to S$ is a smooth family over $M$, which implies (by definition) that the projection $\text{Isom}(\pi, \phi) \to \text{Spec}(A)$ is smooth. In particular, in the following diagram, there exists a map $\text{Spec}(A) \to \text{Isom}(\pi, \phi)$ such that the composition $\text{Spec}(A) \to \text{Isom}(\pi, \phi) \to \text{Spec}(A)$ is identity,

\[
\begin{array}{ccc}
\text{Spec}(A/I) & \xleftarrow{=} & \text{Isom}(\pi, \phi) \\
\downarrow & & \downarrow \\
\text{Spec}(A) & \xrightarrow{id} & \text{Spec}(A) \\
\end{array}
\]

Therefore, there exists a diagram,

\[
\begin{array}{ccc}
Y & \xleftarrow{id} & Y \\
\downarrow & & \downarrow \\
\text{Spec}(A) & \to & \text{Isom}(\pi, \phi) \\
\end{array}
\]

Here the bigger rectangle is a fibred product, by identity map $\text{Spec}(A) \to \text{Spec}(A)$ and the 2nd square is a fibred product under the projection $\text{Isom}(\pi, \phi) \to \text{Spec}(A)$. The first square is a fibered product by lemma 2.6.

Composing the first square with the projection induced by $\text{Isom}(\pi, \phi) \to S$, we get morphisms of families,

\[
\begin{array}{ccc}
Y & \to & (X, Y) & \to & X \\
\downarrow & & \downarrow & & \downarrow \\
\text{Spec}(A) & \to & \text{Isom}(\pi, \phi) & \to & S \\
\end{array}
\]

which completes the diagram (4) as claimed.

We will now show the converse. That is suppose for every diagram as below,

\[
\begin{array}{ccc}
Y_0 & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow \\
\text{Spec}(A/I) & \xrightarrow{\pi} & S \\
\end{array}
\]

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow \\
\text{Spec}(A/I) & \xrightarrow{\pi} & S \\
\end{array}
\]
where solid squares are morphism between families of curves, one can find maps along dotted lines, making it a morphism between families of curves. Then, we claim that $\pi : X \to S$ is a smooth family over $M$.

To show smoothness of $\pi$ over $M$, we need to show (by definition) that for any other family $\pi' : X' \to S'$, the projection $\text{Isom}(\pi, \pi') \to S'$ is smooth. We show this using the criteria for smoothness. So suppose we are given another family $\pi' : X' \to S'$ and the diagram,

$$
\begin{array}{ccc}
\text{Spec}(A/I) & \to & \text{Isom}(\pi, \pi') \\
\downarrow & & \downarrow \\
\text{Spec}(A) & \to & S'
\end{array}
$$

we want to show the existence of the map along dotted line.

To use the assumption, we need families $Y \to \text{Spec}(A)$ and $Y_0 \to \text{Spec}(A/I)$ which can be fit inside the diagram (5),

We get the family $Y_0 \to \text{Spec}(A/I)$ as fiber product induced by the map $\text{Spec}(A/I) \to \text{Isom}(\pi, \pi')$,

$$
\begin{array}{ccc}
Y_0 & \to & (X, X') \\
\downarrow & & \downarrow \\
\text{Spec}(A/I) & \to & \text{Isom}(\pi, \pi')
\end{array}
$$

We get the family $\phi : Y \to \text{Spec}(A)$ as fiber product induced by the map $\text{Spec}(A) \to S'$,

$$
\begin{array}{ccc}
Y & \to & X' \\
\phi \downarrow & & \downarrow \\
\text{Spec}(A) & \to & S'
\end{array}
$$

Composition of maps $\text{Spec}(A/I) \to \text{Spec}(A) \to S'$ along with lemma 2.6, show that these 2 families, fit into following diagram consisting of morphisms between families of curves,

$$
\begin{array}{ccc}
Y_0 & \to & Y & \to & X' \\
\downarrow & & \downarrow & & \downarrow \\
\text{Spec}(A/I) & \to & \text{Spec}(A) & \to & S'
\end{array}
$$
Thus we have diagram,

\[
\begin{array}{c}
Y_0 \\
\downarrow \\
Y \\
\downarrow \phi \\
\text{Spec}(A) \\
\downarrow \\
\text{Spec}(A/I) \\
\downarrow \pi \\
X \\
\downarrow \\
\text{Spec}(A) \\
\downarrow \\
S
\end{array}
\]

which by assumption, can be completed to give,

\[
\begin{array}{c}
Y_0 \\
\downarrow \\
Y \\
\downarrow \phi \\
\text{Spec}(A) \\
\downarrow \\
\text{Spec}(A/I) \\
\downarrow \pi \\
X \\
\downarrow \\
\text{Spec}(A) \\
\downarrow \\
S
\end{array}
\]

Thus the family \( \phi : Y \to \text{Spec}(A) \) maps to families \( \pi \) and \( \pi' \). By universal property, the map \( \phi \to \pi' \) must factor via product family. In particular, there exists a map \( \text{Spec}(A) \to \text{Isom}(\pi, \pi') \) factoring the map \( \text{Spec}(A) \to S' \). This completes the criteria for smoothness and hence proves the claim.

**References**