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Simultaneous Surface Resolution in Quadratic and Biquadratic Galois Extensions

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ABSTRACT. We show that simultaneous surface resolution is always possible in a quadratic extension, and if the characteristic is different from two then in every compositum of such extensions. We also construct examples to show that the latter is not always possible if the characteristic is two.

1. Introduction

Let K be a two dimensional algebraic function field over an algebraically closed ground field k. Recall that K/k has a minimal model means that amongst all the nonsingular projective models of K/k there is one which is dominated by all others (basic reference [**A09**]). Also recall that K/k has a minimal model if and only if it is not a ruled function field, i.e., K is not a simple transcendental field extension of a one dimensional algebraic function field over k (see [**Z02**]). A finite algebraic field extension L/K is said to have a simultaneous resolution if there exist nonsingular projective models V and W of K/k and L/k, respectively, such that W is the normalization of V in L. Given any prime number $q \neq \text{char}(K)$, where char denotes characteristic, in [**A02**] it was shown that if $q \leq 3$ and L/K is a cyclic Galois extension of degree q then it has a simultaneous resolution, whereas if $q > 3$ and K/k has a minimal model then there exists a cyclic Galois extension L/K of degree q which has no simultaneous resolution. At the September 2003 Galois Theory Conference in Banff (Canada), Ted Chinberg asked whether simultaneous resolution was always possible if L/K was Galois with Galois group a direct sum of any finite number of copies, say m, of a cyclic group $\mathbb{Z}_2$ of order 2. The purpose of this note is to prove yes if either $\text{char}(K) \neq 2$ or $m = 1$, and no if $\text{char}(K) = 2 = m$ and K/k has a minimal model. This also provides a negative answer to the question which David Harbater raised at that conference and which asks if a positive answer for two Galois groups implies a positive answer for their direct sum. It may be noted that our yes answer remains valid also in the arithmetic case and in fact for surfaces over any excellent domain.

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NOTATION. By $M(R)$ we denote the maximal ideal in a (always noetherian) local ring R. By a quadratic transform of a regular local domain R, with maximal ideal $M(R)$ and quotient field K, we mean a domain R' which, for some nonzero element X in $M(R)$, can be expressed as a localization of $R[(Y/X)_{Y\in M(R)}]$ at some prime ideal containing $M(R)$; note that then R' is a regular local domain with quotient field K and the dimension of R' is $\le$ the dimension of R. By a two dimensional quadratic sequence we mean an infinite sequence $(R_i)_{i=0,1,2,\dots}$ of two dimensional regular local domains such that R_i is a quadratic transform of R_{i-1} for all $i > 0$. By a two dimensional regular semilocal domain we mean a noetherian domain S having at least one and most a finite number of maximal ideals such that the localization of S at any maximal ideal in it is a two dimensional regular local domain. For any element A in a local ring R we put $\mathrm{ord}_R A = \infty$ if $A = 0$, and if $A \neq 0$ then

$$\mathrm{ord}_R A = \text{the largest nonnegative integer } e \text{ with } A \in M(R)^e.$$

Moreover, for any polynomial

$$F(Z) = \sum_i A_i Z^i$$

in an indeterminate Z with coefficients A_i in R we put $\mathrm{ord}_R F(Z) = \infty$ if $A_i = 0$ for all i, and if $A_i \neq 0$ for some i then

$$\mathrm{ord}_R F(Z) = \min(i + \mathrm{ord}_R A_i)$$

where the min is taken over all those i for which $A_i \neq 0$.

A ring (always commutative with 1) R is said to be pseudogeometric if for any prime ideal P in R, the integral closure of R/P in any finite algebraic field extension of its quotient field is a finite (= finitely generated) (R/P)-module. Obviously any field is pseudogeometric and any homomorphic image of any pseudogeometric ring is pseudogeometric. It is also well-known that: every affine ring over (= finitely generated ring extension of) a pseudogeometric ring is pseudogeometric, the localization of any pseudogeometric ring at any multiplicative set in it is pseudogeometric, and every complete local ring is pseudogeometric

2. Local Theory

NOTATION FOR LEMMA 1 AND THEOREM 1. Let R be a two dimensional regular local domain with quotient field K and $M(R) = (X,Y)R$. For $1 \le j \le n$, where n is a positive integer, let

$$F_j = F_j(Z) = Z^2 + A_j Z + B_j$$

be a monic quadratic polynomial in an indeterminate Z with A_j, B_j in R. Let

$$C_j = A_j^2 - 4B_j$$

and

$$J = \{j : 1 \le j \le n \text{ with } C_j \neq 0\}$$

with

$$C = \prod_{j\in J} C_j$$

and

$$F = F(Z) = \prod_{1\le j\le n} F_j(Z).$$

Let L be a splitting field of F over K, and let S be the integral closure of R in L.

LEMMA 1. For $1 \le j \le n$ let L_j be the splitting field of F_j over K in L, and let S_j be the integral closure of R in L_j. Then we have the following.

(1.0) Assume that $\mathrm{char}(K) \neq 2$ and for every $j \in J$ we have

$$C_j = D_j X^{r_j} Y^{s_j}$$

with $D_j \in R \setminus M(R)$ and nonnegative integers r_j, s_j. For any integer r, let $\overline{r}$ denote the residue of r modulo 2, i.e., $\overline{r}$ is the unique integer in $\{0,1\}$ such that $r - \overline{r}$ is even. Then for every $j \in J$ there exists $H_j \in L$ with

$$H_j^2 = D_j X^{\overline{r_j}} Y^{\overline{s_j}}$$

and, for any such H_j, upon letting

$$\begin{cases} J' = \{j \in J : (\overline{r_j}, \overline{s_j}) = (0,1)\} \\ J'' = \{j \in J : (\overline{r_j}, \overline{s_j}) = (1,0)\} \\ J''' = \{j \in J : (\overline{r_j}, \overline{s_j}) = (1,1)\} \end{cases}$$

we have the following:

(i) If $J' \cup J'' \cup J''' = \emptyset$ then S is a two dimensional regular semilocal domain and for its localization T at any maximal ideal in it we have $M(T) = (X, Y)T$.

(i′) For every $j \in J'$ we have $S_j = R[H_j] =$ a two dimensional regular local domain with $M(S_j) = (X, H_j)$.

(i″) For every $l \in J''$ we have $S_l = R[H_l] =$ a two dimensional regular local domain with $M(S_j) = (H_l, Y)$.

(ii′) If $J' \neq \emptyset = (J'' \cup J''')$ then S is a two dimensional regular semilocal domain and for its localization T at any maximal in it we have with $M(T) = (X, H_j)$ where j is any element of J'.

(ii″) If $J'' \neq \emptyset = (J' \cup J''')$ then S is a two dimensional regular semilocal domain and for its localization T at any maximal ideal in it we have with $M(T) = (H_l, Y)$ where l is any element of J''.

(iii) If $J' \neq \emptyset \neq J''$ then S is a two dimensional regular semilocal domain and for its localization T at any maximal ideal in it we have with $M(T) = (H_l, H_j)$ where j and l are any elements of J' and J'' respectively.

(iii′) If $J' \neq \emptyset \neq J'''$ then S is a two dimensional regular semilocal domain and for its localization T at any maximal ideal in it we have with $M(T) = (H_u/H_j, H_j)$ where j and u are any elements of J' and J''' respectively.

(iii″) If $J'' \neq \emptyset \neq J'''$ then S is a two dimensional regular semilocal domain and for its localization T at any maximal ideal in it we have with $M(T) = (H_l, H_u/H_l)$ where l and u are any elements of J'' and J''' respectively.

(iv) If $(J' \cup J'') \neq \emptyset$ or $J''' = \emptyset$ then S is a two dimensional regular semilocal domain.

(v) If $(J' \cup J'') = \emptyset$ and $J''' \neq \emptyset$ and R' is any two dimensional quadratic transform of R, then the integral closure S' of R' in L is a two dimensional regular semilocal domain.

(1.1) Assume that $\mathrm{char}(K) \neq 2$ and for every $j \in J$ we have

$$C_j = D_j X^{r_j} Y^{s_j}$$

with $D_j \in R \setminus M(R)$ and nonnegative integers r_j, s_j. Then either S is a two dimensional regular semilocal domain, or for every two dimensional quadratic transform

R' of R we have that the integral closure S' of R' in L is a two dimensional regular semilocal domain.

(1.2) Assume that $n = 1$ and there exists $0 \neq \alpha \in K$ together with $\beta \in K$ such that for

$$F_1' = F_1'(Z) = \alpha^{-2} F_1(\alpha Z + \beta) = Z^2 + A'Z + B'$$

we have $F_1'(Z) \in R[Z]$ with $\mathrm{ord}_R F_1'(Z) = 1$. Then S is two dimensional regular semilocal domain.

(1.3) Assume that $\mathrm{char}(K) = 2 = n$ and

$$A_1 = XY^a \quad \text{and} \quad A_2 = XY^aE \quad \text{with} \quad B_1 = B_2 = Y$$

where

$$a = \text{a positive integer}$$

and

$$E = X^tY^u \text{ with nonnegative integers } t, u \text{ at least one of which is positive.}$$

Let H be a root of F_1 in L_1. Then S_1 is a two dimensional regular local domain with $M(S_1) = (X, H)S_1$, and S is a two dimensional nonregular local domain. Moreover, L/K is Galois with Galois group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

PROOF. To prove (1.0) note that $L_j = K(H_j)$. Also note that (i) follows from the discriminant theory given in [**A03**], (i$'$) and (i$''$) are straightforward, and, in view of (i$'$) (resp: (i$''$)), (ii$'$) (resp: (ii$''$)) follows by taking

$$\begin{cases} (R_j, L_j, X, H_j) \text{ and } (F_1, \dots, F_{j-1}, F_{j+1}, \dots, F_n) \\ (\text{resp: } (R_l, L_l, H_l, Y) \text{ and } (F_1, \dots, F_{l-1}, F_{l+1}, \dots, F_n)) \end{cases}$$

for (R, K, X, Y) and $(F_1, \dots, F_n)$ in (i). Likewise, in view of (i$'$), (iii) follows by taking

$$(R_j, L_j, X, H_j) \text{ and } (F_1, \dots, F_{j-1}, F_{j+1}, \dots, F_n)$$

for (R, K, X, Y) and $(F_1, \dots, F_n)$ in (i$''$). Likewise, in view of (i$'$) (resp: (i$''$)), (iii$'$) (resp: (iii$''$)) follows by taking

$$\begin{cases} (R_j, L_j, X, H_j) \text{ and } (F_1, \dots, F_{j-1}, F_{j+1}, \dots, F_n) \\ (\text{resp: } (R_l, L_l, H_l, Y) \text{ and } (F_1, \dots, F_{l-1}, F_{l+1}, \dots, F_n)) \end{cases}$$

for (R, K, X, Y) and $(F_1, \dots, F_n)$ in (ii$''$) (resp: (ii$'$)). (iv) follows from (ii$'$), (ii$''$), (iii), (iii$'$), and (iii$''$). To prove (v) suppose that $(J' \cup J'') = \emptyset$ and $J''' \neq \emptyset$ and let R' is any two dimensional quadratic transform of R. Suitably relabelling X, Y we may assume that $Y/X \in R'$. Now if $Y/X \in M(R')$ then $M(R') = (X, Y/X)R'$ and for all $j \in J$ we have

$$C_j = D_j X^{r_j'} Y^{s_j'}$$

where $D_j \in R' \setminus M(R')$ with $r_j' = r_j + s_j$ and $s_j' = s_j$, and hence $\overline{r_j'} = 0$ with $\overline{s_j'} = 0$ or 1, and so we are reduced to (iv). Likewise if $Y/X \notin M(R')$ then $M(R') = (X, Y')R'$ for some $Y' \in M(R')$, and for all $j \in J$ we have

$$C_j = D_j' X^{r_j'} (Y')^{s_j'}$$

where $D_j' = D_j(Y/X)^{s_j} \in R' \setminus M(R')$ with $r_j' = r_j + s_j$ with $s_j' = 0$, and hence $\overline{r_j'} + \overline{s_j'} = 0$, and so we are again reduced to (iv).

(1.1) follows from parts (iv) and (v) of (1.0).

To prove (1.2) it suffices to note that L is a splitting of F_1' over K.

To prove (1.3), since $A_1 \in M(R)$ with $B_1 = Y \in M(R) \setminus M(R)^2$, we see that S_1 is a two dimensional regular local domain with $M(S_1) = (X, H)S_1$. Also $Y = H^2 + XY^aH$ and substituting this in

$$F_2'(Z) = F_2(Z + H) = Z^2 + XY^aEZ + (Y + H^2 + XY^aEH)$$

we get

$$F_2'(Z) = Z^2 + XY^aEZ + XY^aH(1 + E)$$

where

$$\begin{cases} Y^a = (H^2 + XY^aH)^a \\ \quad = (H^2 + X[H^2 + XY^aH]^aH)^a \\ \quad = (H^2 + X[H^2 + X(H^2 + XY^aH)^aH]H)^a \\ \quad = H^{2a}D^* \quad \text{with} \quad D^* \in R \setminus M(R) \end{cases}$$

and hence

$$F_2'(Z) = Z^2 + XH^{2a}ED^*Z + XH^{2a+1}D^{**} \text{ with } D^{**} \in R \setminus M(R)$$

and therefore

$$F_2''(Z) = H^{-2a}F_2'(ZH^a) = Z^2 + XH^aED^*Z + XHD^{**}.$$

Consequently "the irreducible surface $F_2''(Z) = 0$ is devoid of singular curves" and hence, by the following Normality Theorem 3, we see that $S = S_1[I]$ where I is a root of F_2'' in L. Since the coefficients of Z^1 and Z^0 in $F_2''(Z)$ belong to $M(R)$ and $M(R)^2$ respectively, it follows that S is a two dimensional nonregular local domain. Since $F_1(Z)$ and $F_2''(Z)$ are irreducible, we also see that L/K is Galois with Galois group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

THEOREM 2. Let $(R_i)_{i=0,1,2,\ldots}$ be a two dimensional quadratic sequence with $R_0 = R$, and let S_i be the integral closure of R_i in L. Then we have the following.

(2.1) If $\mathrm{char}(K) \neq 2$ and R is pseudogeometric, then S_i is a two dimensional semilocal regular domain for infinitely many i.

(2.2) If $n = 1$ with R pseudogeometric and $R/M(R)$ algebraically closed, then S_i is a two dimensional semilocal regular domain for infinitely many i.

(2.3) If $\mathrm{char}(K) = 2 = n$ with F_1 and F_2 as in Lemma (1.3) and for $i = 0, 1, 2, \ldots$ we have $R_i = R[X/Y^i]_{P_i}$ where P_i is the prime ideal in $R[X/Y^i]$ generated by X/Y^i and Y, then S_i is a two dimensional nonregular local domain for every i, and L/K is Galois with galois group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

PROOF. To prove (2.1), by applying the following Total Embedded Curve Resolution Theorem 4 to "the plane curve $C = 0$," for all sufficiently large i we can write $C = DX_i^rY_i^s$, with $D \in R_i \setminus M(R_i)$ and nonnegative integers r, s, where $M(R_i) = (X_i, Y_i)R_i$. This amounts to writing $C_j = D_jX_i^{r_j}Y_i^{s_j}$ for all $j \in J$ with $D_j \in R_i \setminus M(R_i)$ and nonnegative integers r_j, s_j. Now we are done by Lemma (1.1).

To prove (2.2), in view of Lemma (1.2) and Theorem (2.1), it suffices to show that, assuming $\mathrm{char}(K) = 2$ with R pseudogeometric and $R/M(R)$ algebraically closed, given any irreducible $F(Z) = Z^2 + AZ + B \in R[Z]$, for infinitely many i there exists $0 \neq \alpha_i \in K$ and $\beta_i \in K$ such that for $F_i'(Z) = \alpha_i^{-2}F(\alpha_iZ + \beta_i)$ we have

$F_i'(Z) \in R_i[Z]$ with $\mathrm{ord}_{R_i} F_i'(Z) = 1$. But this is Abhyankar's Thesis Theorem 5 cited below.

To prove (2.3) note that, for every i, relative to the basis $(X/Y^i, Y)$ of $M(R_i)$ we have

$$A_1 = (X/Y_i)Y^{a+i} \quad \text{and} \quad A_2 = (X/Y_i)Y^{a+i}E$$

with $0 \neq E \in M(R_i)$ and positive integer $a+i$. So we are done by Lemma (1.3).

NORMALITY THEOREM 3. This refers to the well-known theorem which says that if N is a nonzero nonunit irreducible element in a regular local domain Q such that, for every height one prime ideal P in $Q/(NQ)$, the localization of $Q/(NQ)$ at P is regular, then $Q/(NQ)$ is normal; for instance see (Q15)(T69), (Q15)(T70), (Q19)(T86), and (Q19)(T88) of Lecture L5 of [**A09**]. In our case $Q =$ the localization of $R[Z]$ at the maximal ideal generated by $M(R)$ and Z, and $N = F_2''(Z)$.

TOTAL EMBEDDED CURVE RESOLUTION THEOREM 4. In (10.7) on page 44 of [**A07**] and again in (5.12) on page 1595 of [**A08**] it is proved that if C is any nonzero element in a two dimensional pseudogeometric local domain R and $(R_i)_{i=0,1,2,\dots}$ is any two dimensional quadratic sequence with $R_0 = R$ then for all large enough i we have $C = DX_i^rY_i^s$, with $D \in R_i \setminus M(R_i)$ and nonnegative integers r, s, where $M(R_i) = (X_i, Y_i)R_i$.

ABHYANKAR'S THESIS THEOREM 5. See §8 and §9 of [**A01**], Proposition 10 of [**A05**], and Theorems 1 to 12 of [**A04**].

3. Global Theory

Let K/k be a two dimensional excellent function field, i.e., K is a finitely generated field extension of the quotient field of an excellent domain k such that the transcendence degree of the said extension plus the (Krull) dimension of k equals two. In [**A05**] and [**A06**] it was shown that then there exists a nonsingular projective model of K/k and moreover, after applying a finite number of successive quadratic transformations to such a model, it can be made to dominate any given projective model of K/k. For the case of algebraically closed ground fields, this was proved in [**Z01**] for zero characteristic and in [**A01**] for nonzero characteristic.

Note that if V' is a model of K/k which is obtained by applying a finite number of successive quadratic transformations to a nonsingular projective model V of K/k then V' is again a nonsingular projective model of K/k. We call V' an iterated quadratic transform of V. Note that in applying a quadratic transformation to V we are permitted to simultaneously blow up a "finite number of points of V." Also note that, since we have adopted the model view point, a "point" of V" actually means a two dimensional regular local domain R whose residue field is $R/M(R)$.

THEOREM 6. For any two dimensional excellent function field K/k we have the following.

(6.1) If $\mathrm{char}(K) \neq 2$ then, given any nonsingular projective model V of K/k and any finite Galois extension L/K whose Galois group is the direct sum of a finite number of copies of a cyclic group of order 2, there exists an iterated quadratic transform V' of V such that the normalization W' of V' in L is a nonsingular projective model of L/k.

(6.2) If $\operatorname{char}(K) = 2$ then, given any nonsingular projective model V of K/k such that the residue field of any point of it is algebraically closed and given any algebraic field extension L/K of degree 2, there exists an iterated quadratic transform V' of V such that the normalization W' of V' in L is a nonsingular projective model of L/k.

(6.3) If $\operatorname{char}(K) = 2$ with algebraically closed ground field k such that K/k has a minimal model then there exists a Galois extension L/K, with Galois group a direct sum of 2 copies of a cyclic group of order 2, such that there does not exist any nonsingular projective model V' of K/k whose normalization W' in L is a nonsingular projective model of L/k.

PROOF. In case of (6.1) and (6.2) let V be the given nonsingular projective model of K/k, and in case of (6.3) let V be the minimal model of K/k. In a moment we shall construct a sequence of nonsingular projective models $(V_i)_{i=0,1,2,\dots}$ of K/k, with $V_0 = V$, such that V_i is an iterated quadratic transform of V_{i-1} for all $i > 0$. In case of (6.1) and (6.2) let L/K be the given Galois extension, and in case of (6.3) let L/K be the Galois extension with Galois group a direct sum of 2 copies of a cyclic group of order 2, which is to be constructed.

In all the cases let $(W_i)_{i=0,1,2,\dots}$ be the sequence of projective models of L/k such that W_i is the normalization of V_i in L for all $i \geq 0$. Let $H(W_i)$ be the set of all singular points of W_i. Let $G(V_i)$ be the set of all those points R of V_i for which there is a point S of $H(W_i)$ such that S dominates R. For each $i \geq 0$, since W_i is a "normal surface," $H(W_i)$ is a finite set, and hence so is $G(V_i)$. For each $i > 0$, we decree that V_i be the quadratic transform of V_{i-1} obtained by quadratically blowing up $G(V_{i-1})$. Note that for each $i > 0$, clearly $G(V_i)$ is contained in the inverse image of $G(V_{i-1})$ under the domination map $V_i \to V_{i-1}$. What we need to show is that, in case of (6.1) and (6.2), $G(V_i)$ is empty for all large enough i and, in case of (6.3), $G(V_i)$ is nonempty for all i.

If $G(V_i)$ is nonempty for all i then we can take a point R_i in $G(V_i)$ such that R_i is a two dimensional quadratic transform of R_{i-1} for all $i > 0$. In case of (6.1) and (6.2) this is impossible by (2.1) and (2.2) respectively.

In case of (6.3) let R be any point of V, let X, Y be any generators of $M(R)$, let F_1 and F_2 be as in Lemma (1.3), let L be a splitting field of F_1F_2 over K, and for $i = 0, 1, 2, \dots$ let $R_i = R[X/Y^i]_{P_i}$ where P_i is the prime ideal in $R[X/Y^i]$ generated by X/Y^i and Y. Then by (2.3) we know that L/K is Galois with Galois group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and, for all i, the integral closure S_i of R_i in L is a two dimensional nonregular local domain for every i. It follows that, for all i, the point R_i belongs to $G(V_i)$ and hence $G(V_i)$ is nonempty.

4. Problems

PROBLEM 7. In (6.2), how far can you remove the assumption that the residue fields are algebraically closed.

PROBLEM 8. In (6.3), how far can you remove the assumption that K/k has a minimal model.

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