ON THE COMPOSITUM OF WILDLY RAMIFIED EXTENSIONS

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Abstract. The ramification filtration on the compositum of two wildly ramified extensions are computed in various cases. Some positive results towards Abhyankar's Inertia conjecture have been also proved.

1. Introduction

The higher ramification filtration of a wildly ramified extension contains vital information about the extension. For instance the degree of the different is encoded in this data (Hilbert's different formula). This in turn helps in computing the genus of a (wildly ramified) cover of a given curve. For more details see [Pri] where Galois action on ramification filtration is also studied. But computing ramification filtration is a difficult task. For cyclic extensions, these computations can be found in [Gar]. In this manuscript, we compute the ramification filtration for the compositum of two p-cyclic and p^2-cyclic extensions in a few of cases (Proposition 3.1, 3.2, 3.3).

For an algebraically closed field k of characteristic p in [Ha1], Harbater showed that every Galois cover of the local field $k((x))$ with Galois group a p-group P can be extended to a P-cover of $\mathbb{P}^1_k$ branched only at $x = 0$. This was extended by Katz to all finite Galois covers of $k((x))$ in [Kat]. We will call such covers of $\mathbb{P}^1$ as the Harbater-Katz-Gabber cover associated to the local cover.

Let G be a quasi-p group, i.e. a group generated by its Sylow-p subgroups, and $I \leq G$ be such that $I = P \rtimes \mathbb{Z}/n\mathbb{Z}$ where P is a p-group whose conjugates generate G and $(n, p) = 1$. Abhyankar's Inertia conjecture asserts that there exists a G-Galois cover $X \to \mathbb{P}^1_k$ branched only at ∞ such that the inertia group at a point of X lying above ∞ is I. It is easy to see that the inertia group at any ramified point of a Galois cover of $\mathbb{P}^1_k$ branched only at ∞ has the above mentioned property. For a pair (G, I) as above, we will say that (G, I) is realizable if Abhyankar's Inertia conjecture is true for (G, I). This conjecture is largely open though there are some results in support of the conjecture. For instance Harbater in [Ha1] showed that if (G, P) is realizable for a p-subgroup P of G and Q is a p-subgroup of G containing P then (G, Q) is realizable. There are some positive results for specific Galois groups, for instance, see [BP], [MP] and [Obu].

It is shown in Corollary 4.6 that if the inertia conjecture holds for every p-subgroup of a quasi-p group G then the inertia conjecture holds for every p-group of $G \times P$ for any p-group P under the hypothesis that there is no epimorphism from G to $\mathbb{Z}/p\mathbb{Z}$. It is also shown in Theorem 4.8 that if (G, P) is realizable where P is a p-subgroup of G then $(G \times \mathbb{Z}/p\mathbb{Z}, Q)$ is also realizable for any index p-subgroup Q of $P \times \mathbb{Z}/p\mathbb{Z}$ such that the projection of Q on P and $\mathbb{Z}/p\mathbb{Z}$ are surjective. As a

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consequence it is shown that if the inertia conjecture holds for every p-subgroup of G then the same is true for $G \times (\mathbb{Z}/p\mathbb{Z})^n$ (Corollary 4.9). Finally in Theorem 4.10 we assume that there is no epimorphism from $G \to \mathbb{Z}/p\mathbb{Z}$. We show that if (G, P) is realizable where P is any p-subgroup of G with an epimorphism $a : P \to \mathbb{Z}/p^r\mathbb{Z}$ then $(G \times \mathbb{Z}/p^r\mathbb{Z}, Q)$ is realizable where Q is any index p subgroup of $P \times_{\mathbb{Z}/p^{r-1}\mathbb{Z}} \mathbb{Z}/p^r\mathbb{Z}$ such that the projection of Q on the two factors are surjective. Also in this case Q is isomorphic to P and the ramification filtration on Q for the new cover can be computed explicitly in terms of the ramification filtration on P for the given G-cover. These results use the computation of the ramification filtration of the compositum of local field extensions, the existence of Harbater-Katz-Gabber covers and the results of [Kum].

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2. Ramification filtration and Artin-Schreier-Witt theory

Let L/K be a Galois extension of local fields with Galois group G. Let v_K and v_L denote the valuation associated to K and L respectively, with the value group $\mathbb{Z}$. As in [Ser], define a decreasing filtration on G by

$$G_i = \{\sigma \in G : v_S(\sigma x - x) \geq i + 1, \forall x \in S\}.$$

Note that $G_{-1} = G$ and G_0 is the called inertia subgroup of G. This filtration is called the lower (numbering) ramification filtration. For every i, G_i is a normal subgroup of G. One extends this filtration to the real line as follows: for $u \in \mathbb{R}, u \geq -1$, let m be the smallest integer such that $m \geq u$ then $G_u = G_m$. The upper (numbering) ramification filtration on G is defined as $G^v = G_{\psi(v)}$ where ψ is the inverse of the Herbrand function ϕ given by

$$\phi(v) = \int_0^v \frac{du}{[G : G_u]}.$$

Note that ϕ is a bijective piece-wise linear function. Let $G^{v+} = \cup_{\epsilon>0} G^{v+\epsilon}$.

A number $u \geq 0$ (respectively $l \geq 0$) is called an upper jump (respectively a lower jump) of the ramification filtration of G if $G^u \neq G^{u+}$ (respectively $G_l \neq G_{l+}$). Let $u_1, \ldots, u_r$ be the upper jumps, $l_1, \ldots, l_r$ be the lower jumps and $s_i = [G : G^{u_i}] = [G : G_{l_i}]$ for $1 \leq i \leq r$. Note that G_0 is a p-group iff $s_1 = 1$, i.e. L/K is purely wildly ramified.

Remark 2.1. A straight forward computation shows that if G_0 is a p-group and if we set $l_0 = u_0 = 0$ then for $i \geq 1$,

$$u_i = \sum_{j=1}^{i} \frac{l_j - l_{j-1}}{s_j} \qquad \text{and} \qquad l_i = \sum_{j=1}^{i} (u_j - u_{j-1}) s_j.$$

Remark 2.2. Note that for $i \geq 0$, $G_i = G$ iff $G^i = G$. Since $\phi(v) \leq v$, $G_v = G^{\phi(v)} \supset G^v$. This explains the "if part". The "only if" is true because for $v \leq i$, $\phi(v) = v$ and hence $\psi(v) = v$.

Lemma 2.3. *Let L/K be a finite Galois extension of local fields with the Galois group G and H be a normal subgroup of G.*

(1) Let $u_1, \dots, u_r$ be upper jumps of G/H then $u_1, \dots, u_r$ are also upper jumps of G.
(2) If $H = G_i$ for some i, and $l_1, \dots, l_r$ are lower jumps of H then $l_1, \dots, l_r$ are also lower jumps of G.

Proof. The statement (1) follows immediately from [Ser, IV, 1, Proposition 14] by noting that if $G^{u+}H/H \subsetneq G^u H/H$ then $G^{u+} \subsetneq G^u$. The statement (2) is a consequence of [Ser, IV, 1, Proposition 2]. □

Corollary 2.4. *Let L/K and M/K be Galois extensions of local fields with the upper jumps of their Galois groups $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots v_m\}$ respectively. Let N be the cardinality of the set $U \cup V$. If $[LM : K] = p^N$ then $U \cup V$ is the set of all the upper jumps of* $\mathrm{Gal}(LM/K)$.

Proof. Let $G = \mathrm{Gal}(LM/K)$. Applying the above lemma with H as $\mathrm{Gal}(LM/L)$ and $\mathrm{Gal}(LM/M)$ we see that all the elements of $U \cup V$ are upper jumps of G. Since $[LM : K] = p^N$, G has at most N upper jumps. □

Corollary 2.5. *Let L/K and M/K be Galois extensions of local fields with the Galois groups G and H respectively. Assume that the upper jumps of G and H are $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots v_m\}$ respectively. If U and V are disjoint then $U \cup V$ is the set of all upper jumps of* $\mathrm{Gal}(LM/K) = G \times H$. *In fact the upper ramification filtration on $G \times H$ is given by $(G \times H)^u = G^u \times H^u$.*

Proof. Note that since U and V are disjoint, L and M are linearly disjoint over K. We will prove the result by induction on $[LM : K]$. By Lemma 2.3 it is clear that the elements of $U \cup V$ are upper jumps of $\Gamma = \mathrm{Gal}(LM/K) = G \times H$. Without loss of generality we may assume that $u_n > v_m$. Note that $\{e\} = H^{u_n} = (\Gamma/G)^{u_n} = \Gamma^{u_n}G/G$. So we have $\Gamma^{u_n} \subset G$ and hence $G^{u_n} = (\Gamma^{u_n}H)/H = \Gamma^{u_n}$. Let $L_1 = L^{G^{u_n}}$. Then the extension L_1/K is Galois with Galois group G/G^{u_n} and the upper jumps are $\{u_1, \dots, u_{n-1}\}$. By induction hypothesis, the upper jumps of L_1M/K are precisely $U \cup V \setminus \{u_n\}$ and $(\Gamma/G^{u_n})^u = (G/G^{u_n})^u \times H^u$. Moreover $L_1M = (LM)^{\Gamma^{u_n}}$, hence u_n is the only other upper jump of Γ. This proves that $U \cup V$ is the set of all upper jumps of Γ.

Since H^{u_n} is trivial, $\Gamma^u = G^u \times H^u$ for $u \geq u_n$. For $u < u_n$, note that

$$\Gamma^u/G^{u_n} = (\Gamma/G^{u_n})^u = (G/G^{u_n})^u \times H^u = (G^u \times H^u)/G^{u_n}.$$

Hence $\Gamma^u = G^u \times H^u$ for $u < u_n$ as well. □

Now on it is assumed that the fields under consideration are of characteristic $p > 0$. For a ring R of characteristic p, $W_r(R)$ denotes the ring of Witt vectors of length r over R. Let $F : W_r(R) \to W_r(R)$ sending $(a_0, \dots, a_{r-1})$ to $(a_0^p, \dots, a_{r-1}^p)$ be the Frobenius homomorphism and let $\mathscr{P}$ denote the group endomorphism $F - Identity$ of $(W_r(R), +)$.

Definition 2.6. Let L/K be a field extension. We say an element $\alpha \in L \setminus K$ is an *AS-element* of L/K if $\alpha^p - \alpha \in K$. Moreover if (K, v_K) is a local field, L/K is totally ramified and $v_K(\alpha^p - \alpha)$ is coprime to p then α is said to be a *reduced AS-element*. Also $\alpha^p - \alpha$ will be called a reduced element of K.

Definition 2.7. Let $(a_0, \dots, a_{n-1}) \in W_n(K)$. We will say that L/K is a field extension corresponding to $(a_0, \dots, a_{n-1})$ if there exist $(\alpha_0, \dots, \alpha_{n-1}) \in W_n(L)$ such that $\mathscr{P}(\alpha_0, \dots, \alpha_{n-1}) = (a_0, \dots, a_{n-1})$ and $L = K(\alpha_0, \dots, \alpha_{n-1})$.

Remark 2.8. Let L/K be a totally ramified extension of local fields with perfect residue field. If α is an AS-element of L/K then there exist $x \in K$ such that $\alpha - x$ is a reduced AS-element of L/K. In fact more generally, for any $(a_0, \ldots, a_{n-1}) \in W_n(K)$ there exist $(a'_0, \ldots, a'_{n-1}) \in W_n(K)$ such that $v_K(a'_i)$ is coprime to p for all i and $(a_0, \ldots, a_{n-1}) -_w (a'_0, \ldots, a'_{n-1}) = \mathscr{P}(x_0, \ldots, x_{n-1})$ for some $(x_0, \ldots, x_{n-1}) \in W_n(K)$ ([Sch], [Tho, Proposition 4.1]). We will say that the Witt vector $(a'_0, \ldots, a'_{n-1})$ is *reduced* if $v_K(a'_i)$ is coprime to p for all i.

Lemma 2.9. *Let L/K be a p^r-cyclic totally ramified extension of local fields with perfect residue field. Then there exists a Witt vector $(a_0, a_1, \ldots, a_{r-1}) \in W_r(L)$ such that $\mathscr{P}(a_0, a_1, \ldots, a_{r-1}) = (\alpha_0, \ldots, \alpha_{r-1}) \in W_r(K)$ is reduced and $L = K(a_0, a_1, \ldots, a_{r-1})$. Let $n_i = -v_K(\alpha_i)$. Then the upper jumps of L/K is n_0, $\max\{n_0p, n_1\}$, $\max\{n_0p^2, n_1p, n_2\}$, $\ldots$, $\max\{n_ip^{r-1-i-1} : 0 \le i \le r-2\}$ and $\max\{n_ip^{r-i-1} : 0 \le i \le r-1\}$.*

Proof. The first statement follows from Artin-Schreier-Witt theory and Remark 2.8. Note that $n_0, n_1, \ldots, n_{r-1}$ are coprime to p. The conclusion on the upper jumps is the content of [Gar, Theorem 1.1] applied to the Witt vectors a_0, (a_0, a_1), $\ldots$, $(a_0, a_1, \ldots, a_{r-2})$ and $(a_0, a_1, \ldots, a_{r-1})$. $\square$

Definition 2.10. Let L/K be a compositum of Artin-Schreier extensions. A subset $\{\alpha_1, \ldots, \alpha_n\} \subset L \setminus K$ is an *AS-generating set* of L/K if $\alpha_1, \ldots, \alpha_n$ are AS-elements and $L = k(\alpha_1, \ldots, \alpha_n)$. Moreover the above set will be called an *AS-basis* if $[L : K] = p^n$.

Lemma 2.11. *Let $L = K(\alpha_1, \ldots, \alpha_n)$ be a compositum of Artin-Schreier extensions, where $\alpha_i \in \bar{K}$, $f_i = \alpha_i^p - \alpha_i \in K$ for all i and $\{\alpha_1, \ldots, \alpha_n\}$ is an AS-basis of L/K. Let $\gamma \in L$ be such that $\gamma^p - \gamma \in K$ then $\gamma = a_1\alpha_1 + \ldots a_n\alpha_n + x$ for some $a_1, \ldots, a_n \in \mathbb{F}_p$ and $x \in K$.*

Proof. Note that $G = \mathrm{Gal}(L/K) = (\mathbb{Z}/p\mathbb{Z})^n$. By Galois theory $\mathrm{Gal}(L/K(\gamma))$ is an index p subgroup of G. Let $\gamma(a_1, \ldots, a_n) = a_1\alpha_1 + \ldots + a_n\alpha_n \in L$ where $a_1, \ldots, a_n \in \mathbb{F}_p$. Then $\gamma(a_1, \ldots, a_n)^p - \gamma(a_1, \ldots, a_n) \in K$. Moreover if some $a_i \neq 0$ then $\gamma(a_1, \ldots, a_n) \in L \setminus K$ because $1, \alpha_1, \ldots \alpha_n$ are linearly independent over K. Hence $\mathrm{Gal}(L/K(\gamma(a_1, \ldots, a_n)))$ is an index p-subgroup of G if $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$. From Artin-Schreier theory, we know that for $b_1, \ldots, b_n \in \mathbb{F}_p$, $K(\gamma(a_1, \ldots, a_n)) = K(\gamma(b_1, \ldots, b_n))$ iff

$$\gamma(a_1, \ldots, a_n)^p - \gamma(a_1, \ldots, a_n) = c(\gamma(b_1, \ldots, b_n)^p - \gamma(b_1, \ldots, b_n)) + x^p - x$$

for some $c \in \mathbb{F}_p$ and $x \in K$. Equivalently, $a_1f_1 + \ldots + a_nf_n = c(b_1f_1 + \ldots + b_nf_n) + x^p - x$. This is equivalent to $x^p - x = (a_1 - cb_1)f_1 + \ldots + (a_n - cb_n)f_n$. Since for each i, $K(\alpha_i)$ and $K(\alpha_j | 1 \le j \le n, j \neq i)$ are linearly disjoint over K, no nontrivial $\mathbb{F}_p$-linear combination of $f_1, \ldots, f_n$ is of the form $x^p - x$. Hence $x^p - x = (a_1 - cb_1)f_1 + \ldots + (a_n - cb_n)f_n$ is equivalent to $(a_1, \ldots, a_n) = c(b_1, \ldots, b_n)$.

So we have shown that there are $(p^n - 1)/(p - 1)$ many distinct p-cyclic intermediate extensions of the form $K(\gamma(a_1, \ldots, a_n))/K$ of L/K. Also there are $(p^n - 1)/(p - 1)$ subgroups of index p in G. Hence by Galois theory $K(\gamma) = K(\gamma(a_1, \ldots, a_n))$ for some $a_1, \ldots, a_n$ not all zero. But this implies $\gamma^p - \gamma = c(\gamma(a_1, \ldots, a_n)^p - \gamma(a_1, \ldots, a_n)) + x^p - x$ for some $c \in \mathbb{F}_p$ and $x \in K$. Simplifying, we obtain

$$(\gamma - (ca_1\alpha_1 + \ldots + ca_n\alpha_n) - x)^p - (\gamma - (ca_1\alpha_1 + \ldots + ca_n\alpha_n) - x) = 0$$

Hence $\gamma = (ca_1\alpha_1 + \ldots + ca_n\alpha_n) + x + d$ for some $d \in \mathbb{F}_p$. □

3. Local theory

The fields considered in this section are local fields of characteristic p with perfect residue field.

Proposition 3.1. *Let L and M be distinct totally ramified Artin-Schreier extensions of K corresponding to f and g respectively, where $f, g \in K$ are reduced. Let i be the upper jump of L/K and j be the upper jump of M/K. The ramification filtration on the Galois group $G = \mathrm{Gal}(LM/K)$ is given as follows:*

1. *If $i < j$ then the upper jumps are i and j with $G^i = G = (\mathbb{Z}/p\mathbb{Z})^2$ and $G^j = \mathbb{Z}/p\mathbb{Z}$.*
2. *LM/K is not totally ramified iff for some $a \in \mathbb{F}_p$ and $x \in K$, $f + ag + x^p - x \in k$. In this case, $i = j$, the inertia group is $\mathbb{Z}/p\mathbb{Z}$ and the only upper jump is at i.*
3. *If $i = j$ and LM/K is totally ramified then there are two cases. If $l = -v(f + ag + x^p - x) < i$ for some $a \in \mathbb{F}_p$ and $x \in K$ then $l \geq 1$ and the upper jumps are l and i with $G^l = G = (\mathbb{Z}/p\mathbb{Z})^2$ and $G^i = \mathbb{Z}/p\mathbb{Z}$. Other wise there is only one upper jump at i and $G^i = G = (\mathbb{Z}/p\mathbb{Z})^2$.*

Proof. Let $\alpha, \beta \in \bar{K}$ be such that $\alpha^p - \alpha = f$ and $\beta^p - \beta = g$. Since the upper jump of L/K is i, $v(f) = -i$. Similarly, $v(g) = -j$.

Note that (1) follows from Lemma 2.9 and Corollary 2.4.

For (2) note that if $f + ag + x^p - x \in k$ for some $a \in \mathbb{F}_p$ and $x \in K$ then $K(\alpha + a\beta)/K$ is an unramified extension of K. Hence LM/K is not totally ramified. Conversely, if LM/K is not totally ramified then there exist $\gamma \in LM \setminus K$ such that $\gamma^p - \gamma \in k$, since $\mathrm{Gal}(LM/K) = (\mathbb{Z}/p\mathbb{Z})^2$ and K is a local field of characteristic p. Now by Lemma 2.11, $\gamma = a\alpha + b\beta + x$ for some $x \in K$ and $a, b \in \mathbb{F}_p$. So $\gamma^p - \gamma = a(\alpha^p - \alpha) + b(\beta^p - \beta) + x^p - x \in k$. Hence $af + bg + x^p - x \in k$. If $a \neq 0$ then dividing by a we get $f + a'g + x'^p - x' \in k$ for some $a' \in \mathbb{F}_p$ and $x' \in K$. Otherwise $\gamma = b\beta + x \notin K$, so $b \neq 0$. Hence $v(\gamma^p - \gamma) = v(bg + x^p - x) < 0$. This contradicts $\gamma^p - \gamma \in k$. The rest of (2) follows.

For (3) note that if $i = j$ and $l = -v(f + ag + x^p - x) < i$ for some $a \in \mathbb{F}_p$ and $x \in K$ then $l \geq 1$. This is because if $l = 0$ then LM/K will not be totally ramified. Let $h = f + ag + x^p - x$ and γ be such that $\gamma^p - \gamma = h$ then $LM = LK(\gamma)$. So we are reduced to (1). Hence the upper jumps are l and i.

Finally in the remaining scenario, if $\gamma \in LM$ is such that $\gamma^p - \gamma \in K$ and $(v(\gamma^p - \gamma), p) = 1$ then in view of Lemma 2.11, $v(\gamma^p - \gamma) = -i$. Hence by [Kum, Proposition 2.7] the first upper jump for LM/K is i. But the highest upper jump for LM/K is the maximum of upper jumps for L/K and M/K. Hence the only upper jump is i. □

We will now consider compositum of two p^2-cyclic extensions.

Proposition 3.2. *Let L and M be linearly disjoint totally ramified p^2-cyclic extensions of K corresponding to the reduced Witt vectors (α_0, α_1) and (β_0, β_1) in $W_2(K)$. Let the upper jumps of L/K and M/K be u_1, u_2 and v_1, v_2 respectively. Let the upper jumps of $G = \mathrm{Gal}(LM/K)$ be $w_1, w_2, \ldots$. The following holds:*

(1)

$$(u_1,u_2)=\begin{cases}(-v_K(\alpha_0),-v_K(\alpha_0)(p-1)), & \text{if } v_K(\alpha_1)\geq v_K(\alpha_0)p\\ (-v_K(\alpha_0),-v_K(\alpha_1)), & \text{otherwise.}\end{cases}$$

$$(v_1,v_2)=\begin{cases}(-v_K(\beta_0),-v_K(\beta_0)(p-1)), & \text{if } v_K(\beta_1)\geq v_K(\beta_0)p\\ (-v_K(\beta_0),-v_K(\beta_1)), & \text{otherwise.}\end{cases}$$

(2) If $u_1 \neq v_1$ then $w_1 = \min(u_1, v_1)$. Moreover if u_1, u_2, v_1, v_2 are all distinct then these are the four upper jumps of G.

(3) Suppose $u_1 = v_1$. If there exist $c \in \mathbb{F}_p$ and $x \in K$ such that $l = -v_K(\alpha_0 + c\beta_0 + x^p - x) < u_1$ then $w_1 = l$ otherwise $w_1 = u_1$. Moreover, if $u_2 \neq v_2$ then l, u_1, u_2, v_2 are the only upper jumps of G in the first case and $w_1 = u_1, u_2, v_2$ are the only upper jumps of G with $G^{w_1+} = (\mathbb{Z}/p\mathbb{Z})^2$ in the latter case.

Proof. Statement (1) is a direct consequence of Lemma 2.9. Without loss of generality, we may assume $u_1 \leq v_1$.

Note that $G = \mathrm{Gal}(L/K)\times\mathrm{Gal}(M/K)$. Let H_1 and H_2 be the p-cyclic subgroups of $\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(M/K)$ respectively. Let $(a_0, a_1) \in W_2(L)$ and $(b_0, b_1) \in W_2(M)$ be such that $\mathscr{P}(a_0, a_1) = (\alpha_0, \alpha_1)$ and $\mathscr{P}(b_0, b_1) = (\beta_0, \beta_1)$. Note that $L^{H_1} = K(a_0)$ and $M^{H_2} = K(b_0)$.

If $u_1 \neq v_1$ then the upper jumps of $\mathrm{Gal}(K(a_0, b_0)/K) = G/(H_1\times H_2)$ are u_1, v_1 by Proposition 3.1. By Lemma 2.3, u_1 is an upper jump of G which implies $w_1 \leq u_1$. Also since u_1 is the first upper jump of $G/(H_1 \times H_2)$ and $(G/(H_1 \times H_2))^{u_1+}$ is non-trivial, we have

$$G^{u_1}(H_1 \times H_2) = G \text{ and}$$
$$H_1 \times H_2 \subsetneq G^{u_1+}(H_1 \times H_2) \subsetneq G.$$

Since w_1 is the first upper jump of G, by [Ser, IV, 2, Corollary 3] G/G^{w_1+} is a group of exponent p. But $G = (\mathbb{Z}/p^2\mathbb{Z})^2$ so $G^{w_1+} \supset H_1 \times H_2$. If $w_1 < u_1$ then $G^{w_1+} \supset G^{u_1}$. But this implies $G^{w_1+} = G$ which contradicts that w_1 is an upper jump of G. So $w_1 = u_1$. The moreover part of the statement (2) follows from Corollary 2.4.

For statement (3), we note that by Proposition 3.1, the upper jumps of $G/(H_1 \times H_2) = \mathrm{Gal}(K(a_0, b_0)/K)$ are l, u_1 if there exist $c \in \mathbb{F}_p$ and $x \in K$ such that $l = -v_K(\alpha_0 + c\beta_0 + x^p - x) < u_1$. So by Lemma 2.3, l, u_1, u_2, v_2 are upper jumps of G. Also since the upper jumps of $G/(H_1 \times H_2)$ are l, u_1, we are in the previous setup. Hence $w_1 = l$. Moreover if $u_1 \neq v_1$ then l, u_1, u_2, v_2 are all distinct. So by Corollary 2.4 these are all the upper jumps of G.

In the case where no such c and x exist, again by Proposition 3.1, the only upper jump of $G/(H_1 \times H_2)$ is u_1. So $(G/(H_1 \times H_2))^{u_1+} = 1$ and $(G/(H_1 \times H_2))^{u_1} = G/H_1 \times H_2$. But this is equivalent to $G^{u_1+} \subset H_1 \times H_2$ and $G^{u_1}(H_1 \times H_2) = G$. Again $G^{w_1+} \supset H_1 \times H_2$ and if $w_1 < u_1$ then $G^{w_1+} \supset G^{u_1}$ which would imply $G^{w_1+} = G$ contradicting that w_1 is an upper jump. Hence $w_1 = u_1$. Also G^{w_1+} has index at most p^2 in G. Moreover u_2 and v_2 are both greater than w_1 and they are upper jumps of G. Hence $G^{w_1+} = H_1 \times H_2$ is exactly of index p^2 and u_1, u_2, v_2 are the only upper jumps of G. □

Proposition 3.3. *Let L and M be totally ramified Galois extensions of K. Assume that $\mathrm{Gal}(L/K) = \mathbb{Z}/p^r\mathbb{Z}$ and $\mathrm{Gal}(M/K) = P$ is a p-group with a normal* *subgroup N such that $P/N \cong \mathbb{Z}/p^r\mathbb{Z}$. Suppose $L \cap M = L \cap M^N$ is a p^{r-1}-cyclic extension of K. Let $(a_0, \ldots, a_{r-2}) \in W_{r-1}(L \cap M)$ be such that $(\alpha_0, \ldots \alpha_{r-2}) = \mathscr{P}(a_0, \ldots, a_{r-2}) \in W_{r-1}(K)$ is a reduced Witt vector and $L \cap M = K(a_0, \ldots, a_{r-2})$. Let $u_1, \ldots, u_r$ and $v_1, \ldots, v_s$ be the upper jumps of L/K and M/K respectively. Then the following holds:*

1. *The Galois group $G = \mathrm{Gal}(LM/K) \cong P \times \mathbb{Z}/p\mathbb{Z}$.*
2. *The upper jumps of $\mathrm{Gal}(L \cap M/K) = \mathbb{Z}/p^{r-1}\mathbb{Z}$ are $u_1, u_2, \ldots, u_{r-1}$.*
3. *Let $a \in L$ and $b \in M$ be such that $L = (L \cap M)(a)$, $M^N = (L \cap M)(b)$ and the vectors $\mathscr{P}(a_0, \ldots, a_{r-2}, a) = (\alpha_0, \ldots, \alpha_{r-2}, \alpha)$ and $\mathscr{P}(a_0, \ldots, a_{r-2}, b) = (\alpha_0, \ldots, \alpha_{r-2}, \beta)$ are reduced Witt vectors in $W_r(K)$. Then $(a - b)^p - (a - b) = \alpha - \beta$. Set $V = -v_K(\alpha - \beta)$. Suppose V is different from v_i for all $1 \leq i \leq s$, then the only upper jumps of G are $v_1, v_2, \ldots, v_s, V$ and the ramification filtration on G is given by $G^w = P^w \times (\mathbb{Z}/p\mathbb{Z})^w$ where $(\mathbb{Z}/p\mathbb{Z})^w = \mathbb{Z}/p\mathbb{Z}$ for $w \leq -v_K(\alpha_1 - c\beta_1)$ and trivial otherwise.*

Proof. Note that $G = \mathrm{Gal}(L/K) \times_{\mathrm{Gal}(L\cap M/K)} \mathrm{Gal}(M/K) \cong P \times_{\mathbb{Z}/p^{r-1}\mathbb{Z}} \mathbb{Z}/p^r\mathbb{Z}$ where $P \to \mathbb{Z}/p^{r-1}\mathbb{Z}$ is the epimorphism corresponding to the field extensions $M/L \cap M/K$. One obtains an explicit isomorphism $P \times_{\mathbb{Z}/p^{r-1}\mathbb{Z}} \mathbb{Z}/p^r\mathbb{Z} \cong P \times \mathbb{Z}/p\mathbb{Z}$ by choosing a section θ of $\mathbb{Z}/p^r\mathbb{Z} \to \mathbb{Z}/p^{r-1}\mathbb{Z}$ and sending $(a, b) \in P \times_{\mathbb{Z}/p^{r-1}\mathbb{Z}} \mathbb{Z}/p^r\mathbb{Z}$ to $(a, b - \theta(\phi(a)))$. Note that $L \cap M = K(a_0, \ldots, a_{r-2})$ is the unique p^{r-1}-cyclic sub-extension of K contained in L. So the only upper jumps of $K(a_0 \ldots, a_{r-2})/K$ are $u_0, \ldots, u_{r-2}$. This proves (2).

Finally for (3), note that $(0, \ldots, 0, \alpha - \beta) = (\alpha_0, \ldots, \alpha_{r-2}, \alpha) -_w (\alpha_0, \ldots, \alpha_{r-2}, \beta)$. But the RHS equals $\mathscr{P}(a_0, \ldots, a_{r-2}, a) -_w \mathscr{P}(a_0, \ldots, a_{r-2}, b)$ which is same as $[(a_0^p, \ldots, a_{r-2}^p, a^p) -_w (a_0^p, \ldots, a_{r-2}^p, b^p)] -_w (0, \ldots, 0, a - b)$. Simplifying further we get, $(a - b)^p - (a - b) = \alpha - \beta$. Hence we obtain that $K(a - b)/K$ is a p-cyclic extension of K and $LM = MK(a - b)$. The hypothesis that V is different from v_i for $1 \leq i \leq s$ implies that the only upper jumps of G are $v_1, v_2, \ldots, v_s$ and V (Corollary 2.5). The last statement also follows from Corollary 2.5. □

Corollary 3.4. *Let the notation and the situation be as in Proposition 3.3(3). Then the Galois group $Q := \mathrm{Gal}(LM/K(a - b))$ is isomorphic to P and the lower jump in the ramification filtration of Q are $l_1, \ldots, l_{i-1}, \hat{l}_i, l_{i+1}, \ldots, l_{s+1}$ where l_j's are as in equation* (3.1) *and the filtration is given by equation* (3.2) *below.*

Proof. Note that $\mathrm{Gal}(M/K) = P$, $\mathrm{Gal}(K(a - b)/K) = \mathbb{Z}/p\mathbb{Z}$, $LM = MK(a - b)$ and $M \cap K(a - b) = K$. Hence the Galois group $Q = \mathrm{Gal}(LM/K(a - b)) \cong P$. The ramification filtration on LM/K is $(P \times \mathbb{Z}/p\mathbb{Z})^w = P^w \times (\mathbb{Z}/p\mathbb{Z})^w$ for any w. Note that $(\mathbb{Z}/p\mathbb{Z})^w = \mathbb{Z}/p\mathbb{Z}$ for $w \leq V$ and trivial otherwise. Let $s_j = [P : P^{v_j}]$ for $1 \leq j \leq s$, $s_{s+1} = |P|$ and set $v_0 = l_0 = 0$. Let i be such that $v_{i-1} < V < v_i$ if $V < v_s$ otherwise set $i = s + 1$. Note that $s_1 = 1$ because P is a p-group. Let $l_1, \ldots, l_{s+1}$ be the lower jumps of $P \times \mathbb{Z}/p\mathbb{Z}$. Using Remark 2.1 we obtain that

$$l_j = \begin{cases} \sum_{h=1}^{j} (v_h - v_{h-1}) s_h \text{ for } 1 \leq j \leq i - 1, \\ l_{i-1} + (V - v_{i-1}) s_i \text{ for } j = i, \\ l_i + (v_i - V) p s_i \text{ for } j = i + 1 \text{ and } i \neq s + 1, \\ l_{i+1} + \sum_{h=i+1}^{j-1} (v_h - v_{h-1}) p s_h \text{ for } i + 1 < j \leq s + 1. \end{cases} \tag{3.1}$$

Finally using Lemma 2.3 one obtains that the lower jumps in the ramification filtration on $Q = \mathrm{Gal}(LM/K(a-b))$ are $l_1, \ldots, l_{i-1}, \hat{l}_i, l_{i+1}, \ldots, l_{s+1}$ with

$$Q_{l_j} = Q \cap (P \times \mathbb{Z}/p\mathbb{Z})_{l_j} = \begin{cases} P^{v_j} \text{ for } 1 \leq j \leq i, \\ P^{v_{j-1}} \text{ for } i+1 \leq j \leq s+1. \end{cases} \tag{3.2}$$

□

4. Global applications

Let G be quasi-p group and $I \leq G$ be a subgroup. Recall that we say the pair (G, I) is realizable if there exists a G-Galois cover $X \to \mathbb{P}^1$ branched only at one point ∞ and the inertia group at a point of X above ∞ is I.

Remark 4.1. Note that if (G, I) is realizable then $I \cong P \rtimes \mathbb{Z}/n\mathbb{Z}$ for n coprime to p and P a p-subgroup whose conjugates in G generate G. This follows from the fact that there are no nontrivial tamely ramified covers of $\mathbb{P}^1$ branched only at ∞.

Let S be a collection of subgroups of G, we will say that the inertia conjecture holds for every subgroup of G which belongs to S if for all $I \in S$ with the property that $I \cong P \rtimes \mathbb{Z}/n\mathbb{Z}$ for some n coprime to p and some p-subgroup P whose conjugates in G generate G, we have (G, I) is realizable. In other words, for all $I \in S$ either (G, I) is realizable or I does not satisfy the hypothesis of Abhyankar's inertia conjecture.

Theorem 4.2. *Suppose (G, I) is realizable by the cover $X \to \mathbb{P}^1$ and let P be a p-group then*

1. *$(G \times P, I \times P)$ is realizable.*
2. *(G, Q) is realizable where Q is any p-subgroup of G containing I_2 if there is no epimorphism from G to any nontrivial quotient of I_2. Here I_2 is the second lower ramification group of the G-cover $X \to \mathbb{P}^1$ at the point $r \in X$ where the inertia subgroup is I.*

Proof. Since (G, I) is realizable, there exists a G-cover $X \to \mathbb{P}^1$ branched only at ∞ and the inertia group at a point $r \in X$ above ∞ is the subgroup I. This implies that the Galois group of the field extension $\mathrm{QF}(\hat{\mathcal{O}}_{X,r})/\mathrm{QF}(\hat{\mathcal{O}}_{\mathbb{P}^1,\infty})$ is I. Since there are infinitely many linearly disjoint P-extension of $\mathrm{QF}(\hat{\mathcal{O}}_{\mathbb{P}^1,\infty})$ [Ha1], there exists a P-extension $\hat{L}/\mathrm{QF}(\hat{\mathcal{O}}_{\mathbb{P}^1,\infty})$ linearly disjoint from $\mathrm{QF}(\hat{\mathcal{O}}_{X,r})/\mathrm{QF}(\hat{\mathcal{O}}_{\mathbb{P}^1,\infty})$. Let $Y \to \mathbb{P}^1$ be the Harbater-Katz-Gabber P-cover $Y \to \mathbb{P}^1$ associated to the P-extension $\hat{L}/\mathrm{QF}(\hat{\mathcal{O}}_{\mathbb{P}^1,\infty})$. Note that $Y \to \mathbb{P}^1$ is linearly disjoint to the cover $X \to \mathbb{P}^1$. Letting U to be the normalization of $X \times_{\mathbb{P}^1} Y$ we note that $U \to \mathbb{P}^1$ is a $G \times P$ cover branched only at ∞. Moreover, the linear disjointness of $\hat{L}$ and $\mathrm{QF}(\hat{\mathcal{O}}_{X,r})$ over $\mathrm{QF}(\hat{\mathcal{O}}_{\mathbb{P}^1,\infty})$ implies that $\mathrm{Gal}(\hat{L}\,\mathrm{QF}(\hat{\mathcal{O}}_{X,r})/\mathrm{QF}(\hat{\mathcal{O}}_{\mathbb{P}^1,\infty})) = I \times P$. By [Kum, Lemma 3.1], the inertia group at the point (r, ∞) of the cover $U \to \mathbb{P}^1$ is $I \times P$.

The second statement is a consequence of [Kum, Theorem 3.7] and [Ha2, Theorem 2]. □

Theorem 4.3. *Suppose there is no epimorphism from G to $\mathbb{Z}/p\mathbb{Z}$ and (G, I) is realizable. Let P and $\tilde{P}$ be p-groups such that P is a quotient of I and $\tilde{P}$. Let Q be the subgroup $I \times_P \tilde{P}$ of $I \times \tilde{P}$. Then $(G \times \tilde{P}, Q)$ is realizable. Moreover, if $\tilde{P} = P$ then $Q \cong I$ and the ramification filtration on Q is same as that of I.*

Proof. As in the previous proof, there exists a G-cover $X \to \mathbb{P}^1$ branched only at ∞ and the inertia group at a point $r \in X$ above ∞ is the subgroup I. The Galois group of the field extension L/K is I where $L = \mathrm{QF}(\hat{\mathcal{O}}_{X,r})$ and $K = \mathrm{QF}(\hat{\mathcal{O}}_{\mathbb{P}^1,\infty}) = k((x^{-1}))$. Let $N \lhd I$ be such that $I/N = P$. Then L^N is a P-Galois extension of K. Let M/L^N be a field extension linearly disjoint from L/L^N such that M/K is Galois with Galois group $\tilde{P}$. Note that such extensions exist because the pro-p part of the absolute Galois group of K is pro-p free of infinite rank ([Sha], also see [MS, Theorem 1]). Let $Y \to \mathbb{P}^1$ be the Harbater-Katz-Gabber cover associated to the local fields extension M/K. So $k(Y)K = M$.

Since there is no epimorphism from $G \to \mathbb{Z}/p\mathbb{Z}$ and $\tilde{P}$ is a p-group, the extensions $k(Y)/k(x)$ and $k(X)/k(x)$ are linearly disjoint. Letting U to be the normalization of $X \times_{\mathbb{P}^1_x} Y$ we note that U is smooth and irreducible. The cover $U \to \mathbb{P}^1_x$ is a $G \times \tilde{P}$-cover branched only at ∞ and the inertia group and the ramification filtration at $r' = (r, \infty_Y) \in U$ is given by the extension of local fields LM/K by [Kum, Lemma 3.1]. Hence the inertia subgroup $Q = \mathrm{Gal}(LM/K) = \mathrm{Gal}(L/K) \times_{\mathrm{Gal}(L\cap M/K)} \mathrm{Gal}(M/K) = I \times_P \tilde{P}$, i.e. (G, Q) is realizable. Moreover, if $\tilde{P} = P$ then Q is isomorphic to $\mathrm{Gal}(L/K) = I$ and $LM = L$. Hence the ramification filtration on Q and I are the same. □

Remark 4.4. Let G and $\tilde{P}$ be as in the above theorem and assume I is a p-sylow subgroup of G. Then (G, I) is realizable. Let $Q \leq I \times \tilde{P}$ be such that $\pi_1(Q) = I$ and $\pi_2(Q) = \tilde{P}$. Then it is clear that the conjugates of Q generate $G \times \tilde{P}$. Also by Goursat's lemma there exist normal subgroups $N_1 \lhd I$ and $N_2 \lhd \tilde{P}$ such that $I/N_1 \cong \tilde{P}/N_2 (= P \text{ say})$ and $Q = I \times_P \tilde{P}$. Hence by the above theorem, (G, Q) is also realizable. Note that there are many subgroups Q of $I \times \tilde{P}$ with $\pi_1(Q) = I$ and $\pi_2(Q) = \tilde{P}$. For instance if I is abelian then the number of such subgroups is at least $|\mathrm{Hom}(\tilde{P}, I)| - 1$.

Corollary 4.5. *Suppose (G, I) is realizable by the cover $X \to \mathbb{P}^1$ and $P \leq G$ be any p-group containing I_2 where I_2 is the second lower ramification subgroup of I associated to this cover. Also assume that there is no epimorphism from G to $\mathbb{Z}/p\mathbb{Z}$. Then $(G \times \mathbb{Z}/p\mathbb{Z}, Q)$ is realizable where $Q = P \times_{\mathbb{Z}/p\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is a subgroup of $P \times \mathbb{Z}/p\mathbb{Z}$ isomorphic to P.*

Proof. It follows from Theorem 4.2 that (G, P) is realizable and Theorem 4.3 implies that $(G \times \mathbb{Z}/p\mathbb{Z}, Q)$ is realizable. □

Corollary 4.6. *Let G be a quasi-p group such that the inertia conjecture holds for every p-subgroup of G and $\tilde{P}$ be any p-group. Also assume that there is no epimorphism from G to $\mathbb{Z}/p\mathbb{Z}$. Then the inertia conjecture holds for every p-subgroup of $G \times \tilde{P}$.*

Proof. Let π_1 and π_2 be the projections of $G \times \tilde{P}$ to G and $\tilde{P}$ respectively. Let Q be a p-subgroup of $G \times \tilde{P}$ such that the conjugates of Q in $G \times \tilde{P}$ generate the whole group. Then the conjugates of $\pi_1(Q)$ in G generate G and $\pi_2(Q) = \tilde{P}$. Hence $(G, \pi_1(Q))$ is realizable by the hypothesis on G and Q is a subgroup of $\pi_1(Q) \times \tilde{P}$ with $\pi_2(Q) = \tilde{P}$. By Goursat's lemma, one obtains that there exist normal subgroups N_1 and N_2 of $\pi_1(Q)$ and $\tilde{P}$ respectively such that $\pi_1(Q)/N_1 \cong \tilde{P}/N_2$ $(= P \text{ say})$ and $Q = \pi_1(Q) \times_P \tilde{P}$. Hence $(G \times \tilde{P}, Q)$ is realizable by Theorem 4.3. □

Corollary 4.7. *Let G be a finite simple quasi-p such that its order is divisible by p but not by p^2 and let P be any p-group. Then the inertia conjecture holds for p-subgroups of $G \times P$. In particular, the inertia conjecture holds for p-subgroups of $PSL_2(\mathbb{F}_p) \times P$ and $A_m \times P$ for $p \leq m < 2p$ and any p-group P.*

Proof. Note that if $G = \mathbb{Z}/p\mathbb{Z}$ then $G \times P$ is a p-group. So the inertia conjecture certainly holds for $G \times P$. If G is not a p-cyclic group then it is a non-abelian simple quasi-p group whose order is not divisible by p^2. So the inertia conjecture holds for p-subgroups of G. Moreover since G is a simple non-abelian group there is no epimorphism from G to $\mathbb{Z}/p\mathbb{Z}$. Hence by the above corollary the inertia conjecture holds for p-subgroups of $G \times P$. □

Theorem 4.8. *Suppose (G, P) is realizable where $P \leq G$ is a p-group and let Q be an index p subgroup of $P \times \mathbb{Z}/p\mathbb{Z} \subset G \times \mathbb{Z}/p\mathbb{Z}$ such that $\pi_1(Q) = P$ where π_1 is the projection from $P \times \mathbb{Z}/p\mathbb{Z}$ to P and the conjugates of Q in $G \times \mathbb{Z}/p\mathbb{Z}$ generate $G \times \mathbb{Z}/p\mathbb{Z}$. Then $(G \times \mathbb{Z}/p\mathbb{Z}, Q)$ is realizable. Moreover, suppose $u_1, \ldots, u_m$ are the upper jumps of P, $u_1 > 1$ and let $s_i = [P : P^{u_i}]$ for $1 \leq i \leq m$. Then the lower jumps of Q are given by $l_1 = 1 + (u_1 - 1)p$, $l_j = l_1 + \sum_{h=2}^{j} (u_h - u_{h-1})ps_h$ for $2 \leq j \leq m$ and $Q_{l_j} = P^{u_j}$ for $1 \leq j \leq m$.*

Proof. Since (G, P) is realizable, there exists a G-cover $X \to \mathbb{P}^1_x$ branched only at $x = \infty$ and the inertia group at a point $r \in X$ above ∞ is the subgroup P. So the Galois group of the field extension $\mathrm{QF}(\hat{\mathcal{O}}_{X,r})/\mathrm{QF}(\hat{\mathcal{O}}_{\mathbb{P}^1,\infty})$ is P. Let $L = \mathrm{QF}(\hat{\mathcal{O}}_{X,r})$ and $K = \mathrm{QF}(\hat{\mathcal{O}}_{\mathbb{P}^1,\infty}) = k((x^{-1}))$.

Note that $\pi_1|_Q$ is an isomorphism and the conjugates of Q generate $G \times \mathbb{Z}/p\mathbb{Z}$ implies that $\pi_2(Q) = \mathbb{Z}/p\mathbb{Z}$ where π_2 is the projection from $P \times \mathbb{Z}/p\mathbb{Z}$ to $\mathbb{Z}/p\mathbb{Z}$. Let $Q' = \ker(\pi_2|_Q)$ and $P' = \pi_1(Q')$. Note that π_1 is an isomorphism and Q' is an index p normal subgroup of Q. Hence P' is an index p normal subgroup of P. Then $L^{P'}$ is an Artin-Schreier extension of K. Let $\alpha \in L$ be a reduced AS-element such that $L^{P'} = K(\alpha)$ and $\beta = \alpha^p - \alpha$. Note that $\beta = c_n x^n + c_{n-1}x^{n-1} + \ldots c_1 x + c_0 + c_{-1}x^{-1} + \ldots$ for some n coprime to p, $c_n, c_{n-1}, \ldots \in k$ and $c_n \neq 0$. Let $\beta' = \beta + cx$ for some nonzero $c \in k$ such that $Z^p - Z - \beta'$ is an irreducible polynomial in $L[Z]$. Let $\alpha' \in \bar{L}$ be such that $\alpha'^p - \alpha' = \beta'$. Then L and $K(\alpha')$ are linearly disjoint over K. Let $Y \to \mathbb{P}^1$ be the p-cyclic Harbater-Katz-Gabber cover associated to the extension of local fields $K(\alpha')/K$ and ∞_Y be the point lying above $x = \infty$. Note that the covers $Y \to \mathbb{P}^1_x$ and $X \to \mathbb{P}^1_x$ are linearly disjoint because $k(Y)K = K(\alpha')$ and $k(X)K = L$.

Let U be the normalization of $X \times_{\mathbb{P}^1_x} Y$. Then U is smooth and irreducible. The cover $U \to \mathbb{P}^1_x$ is a $G \times \mathbb{Z}/p\mathbb{Z}$ cover branched only at $x = \infty$. Moreover by [Kum, Lemma 3.1] the inertia group and the ramification filtration at the point $r' = (r, \infty_Y) \in U$ is given by the extension of local fields $L(\alpha')/K$. So the inertia group is $\mathrm{Gal}(L(\alpha')/K) = P \times \mathbb{Z}/p\mathbb{Z}$. Note that $L(\alpha')^{\ker(\pi_2)} = K(\alpha')$ and $L^{P'} = K(\alpha)$. Hence $L(\alpha')^{Q'} = K(\alpha, \alpha')$. Set $\gamma = \alpha' - \alpha \in L(\alpha')$. Then $\gamma^p - \gamma = cx$. Since $K(\gamma) \subset L(\alpha')$, there is an induced epimorphism on the Galois groups $\phi : P \times \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ such that $L(\alpha')^{\ker \phi} = K(\gamma)$. Note that $\ker(\phi)$ is an index p subgroup of $P \times \mathbb{Z}/p\mathbb{Z}$ such that $\pi_1(\ker(\phi)) = P$ and $\pi_2(\ker(\phi)) = \mathbb{Z}/p\mathbb{Z}$. Since $K(\gamma) \subset K(\alpha, \alpha')$, we have $Q' \subset \ker(\phi)$. Also notice that $Q' \leq P \times 0 \leq P \times \mathbb{Z}/p\mathbb{Z}$ and $(P \times \mathbb{Z}/p\mathbb{Z})/Q' \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Hence after applying an automorphism on the second factor $\mathbb{Z}/p\mathbb{Z}$ if necessary, we obtain that $Q = \ker(\phi)$.

We note that $k(x)(\gamma)$ and $k(U)$ are linearly disjoint over $k(x)$. To prove this, let us assume the contrary. Then $k(x)(\gamma) \subset k(U) = k(X)k(Y)$ which induces an epimorphism on Galois groups $\Phi : G \times \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$. By construction of ϕ and Galois theory $\Phi|_{P\times\mathbb{Z}/p\mathbb{Z}} = \phi$. It follows that the conjugates of $Q = \ker(\phi)$ in $G \times \mathbb{Z}/p\mathbb{Z}$ are contained in $\ker(\Phi)$. This contradicts the assumption on Q.

Let $V \to \mathbb{P}^1_x$ be the p-cyclic cover corresponding to the extension $k(x)(\gamma)/k(x)$ and W be the normalization of $U \times_{\mathbb{P}^1_x} V$. Let $\infty_V \in V$ be the point lying above $x = \infty$ and $r'' = (r', \infty_V)$. By [Kum, Proposition 3.5], the inertia group of the cover $W \to V$ at r'' is Q. Since $k(U)$ and $k(V)$ are linearly disjoint over $k(x)$, we get that W is connected and $\mathrm{Gal}(k(W)/k(V)) = \mathrm{Gal}(k(U)/k(x)) = G \times \mathbb{Z}/p\mathbb{Z}$. Moreover, $W \to V$ is branched only at ∞_V. Finally, since V is isomorphic to $\mathbb{P}^1$, we get that $(G \times \mathbb{Z}/p\mathbb{Z}, Q)$ is realizable.

Note that the inertia subgroup Q is the inertia group of the local field extension $L(\alpha')/K(\gamma = \alpha' - \alpha)$, $\mathrm{Gal}(L/K) = P$ and $\mathrm{Gal}(K(\alpha')/K) = \mathbb{Z}/p\mathbb{Z}$. Since $u_1 > 1 = -v_K(\gamma^p - \gamma)$, none of the upper jumps are equal to 1. So applying Corollary 3.4 with r, s, $v_1, \dots, v_s$ and V in the corollary as 1, m, $u_1, \dots, u_m$ and 1 respectively, we note that $i = 1$ in the notation of the corollary. So by the formula (3.1) in Corollary 3.4 the lower jumps of $L(\alpha')/K$ are $\tilde{l}_i = \tilde{l}_1 = \tilde{l}_0 + (1 - u_0)s_1 = 1$ as $\tilde{l}_0 = u_0 = 0$ and $s_1 = 1$, $\tilde{l}_{i+1} = \tilde{l}_2 = \tilde{l}_1 + (u_1 - 1)ps_1 = 1 + (u_1 - 1)p$ and $\tilde{l}_j = \tilde{l}_2 + \sum_{h=2}^{j-1}(u_h - u_{h-1})$ for $2 < j \leq m + 1$. Hence the upper jumps of Q are $l_1 = \tilde{l}_2 = 1 + (u_1 - 1)p$ and $l_j = \tilde{l}_{j+1} = l_1 + \sum_{h=2}^{j}(u_h - u_{h-1})$ for $2 \leq j \leq m$. □

Corollary 4.9. *Let G be a quasi-p group such that the inertia conjecture holds for every p-subgroup of G. Then the inertia conjecture holds for every p-subgroup of $G \times (\mathbb{Z}/p\mathbb{Z})^n$ for any $n \geq 0$.*

Proof. By induction it is enough to prove for $n = 1$. Let Q be a p-subgroup of $G \times \mathbb{Z}/p\mathbb{Z}$ such that the conjugates of Q generate $G \times \mathbb{Z}/p\mathbb{Z}$. Let π_1 and π_2 be the projection of $G \times \mathbb{Z}/p\mathbb{Z}$ onto G and $\mathbb{Z}/p\mathbb{Z}$ respectively. Let $P = \pi_1(Q)$. Since the conjugates of Q generate $G \times \mathbb{Z}/p\mathbb{Z}$, conjugates of P generate G. So by assumption (G, P) is realizable. If $Q = P \times \mathbb{Z}/p\mathbb{Z}$ then $(G \times Z/p\mathbb{Z}, Q)$ is realizable by Theorem 4.2. Otherwise Q is an index p subgroup of $P \times \mathbb{Z}/p\mathbb{Z}$ and in this case $(G \times Z/p\mathbb{Z}, Q)$ is realizable by Theorem 4.8 as $\pi_1(Q) = P$ and the conjugates of Q generate $G \times \mathbb{Z}/p\mathbb{Z}$. □

Theorem 4.10. *Let P be a p-subgroup of G and suppose that there is no epimorphism from $G \to \mathbb{Z}/p\mathbb{Z}$. Further assume that (G, P) is realizable and that P has a p^r-cyclic quotient $a : P \to \mathbb{Z}/p^r\mathbb{Z}$. Let Q be an index p subgroup of $P \times_{\mathbb{Z}/p^{r-1}\mathbb{Z}} \mathbb{Z}/p^r\mathbb{Z} \subset G \times \mathbb{Z}/p^r\mathbb{Z}$ such that $\pi_1(Q) = P$ and $\pi_2(Q) = \mathbb{Z}/p^r\mathbb{Z}$ where π_1 and π_2 are the projection maps from $P \times_{\mathbb{Z}/p^{r-1}\mathbb{Z}} \mathbb{Z}/p^r\mathbb{Z}$ to P and $\mathbb{Z}/p^r\mathbb{Z}$ respectively. Then $(G \times \mathbb{Z}/p^r\mathbb{Z}, Q)$ is realizable. Moreover, suppose $u_1, \dots, u_m$ are the upper jumps of P, $u_1 > 1$ and $s_i = [P : P^{u_i}]$ for $1 \leq i \leq m$. Then the lower jumps of Q are given by $l_1 = 1 + (u_1 - 1)p$, $l_j = l_1 + \sum_{h=2}^{j}(u_h - u_{h-1})ps_h$ for $2 \leq j \leq m$ and $Q_{l_j} \cong P^{u_j}$ for $1 \leq j \leq m$.*

Proof. As in the previous proof, there exists a G-cover $X \to \mathbb{P}^1$ branched only at ∞ and the inertia group at a point $\tau \in X$ above ∞ is the subgroup P. The Galois group of the field extension L/K is P where $L = \mathrm{QF}(\hat{\mathcal{O}}_{X,\tau})$ and $K = \mathrm{QF}(\hat{\mathcal{O}}_{\mathbb{P}^1,\infty}) = k((x^{-1}))$. Let $q : \mathbb{Z}/p^r\mathbb{Z} \to \mathbb{Z}/p^{r-1}\mathbb{Z}$ be the quotient map. Note that $\pi_1|_Q$ is an isomorphism. Let $Q' = \ker(a \circ \pi_1|_Q)$ and $P' = \pi_1(Q') =$

$ker(a)$. Then $L^{P'}$ is an Artin-Schreier-Witt extension of K corresponding to a reduced Witt vector of length r. Let $(\alpha_0, \alpha_1, \ldots, \alpha_{r-1}) \in W_r(L)$ be such that $\mathscr{P}(\alpha_0, \ldots, \alpha_{r-1}) = (\beta_0, \ldots, \beta_{r-1}) \in W_r(K)$ and $L^{P'} = K(\alpha_0, \ldots, \alpha_{r-1})$. Let $\beta'_{r-1} = \beta_{r-1} + cx$ for some nonzero $c \in k$ such that the Artin-Schreier-Witt extension of K corresponding to the Witt vector $(\beta_0, \ldots, \beta_{r-2}, \beta'_{r-1})$ is a $\mathbb{Z}/p^r\mathbb{Z}$ extension of K different from $K(\alpha_0, \ldots, \alpha_{r-1})$. Let α'_{r-1} be such that $\mathscr{P}(\alpha_0, \ldots, \alpha_{r-2}, \alpha'_{r-1}) = (\beta_0, \ldots, \beta_{r-2}, \beta'_{r-1})$ then $K(\alpha_0, \ldots, \alpha_{r-1})$ and $K(\alpha_0, \ldots, \alpha_{r-2}, \alpha'_{r-1})$ are linearly disjoint over $K(\alpha_0, \ldots, \alpha_{r-2})$. Let $Y \to \mathbb{P}^1$ be the Harbater-Katz-Gabber cover associated to the local field extension $K(\alpha_0, \ldots, \alpha_{r-2}, \alpha'_{r-1})/K$. So $k(Y)K = K(\alpha_0, \ldots, \alpha_{r-2}, \alpha'_{r-1})$.

Since there is no epimorphism from $G \to \mathbb{Z}/p\mathbb{Z}$, the extensions $k(Y)/k(x)$ and $k(X)/k(x)$ are linearly disjoint. Letting U to be the normalization of $X \times_{\mathbb{P}^1_x} Y$ we note that U is smooth and irreducible. The cover $U \to \mathbb{P}^1_x$ is a $G \times \mathbb{Z}/p^r\mathbb{Z}$ cover branched only at ∞ and the inertia group and the ramification filtration at $\tau' = (\tau, \infty_Y) \in U$ is given by the extension of local fields $L(\alpha'_{r-1})/K$. Hence the inertia group is $\mathrm{Gal}(L(\alpha'_{r-1})/K) = P \times_{\mathbb{Z}/p^{r-1}\mathbb{Z}} \mathbb{Z}/p^r\mathbb{Z}$. By Proposition 3.3(3) $-v_K(\beta_{r-1} - \beta'_{r-1}) = 1$ is an upper jump of $\mathrm{Gal}(L(\alpha'_{r-1})/K)$. Moreover, letting $\gamma = \alpha_{r-1} - \alpha'_{r-1}$, we note that $\gamma^p - \gamma = \beta_{r-1} - \beta'_{r-1} \in K$ and $\gamma \notin K$. So by Galois theory, the tower of field extensions $L(\alpha'_{r-1})/K(\gamma)/K$ induces an epimorphism $\phi : P \times_{\mathbb{Z}/p^{r-1}\mathbb{Z}} \mathbb{Z}/p^r\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ such that $L(\alpha'_{r-1})^{\ker(\phi)} = K(\gamma)$. Note that $L(\alpha'_{r-1})^{Q'} = K(\alpha_0, \ldots, \alpha_{r-1}, \alpha'_{r-1})$ hence $Q' \subset \ker(\phi)$. Moreover, note that $K \subset L^{\ker(\phi)} \subset K(\gamma)$ and $\gamma \notin L$, hence $L^{\ker(\phi)} = K$. Similarly $K(\alpha_0, \ldots, \alpha_{r-2}, \alpha'_{r-1})^{\ker(\phi)} = K$. Hence by Galois theory $\pi_1(\ker(\phi)) = P$ and $\pi_2(\ker(\phi)) = \mathbb{Z}/p^r\mathbb{Z}$. Note that both Q/Q' and $\ker(\phi)/Q'$ are index p subgroups of $\mathbb{Z}/p^r\mathbb{Z} \times_{\mathbb{Z}/p^{r-1}\mathbb{Z}} \mathbb{Z}/p^r\mathbb{Z}$ such that the two projections restricted to Q and $\ker(\phi)$ are surjective. Hence there exists an automorphism θ of $\mathbb{Z}/p^r\mathbb{Z}$ with $q \circ \theta = q$ such that $\tilde{\theta}(Q/Q') = \ker(\phi)/Q'$ where $\tilde{\theta}$ is the automorphism induced by θ on $\mathbb{Z}/p^r\mathbb{Z} \times_{\mathbb{Z}/p^{r-1}\mathbb{Z}} \mathbb{Z}/p^r\mathbb{Z}$. Hence we may assume $Q = ker(\phi)$.

Note that $K(\gamma)/K$ is a p-cyclic extension with the lower jump at 1. So the Harbater-Katz-Gabber cover $V \to \mathbb{P}^1_x$ associated to this extension has the property that V is isomorphic to $\mathbb{P}^1$. Also $k(U) = k(X)k(Y)$ and $k(V)$ are linearly disjoint over $k(x)$. To see this, assume the contrary. Then $k(V) \subset k(U)$ and since there are no epimorphism from $G \to \mathbb{Z}/p\mathbb{Z}$, $k(V) \subset k(Y)$. But this implies $\gamma \in K(\alpha_0, \ldots, \alpha_{r-2}, \alpha'_{r-1})$, a contradiction. Let W be the normalization of $U \times_{\mathbb{P}^1_x} V$ then W is smooth and irreducible. Again we apply [Kum, Proposition 3.5] to conclude that $(G \times \mathbb{Z}/p^r\mathbb{Z}, Q)$ is realizable.

The moreover part also follows in the same way as in proof of Theorem 4.8. $\square$

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