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EMBEDDING PROBLEMS FOR OPEN SUBGROUPS OF THE FUNDAMENTAL GROUP

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ABSTRACT. Let C be a smooth irreducible affine curve over an algebraically closed field of positive characteristic and let $\pi_1(C)$ be its fundamental group. We study various embedding problems for $\pi_1(C)$ and its subgroups.

Problèmes d'embedding pour les sous-groupes ouverts du groupe fondamental

RÉSUMÉ. Soit C une courbe affine irrèductible lisse sur un corps algèbriquement fermé de caractéristique positive et soit $\pi_1(C)$ son groupe fondamental. Nous étudions divers problèmes d'embedding pour $\pi_1(C)$ et ses sous-groupes.

1. Introduction

Let C be a smooth irreducible affine curve over an algebraically closed field k of characteristic $p > 0$. The existence of wild ramification causes the structure of the étale fundamental group $\pi_1(C)$ to be complicated. There have been some attempts to understand this group. One step towards this goal was Raynaud's [14] and Harbater's [4] proof of Abhyankar's conjecture which states that a finite group G is a quotient of $\pi_1(C)$ if and only if the maximal prime-to-p quotient $G/p(G)$ of G is generated by $2g+r-1$ elements where g is the genus of the smooth completion of C and r is the number of points in the boundary.

Though this gives a complete description of the finite quotients of $\pi_1(C)$, it does not say how these groups fit together in the inverse system for $\pi_1(C)$.

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A possible way to understand this is by analyzing which finite embedding problems for $\pi_1(C)$ have a solution (see the "Notation" subsection at the end of this introduction for definition). One crucial result in this direction is by Pop [12] (and independently proved by Harbater as well [7, Theorem 5.3.4]) which says that given an embedding problem $\mathcal{E} = (\alpha : \Pi \to G, \phi : \Gamma \to G)$ where $H = \ker(\phi)$ is a quasi-p group, there is a proper solution $\lambda : \Pi \twoheadrightarrow \Gamma$ to $\mathcal{E}$. This clearly strengthens Raynaud's and Harbater's result. When H is not a quasi-p group then it is clear from the Abhyankar's conjecture that the above embedding problem may not have a solution. But there are finite index open subgroups of $\pi_1(C)$ for which these embedding problems have a solution.

Definition 1. — *Given an embedding problem $\mathcal{E} = (\alpha : \pi_1(C) \to G, \phi : \Gamma \to G)$ and a finite index subgroup Π of $\pi_1(C)$, we say the embedding problem* restricts *to Π if $\alpha(\Pi) = G$. Moreover if the restricted embedding problem has a solution then we say Π is* effective *for the embedding problem $\mathcal{E}$.*

In [8], it was shown that given any finite embedding problem for $\pi_1(C)$ there exists a finite index effective subgroup for the embedding problem. In [2, Theorem 1.3] it was shown that it is even possible to find an index p subgroup of $\pi_1(C)$ which is effective for the embedding problem. Our objective is to find some necessary and some sufficient conditions for a subgroup of $\pi_1(C)$ to be an effective subgroup for a given embedding problem.

Suppose Π is a finite index subgroup of $\pi_1(C)$. This corresponds to a cover $D \to C$. Let $Z \to X$ be the corresponding morphism between their smooth completions and let n_D denote $2g_Z + r_D - 1$. A necessary condition for Π to be effective for the embedding problem $\mathcal{E} = (\alpha : \pi_1(C) \to G, \phi : \Gamma \to G)$ is that $\mathcal{E}$ restricts to Π and the rank of $\Gamma/p(\Gamma)$ is at most n_D. Indeed Π is effective implies there is a surjection from $\pi_1(D) = \Pi \twoheadrightarrow \Gamma$. Hence there is a surjection from prime-to-p part of $\pi_1(D)$ to the maximal prime-to-p quotient of Γ, denoted $\Gamma/p(\Gamma)$. So $\Gamma/p(\Gamma)$ is generated by n_D elements. Also, [8, Theorem 5] can be rephrased in terms of sufficient conditions for $\Pi = \pi_1(D)$ to be an effective subgroup of $\pi_1(C)$.

Proposition 2. — *Let $C \subset X$, Π, $\mathcal{E}$, and $D \subset Z$ be as above. Let $\Psi_X : V_X \to X$ be the G-cover corresponding to the embedding problem $\mathcal{E}$ and assume $\mathcal{E}$ restricts to Π. Let $\tilde{\Psi} : V_X \times_X Z \to Z$ be the pull-back of Ψ_X. Then Π is effective if the number of points in $Z \setminus D$ where the morphism $\tilde{\Psi}$ is not branched, is at least the relative rank of* $\ker(\phi : \Gamma \to G)$ *in* Γ *(see "Notation" subsection for definition of relative rank).*

The subset of $Z \setminus D$ for which the morphism Ψ is unramified can be thought of as points available for branching in any cover dominating Ψ. In the theorem above, having sufficiently many available branch points allows the embedding problem to be solved. A natural question is that can the condition on the number of such potential branch points in $Z \setminus D$ be relaxed? For example, could it be replaced by a condition on the genus of Z or n_D? It is also worth noting that in the above proposition the degree of the cover $Z \to X$ must be large to ensure that the number of points in $Z \setminus D$ where the morphism $\tilde{\Psi}$ is not branched is sufficiently large. Hence the index of Π in $\pi_1(C)$ is also large.

The two main results of this paper, Theorem 15 and Proposition 19, use genus and branch points to obtain effective subgroups. Let H be the kernel of the homomorphism $\phi : \Gamma \to G$ from the embedding problem $\mathcal{E} = (\alpha : \pi_1(C) \to G, \phi : \Gamma \to G)$, and let $H/p(H)$ be the maximal prime to p quotient of H. The main tool in this paper is Theorem 5.1 which investigates the relationship between solving embedding problems for $\pi_1(C)$ and the relative rank of $H/p(H)$ in $\Gamma/p(H)$. Roughly speaking, it shows that if the G-Galois cover $\Psi_X : V_X \to X$ corresponding to α has a deformation which is sufficiently degenerate (in terms of having many components that are trivial G covers), then there exists a proper solution to $\mathcal{E}$. Theorem 15 restricts to the case that C is the affine line and Π has index p in $\pi_1(\mathbb{A}^1_k)$. It uses degenerations and Theorem 5.1 to show that if the curve Z in the $\mathbb{Z}/p\mathbb{Z}$-cover $Z \to \mathbb{P}^1_k$ corresponding to $\Pi \subset \pi_1(\mathbb{A}^1_k)$ has genus g_Z greater than the relative rank of (H/pH) in $\Gamma/p(H)$ and a technical condition which holds for most values of g_Z (see Corollary 16, Corollary 17 and Remark 18) then Π is effective for $\mathcal{E}$. In particular these results provide sufficient conditions for subgroups of $\pi_1(C)$ to be effective. In Section 7 some of the results proved for affine line case are generalized to general curves though the conclusions obtained are slightly weaker (Proposition 19 and Corollary 21).

The techniques involved in proving these results include formal patching and deformations within families of covers. Each result requires the construction of a Galois cover of a degenerate curve. Then a formal patching argument is used to obtain a cover of smooth curves over a complete local ring. Finally, a deformation argument similar to [8, Proposition 4] is used to ensure that the original embedding problem $\mathcal{E}$ restricts to the subgroup Π of $\pi_1(C)$ that is obtained in the construction. The paper is organized as follows: At the end of the Introduction there is a list of notation. Section 2 looks at some of the group theoretic properties and examples of effective groups. Section 3 defines deformations and degenerations of covers and uses a formal patching result of Harbater to solve the embedding problems in the presence of a sufficiently degenerate deformation of the original cover. Section 4 proves a globalization and specialization result using a Lefschetz type principle and Abhyankar's lemma. Section 5 applies the formal patching and Lefschetz-Abhyankar result to obtain Theorem 5.1. Finally in Section 6, Theorem 15 is proved, and in Section 7, Proposition 19 and Corollary 21 is proved.

NOTATION. — *If G is a finite group and p is a prime number, let $p(G)$ denote the subgroup of G generated by its p-subgroups. This is a characteristic subgroup of G, and $G/p(G)$ is the maximal prime-to-p quotient of G. A finite group G is called* quasi-p *if $G = p(G)$.*

Let Γ be any finite group and let H be a subgroup of Γ. A subset $S \subset H$ will be called a relative generating set *for H in Γ if for every subset $T \subset \Gamma$ such that $H \cup T$ generates Γ, the subset $S \cup T$ also generates Γ. We define* the relative rank of H in Γ *to be the smallest non-negative integer $\mu := \mathrm{rank}_\Gamma(H)$ such that there is a relative generating set for H in Γ consisting of μ elements. Every generating set for H is a relative generating set, so $0 \leqslant \mathrm{rank}_\Gamma(H) \leqslant \mathrm{rank}(H)$. Also, $\mathrm{rank}_\Gamma(H) = \mathrm{rank}(H)$ if H is trivial or $H = \Gamma$, while $\mathrm{rank}_\Gamma(H) = 0$ if and only if H is contained in the Frattini subgroup $\Phi(\Gamma)$ of Γ* [5, p. 122].

A finite embedding problem $\mathcal{E}$ for a group Π is a pair of surjections $(\alpha : \Pi \to G, \phi : \Gamma \to G)$, where Γ and G are finite groups. If $H = \ker(\phi)$, the embedding problem $\mathcal{E}$ can be summarized by

$$\begin{tikzcd} & & & \Pi \arrow[dl, dashed] \arrow[d, "\alpha"] & \\ 1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r, "\phi"] & G \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

A weak solution *to $\mathcal{E}$ is a homomorphism $\gamma : \Pi \to \Gamma$ such that $\phi \circ \gamma = \alpha$. We call γ a* proper solution *to $\mathcal{E}$ if in addition it is surjective.*

REMARK 3. — *Let Π be a profinite group and let $\mathcal{E} = (\alpha : \Pi \to G, \phi : \Gamma \to G)$ be a finite embedding problem for Π. Suppose that the epimorphism $\alpha : \Pi \to G$ factors as $r\alpha'$, where $\alpha' : \Pi \to G'$ and $r : G' \to G$ are epimorphisms, for some finite group G'. We consider the* induced embedding problem *$\mathcal{E}_{\alpha'} = (\alpha' : \Pi \to G', \phi' : \Gamma' \to G')$ by taking $\Gamma' = \Gamma \times_G G'$ and letting $\phi' : \Gamma' \to G'$ be the second projection map. Here ϕ' is surjective* *because ϕ is; and so $\mathcal{E}_{\alpha'}$ is a finite embedding problem. Here $\mathcal{E}$ and $\mathcal{E}_{\alpha'}$ have isomorphic kernels; indeed $\ker(\phi') = \ker(\phi) \times 1 \subset \Gamma \times_G G' = \Gamma'$. Note that the first projection map $q : \Gamma' = \Gamma \times_G G' \to \Gamma$ is surjective since $r : G' \to G$ is surjective; and $\phi q = r\phi'$.*

In this situation, every proper solution $\lambda' : \Pi \to \Gamma'$ of $\mathcal{E}_{\alpha'}$ induces a proper solution $\lambda := q\lambda' : \Pi \to \Gamma$ of $\mathcal{E}$; viz. $\phi\lambda = \phi q\lambda' = r\phi'\lambda' = r\alpha' = \alpha$, and λ is surjective because q and λ' are. So we obtain a map $\mathrm{PS}(\mathcal{E}_{\alpha'}) \to \mathrm{PS}(\mathcal{E})$, where PS *denotes the set of proper solutions to the embedding problem.*

In this paper we consider curves over an algebraically closed field k of characteristic $p > 0$. A cover of k-curves is a morphism $\Phi : D \to C$ of smooth connected k-curves that is finite and generically separable. If $\Phi : D \to C$ is a cover, its Galois group $\mathrm{Gal}(D/C)$ *is the group of k-automorphisms σ of D satisfying $\Phi \circ \sigma = \Phi$. If G is a finite group, then a* G-Galois cover *is a cover $\Phi : D \to C$* together *with an inclusion $\rho : G \hookrightarrow$* $\mathrm{Gal}(D/C)$ *such that G acts simply transitively on a generic geometric fiber of $\Phi : D \to C$. If we fix a geometric point of C to be a base point, then the pointed G-Galois étale covers of C correspond bijectively to the surjections $\alpha : \pi_1(C) \to G$, where $\pi_1(C)$ is the algebraic fundamental group of C with the chosen geometric point as the base point. The proper solutions to an embedding problem $\mathcal{E} = (\alpha : \pi_1(C) \to G, \phi : \Gamma \to G)$ for $\pi_1(C)$ then are in bijection to the pointed Γ-Galois covers $E \to C$ that dominate the pointed G-Galois cover $\Phi : D \to C$ corresponding to α. In the case that X is the smooth completion of the affine k-curve C, denote by g_X the genus of X, and define $r_C = \#(X - C)$ and $n_C = 2g_X + r_C - 1$.*

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2. Group theory results and examples of effective subgroups

In this section we analyze when a subgroup of an effective subgroup for an embedding problem is effective using some group theory and Galois theory.

We start with an embedding problem $\mathcal{E}(2.1)$ for Π.

$$
\begin{tikzcd}
& & & \Pi \arrow[d, "\alpha"] \arrow[dl, dashed] & \\
1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r, "\phi"] & G \arrow[r] \arrow[d] & 1 \\
& & & 1 &
\end{tikzcd}
\tag{2.1}
$$

The following remark is easy to see.

Remark 4. — *Suppose Π_1 is an effective subgroup for the embedding problem $\mathcal{E}(2.1)$ and Π_2 is a finite index subgroup of Π_1. Suppose for some solution ψ of the embedding problem $\mathcal{E}(2.1)$ restricted to Π_1, the index of $\ker(\psi) \cap \Pi_2$ in Π_2 is $|\Gamma|$ then Π_2 is also an effective subgroup for $\mathcal{E}(2.1)$. Note that $\ker(\psi|_{\Pi_2}) = \ker(\psi) \cap \Pi_2$ has index $|\Gamma|$ in Π_2. Hence $\psi|_{\Pi_2}$ indeed surjects onto Γ.*

Corollary 5. — *Suppose Π_1 is an effective subgroup for the embedding problem $\mathcal{E}(2.1)$ and Π_2 is a finite index subgroup of Π_1 such that the embedding problem $\mathcal{E}(2.1)$ restricts to Π_2 and $[\Pi_1 : \Pi_2]$ is coprime to $|H|$. Then Π_2 is also effective for $\mathcal{E}(2.1)$.*

Proof. — Let ψ be a solution to the embedding problem $\mathcal{E}(2.1)$ restricted to Π_1. Note that $\ker(\psi) \subset \ker(\alpha|_{\Pi_1})$ and $[\ker(\alpha|_{\Pi_1}) : \ker(\psi)] = |H|$. Since $[\Pi_1 : \Pi_2]$ is coprime to $|H|$, $[\ker(\alpha|_{\Pi_1}) : \ker(\alpha|_{\Pi_2})]$ is also coprime to $|H|$. So the index $[\ker(\alpha|_{\Pi_2}) : \ker(\psi) \cap \ker(\alpha|_{\Pi_2})] = |H|$. That the embedding problem restricts to Π_2 means that $[\Pi_2 : \ker(\alpha|_{\Pi_2})] = |G|$. So we obtain that $[\Pi_2 : \ker(\psi) \cap \ker(\alpha|_{\Pi_2})] = |G||H| = |\Gamma|$. Also note that $\ker(\psi) \cap \ker(\alpha|_{\Pi_2}) = \ker(\psi) \cap \ker(\alpha|_{\Pi_1}) \cap \Pi_2 = \ker(\psi) \cap \Pi_2$. Hence the result is obtained from the previous remark. $\square$

In the above corollary the hypothesis guaranteed that if Π_1 is an effective subgroup for the given embedding problem and ψ is a solution of the embedding problem restricted to Π_1 then $\psi|_{\Pi_2}$ is a solution to the embedding problem restricted to Π_2. Hence Π_2 is also an effective subgroup. But this does not hold unconditionally as the following examples show.

Example. — *Let Π be the absolute Galois group of the reals, let $\Gamma = H = \mathbb{Z}/2\mathbb{Z}$, and let $G = 1$. Then the given embedding problem has a proper solution (the complex numbers). So $\Pi_1 = \Pi$ itself is effective. But if we pull back from $\mathbb{R}$ to $\mathbb{C}$ (corresponding to taking the trivial subgroup Π_2 of Π) then the embedding problem (which has trivial cokernel) restricts to Π_2. But it no longer has a proper solution. Note here $|H| = [\Pi : \Pi_2] = 2$.*

On the contrary, in geometric setting, even if the given solution does not pull back to a proper solution, there might be some other proper solution over the pullback.

Example. — *Let C be the affine x-line minus 0 in characteristic 0, let $\Pi = \pi_1(C)$, $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $H = \mathbb{Z}/3\mathbb{Z}$, and $G = \mathbb{Z}/2\mathbb{Z}$. Then there is a G-cover C_1 of C given by $y^2 = x$. Over that there's a proper solution D to the $\mathcal{E}(2.1)$, given by $z^3 = y$. So again $\Pi_1 = \Pi$ is an effective subgroup of Π for $\mathcal{E}(2.1)$. Here D is the fiber product of C_1 with the curve C_2 given by $w^3 = x$ (where $w = z^2$, and $z = y/w$). Now pull back everything by the degree 3 cover C_2 of C. This is linearly disjoint from C_1, so $\mathcal{E}(2.1)$ restricts to the subgroup $\pi_1(C_2)$ of Π. But the degree is not relatively prime to $|H|$, the cover C_2 is not linearly disjoint from D, and the solution to the given $\mathcal{E}(2.1)$ does not restrict to a proper solution to $\mathcal{E}(2.1)$ restricted to $\pi_1(C_2)$. But $\mathcal{E}(2.1)$ restricted to $\pi_1(C_2)$ does have a proper solution, given by $v^3 = z$.*

3. Formal Patching Results

In this section we develop some formal patching results which are used in later sections to find solutions to various embedding problems. Proposition 11 is the main result of this section and one of the main technical results of this paper.

Notation. — *Given a scheme X, denote by $\mathcal{M}(X)$ the category of coherent sheaves of $\mathcal{O}_X$-modules, $\mathcal{AM}(X)$ the category of coherent sheaves of $\mathcal{O}_X$-algebras and $\mathcal{SM}(X)$ the subcategory of $\mathcal{AM}(X)$ for which the sheaves of algebras are generically separable and locally free. Given a finite group G denote by $G\mathcal{M}(X)$ the category of generically separable coherent locally free sheaves of $\mathcal{O}_X$-algebras S together with a G-action which is transitive on the geometric generic fibers of $\operatorname{Spec}_{\mathcal{O}_X}(S) \to X$. Given categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and functors $\mathfrak{F} : \mathcal{A} \to \mathcal{C}$ and $\mathfrak{G} : \mathcal{B} \to \mathcal{C}$, denote by $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ the associated fiber category.*

The following result is due to Harbater [7, Theorem 3.2.12].

Theorem 3.1. — *Let $(A, \mathfrak{p})$ be a complete local ring and T a proper A-scheme. Let $\{\tau_1, \cdots, \tau_N\}$ be a finite set of closed points of T and $U = T \setminus \{\tau_1, \cdots, \tau_N\}$. Denote by $\hat{\mathcal{O}}_{T,\tau_i}$ the completion of the local ring $\mathcal{O}_{T,\tau_i}$ and let $T_i = \operatorname{Spec}(\hat{\mathcal{O}}_{T,\tau_i})$. Let U^* be the $\mathfrak{p}$-adic completion of U and $\mathcal{K}_i$ the*

$\mathfrak{p}$-adic completion of $T_i \setminus \{\tau_i\}$. Then the base change functor

$$\mathcal{M}(T) \to \mathcal{M}(U^*) \times_{\mathcal{M}(\bigcup_{i=1}^N \mathcal{K}_i)} \mathcal{M}\left(\bigcup_{i=1}^N T_i\right)$$

is an equivalence of categories. The same remains true if we replace $\mathcal{M}$ by $\mathcal{AM}$, $\mathcal{SM}$ or $G\mathcal{M}$ for a fixed finite group G.

Next we state a lemma useful for putting covers into a situation where the theorem above holds. Its proof is just an application of a generalization of the Noether Normalization Lemma.

LEMMA 6. — *Let C be an affine k-curve and X be its smooth projective completion. Then there exists a finite morphism $\Theta_X : X \to \mathbb{P}^1_x$ such that Θ_X is étale at $\Theta_X^{-1}(\{x = 0\})$ and such that $\Theta_X^{-1}(\{x = \infty\}) = X \setminus C$.*

Proof. — By a stronger version of Noether normalization (*cf.* [3, Corollary 16.18]), there exists a finite generically separable k-morphism from $C \to \mathbb{A}^1_x$ where $\mathbb{A}^1_x$ is the affine k-lines with local coordinate x. The branch locus of this morphism is of codimension 1, and hence it is étale away from finitely many points. By translation we may assume that $x = 0$ is not a branch point of $C \to \mathbb{A}^1_x$. This morphism extends to a finite proper morphism $\Theta_X : X \to \mathbb{P}^1_x$. Note that Θ_X is étale at $\Theta_X^{-1}(\{x = 0\})$ and that $\Theta_X^{-1}(\{x = \infty\}) = X \setminus C$. □

The patching results will be applied to covers of reducible curves where there are sufficiently many components of the base over which the cover is trivial. First, we need some terminology.

DEFINITION 7. — *Let $\Phi : V \to X$ be a G-cover of smooth irreducible projective curves over k. Assume X has r marked points and Φ is étale away from these r points. We say that a G-cover of connected projective curves $\Phi' : V' \to X'$ with r marked points on X' is a* deformation *of $V \to X$ if there exist a smooth irreducible k-scheme S, a cover of S-curves $\Phi_S : V_S \to X_S$, r sections $p_1, \ldots, p_r : S \to X_S$ and closed points s and s' in S such that the following three conditions hold.*

1. *Φ_S induces Φ and Φ' at s and s' respectively.*
2. *$\{p_1(s), \ldots, p_r(s)\}$ and $\{p_1(s'), \ldots, p_r(s')\}$ are the marked points of X and X' respectively.*
3. *Φ_S is a G-cover étale away from the sections $p_1, \ldots, p_r$.*

NOTATION. — *Given a $\Phi : V \to X$ and a deformation $\Phi' : V' \to X'$, we will call $(S; \Phi_S : V_S \to X_S; p_1, \ldots, p_r; s; s')$ the associated data of the deformation.*

DEFINITION 8. — *The deformation $V' \to X'$ will be called* a SNC deformation *if all irreducible components of X' are smooth and they intersect transversely. Moreover, it will be called* a degeneration *if it is SNC and for some irreducible component X_1 of X', the restriction of the G-cover to X_1 is induced from a trivial cover, i.e. $V' \times_{X'} X_1 \cong \mathrm{Ind}_e^G X_1$ as G-covers of X_1. In this situation X_1 will be called a trivial component of Φ'.*

REMARK 9. — *Let $\Phi : V_X \to X$ be a G-cover. Suppose Φ has a deformation $\Phi' : V'_X \to X'$ with associated data $(S; \Phi_S : V_S \to X_S; p_1, \dots, p_r; s; s')$. By taking a smooth irreducible k-curve in S passing through s and s' and pulling back $\Phi_S : V_S \to X_S$ to this curve, we may assume S is a smooth k-curve. In other words, if Φ' is a deformation of Φ then we may assume Φ can be deformed to Φ' along a smooth curve.*

EXAMPLE. — *Let $\Phi : V_X \to X$ be a G-cover of irreducible smooth projective curves. Let τ be a closed point in X. Let $S = \mathbb{A}^1_t$ and X_S be the the blowup of $(\tau, t = 0)$ in $X \times S$ and X' be the total transform of the zero locus of $t = 0$ in $X \times S$. Note that X' has two irreducible components, a copy of X and the exceptional divisor isomorphic to $\mathbb{P}^1$ intersecting at τ. One can obtain a G-cover $\Phi_S : V_{X_S} \to X_S$ obtained by pullback along $X_S \to X \times S \to X$. The fiber over $t = 0$ induces a G-cover $\Phi' : V_{X'} \to X'$. The exceptional divisor $\mathbb{P}^1$ is the trivial component of Φ'.*

LEMMA 10. — *Let $\Phi : V \to S$ be a flat family of reduced projective irreducible curves in which S is a smooth connected variety and V is a normal variety. Suppose for every point $s \in S$ the normalization of the fiber V_s has the same genus. Then for every closed point $s \in S$ the fiber V_s is smooth.*

Proof. — Let η be the generic point of S. Note that V_η being a localization of V is normal. Hence there exists a nonempty open subset U of S such that for all closed point $s \in U$, V_s is a normal k-curve and hence smooth. Let g be the genus of such a curve. Suppose there exists $s' \in S$ a closed point such that $V_{s'}$ is singular. Then the arithmetic genus of $V_{s'}$ is g, since Φ is a flat family. But $V_{s'}$ is singular so the geometric $p_g(V_{s'}) < g$. But $p_g(V_{s'})$ is same as the genus of the normalization of $V_{s'}$. This contradicts the hypothesis that the normalization of every fiber has the same genus. □

PROPOSITION 11. — *Let Γ be a finite group. Let G be a subgroup of Γ and let $H_1, \dots, H_m$ be subgroups of Γ of order prime-to-p. Assume that $G, H_1, \dots, H_m$ generate Γ. Let C be an affine k curve with smooth completion X. Let $\Phi : V_X \to X$ be a G-Galois cover of X étale over C. Let*

$B_X = X - C$, $r = \#(B_X)$, and consider the set B_X to be r marked points on X. Suppose Φ has a degeneration $\Phi' : V'_X \to X'$ with an associated data $(S; \Phi_S : V_S \to X_S; p_1, \dots, p_r; s; s')$ such that S is a smooth curve and $X_1, \dots, X_m$ are the trivial components of X'. Let B be the union of the images of the sections $p_1, \dots, p_r : S \to X_S$ and let r_i be the number of smooth marked points of X' lying on X_i for $i = 1, \dots, m$. Further assume that for each $1 \leqslant i \leqslant m$, there exists an H_i-cover $\Phi_{X_i} : W_{X_i} \to X_i$ étale away from these r_i points. Let $T = X_S \times_S \hat{S}_{s'}$ where $S_{s'}$ is the completion of S at s' and $\Phi_T : V_T \to T$ be the pullback of Φ_S to T. Then there exists a Γ-cover $\Psi : W \to T$ such that Ψ is étale away from $B \times_{X_S} T$. Moreover, if $\Gamma = H \rtimes G$ and $H_1, \dots, H_m \leqslant H$ then Ψ dominates the G-cover Φ_T.

Proof. — By Remark 9 we may assume that there exists an associated data $(S; \Phi_S : V_S \to X_S; p_1, \dots, p_r; s; s')$ for the degeneration of Φ to Φ' such that S is a smooth curve. Let t be the local coordinate of S at s'. Then $\hat{S}_{s'} = \mathrm{Spec}(k[[t]])$. By construction, the closed fiber of T is X'. Let X_0 be the closure of $X' \backslash \cup_{i=1}^m X_i$ in X'. So X_0 is made up of the nontrivial components of Φ'. Since Φ' is a degeneration of Φ it is a SNC deformation, so all irreducible components of X' are smooth and they intersect transversely. Let $\tau_1, \dots, \tau_N$ be the closed points of T where X_i and X_j intersect for some $0 \leqslant i \neq j \leqslant m$. Let $X_i^o = X_i \setminus (\{\tau_1, \dots, \tau_N\} \cap X_i)$ for $0 \leqslant i \leqslant m$. Note that $\tau_1, \dots, \tau_N$ will be used to denote the points of T, X' and various components of X' as well, but this should not lead to any confusion.

Let $T^o = T \setminus \{\tau_1, \dots, \tau_N\}$ and $\tilde{T}^o$ be the formal scheme obtained by the completion of T^o along the closed fiber $(t = 0)$ (i.e. the (t)-adic completion). Let $T_i^o = X_i^o \times_k \mathrm{Spec}(k[[t]])$ and $\tilde{T}_i^o$ be the (t)-adic completion of T_i^o (i.e. along X_i^o) for $i = 0, \dots, m$. Since the closed fiber of T^o is the disjoint union of $X_0^o, X_1^o, \dots, X_m^o$,

$$\tilde{T}^o = \tilde{T}_0^o \cup \tilde{T}_1^o \cup \ldots \cup T_m^o$$

By base change of Φ_T to $\tilde{T}^o$ we obtain a G-cover $\Phi_{\tilde{T}^o} : \tilde{V}_{T^o} \to \tilde{T}^o$ and hence a G-cover of the component $\tilde{T}_0^o$ which we will denote by $\Phi_{T_0} : \tilde{V}_0 \to \tilde{T}_0^o$. Note that Φ_T restricted to the closed fiber is the G-cover $\Phi' : V'_X \to X'$. Since X_i, for $i = 1, \dots, m$, are the trivial components of the cover $\Phi' : V'_X \to X'$, the G-covers of the components $\tilde{T}_i^o$ are induced from the trivial cover. Let $\Phi_{T_i} : \tilde{V}_i \to \tilde{T}_i^o$ be the H_i-cover obtained by pulling back $\Phi_{X_i} : W_{X_i} \to X_i$ along the composition of morphisms $\tilde{T}_i^o \to T_i^o \to X_i^o \to X_i$, for $i = 1, \dots, m$. Let $\Phi^o : \mathrm{Ind}_G^\Gamma \tilde{V}_0 \cup \mathrm{Ind}_{H_1}^\Gamma \tilde{V}_1 \cup \ldots \cup \mathrm{Ind}_{H_m}^\Gamma \tilde{V}_m \to \tilde{T}^o$ be the Γ-cover of $\tilde{T}^o$ obtained from Φ_{T_i} for $i = 0, \dots, m$.

Let $\hat{T}_{\tau_j} = \mathrm{Spec}(\hat{\mathcal{O}}_{T,\tau_j})$ be the formal neighbourhood of τ_j in T and $\hat{\mathcal{K}}_j$ be the (t)-adic completion of $\hat{T}_{\tau_j} \setminus \{\tau_j\}$ for $1 \leqslant j \leqslant N$, i.e. $\hat{\mathcal{K}}_j$ is the (t)-adic completion of the punctured formal neighbourhood of τ_j in T. We have a natural morphism $\hat{\mathcal{K}}_j \to \tilde{T}^o$. Note that $\Phi' : V'_X \to X'$ is étale at $\tau_1, \ldots, \tau_N \in X'$, since $\tau_1, \ldots, \tau_N$ lie in X_i for some $1 \leqslant i \leqslant m$ and over X_i, Φ' is induced from a trivial cover. Since Φ' is the restriction of Φ_T to the closed fiber, Φ_T is étale over the points $\tau_1, \ldots, \tau_N$ in T. Also note that Φ_{T_i} is the pullback of the H-cover $\Phi_{X_i} : W_{X_i} \to X_i$ which is étale over $\tau_1, \ldots, \tau_N$. Hence the pullback of Φ^o to $\hat{\mathcal{K}}_j$ is the Γ-cover of $\hat{\mathcal{K}}_j$ induced from the trivial cover for $1 \leqslant j \leqslant N$.

Apply Theorem 3.1 to obtain a Γ-cover $\Psi : W \to T$ which induces the Γ-cover Φ^o of $\tilde{T}^o$ and trivial Γ-cover over $\cup_{j=1}^N \hat{T}_{\tau_j}$. The cover W is connected because Γ is generated by $G, H_1, \ldots, H_m$ and Γ acts transitively on the fibers of $\mathrm{Ind}_G^\Gamma \tilde{V}_0 \cup \mathrm{Ind}_{H_1}^\Gamma \tilde{V}_1 \cup \ldots \cup \mathrm{Ind}_{H_m}^\Gamma \tilde{V}_m$.

Recall that B is the union of the images of the sections $p_1, \ldots, p_r : S \to X_S$. The branch locus of Ψ is clearly contained in T^o because in the formal neighbourhood of τ_i's, Ψ restricts to the Γ-cover induced from the trivial cover. Since the pullback of Ψ to $\tilde{T}^o$ is Φ^o, the branch locus of Φ^o maps to the branch locus of Ψ under the morphism $\tilde{T}^o \to T^o$. Note that for $i = 1, \ldots, m$, Φ_{T_i} is étale away from $B_i \times_{X_i^o} \tilde{T}_i^o$ in $\tilde{T}_i^o$ where B_i is the set of r_i smooth marked points of X' lying on X_i. Moreover, the image of $B_i \times_{X_i^o} \tilde{T}_i^o$ under the morphism $\tilde{T}_i^o \to T$ is contained in $B \times_{X_S} T$.

Note that $\Phi_{T_0} : \tilde{V}_0 \to \tilde{T}_0$ is the pullback of the G-cover $\Phi_S : V_S \to X_S$ under the morphism $\tilde{T}_0 \to \tilde{T}^o \to T^o \to T \to X_S$. Also the branch locus of Φ_S is contained in B. So combining all these we see that Ψ is étale away from $B \times_{X_S} T$.

Finally if $\Gamma = H \rtimes G$ then H is a normal subgroup of Γ and by quotienting one obtains a G-cover $W/H \to T$. Since $H_1, \ldots H_m \leqslant H$ the pullback of W/H on $\tilde{T}^o$ is $\tilde{V}_{T^o}$. Also the pullback over $\hat{T}_{\tau_j}$ of W/H is a trivial cover for $1 \leqslant j \leqslant N$. Hence by Theorem 3.1 the G-covers $W/H \to T$ and $V_T \to T$ are isomorphic. So Ψ dominates Φ_T. □

4. Lefschetz-Abhyankar Result

Now we can use a Lefschetz type principle and Abhyankar's lemma to obtain a Γ-cover of X étale over C and dominating the given G-cover. This is a deformation argument similar to [8, Proposition 4].

Proposition 12. — *In the context of Proposition 11, there exists a* Γ*-cover* $W_s \to X$ *dominating the* G*-cover* $\Phi : V_X \to X$ *which is étale over* C*.*

Proof. — From the conclusion of Proposition 11, there is a Γ-cover $\Psi : W \to T$ of $\hat{S}_{s'}$-curves dominating $\Phi_T : V_T \to T$ where $V_T = V_S \times_S \hat{S}_{s'}$. As in the proof of Proposition 11, let t be local coordinate of S at s' so that $\hat{S}_{s'} = \operatorname{Spec}(k[[t]])$. By hypothesis $T = X_S \times_S \hat{S}_{s'}$, $V_S \to X_S$ is a G-cover and its fiber at $s \in S$ is the G-cover $\Phi : V_X \to X$. Also Ψ is étale away from $B \times_{X_S} T$. Since Ψ is a finite morphism, there exists a finitely generated $\mathcal{O}_S$-algebra $R \subset k[[t]]$ such that the morphisms $W \to V_T \to T$ descend to the morphisms of $S_0 = \operatorname{Spec}(R)$-curves $W_{S_0} \to V_{S_0} \to T_{S_0}$. Note that the morphism $\hat{S}_{s'} \to S$ is the composition of the structure morphism $\pi : S_0 \to S$ and the morphism $\hat{S}_{s'} \to S_0$ induced by the inclusion $R \subset k[[t]]$. So $T_{S_0} = X_S \times_S S_0$ and $V_{S_0} = V_S \times_S S_0$, since T and V_T were base change of X_S and V_S respectively to $\hat{S}_{s'}$. By shrinking S_0 we may assume that S_0 is smooth and for every point $\tau \in S_0$ the fiber of the cover $W_{S_0} \to T_{S_0}$ is a smooth irreducible Γ-cover $W_\tau \to X_{\pi(\tau)}$ which is étale away from $\{p_1(\pi(\tau)), \ldots, p_r(\pi(\tau))\}$ and dominates the G-cover $V_\tau \to X_{\pi(\tau)}$.

Since S_0 is an affine S-scheme, we choose an embedding $S_0 \hookrightarrow \mathbb{A}^n_S$, define $\bar{S}_0$ to be the closure of S_0 in $\mathbb{P}^n_S$ and let $\bar{\pi} : \bar{S}_0 \to S$ be the structure morphism. Since $\bar{\pi}$ is dominating and projective, it is surjective onto S. Let $T_{\bar{S}_0} = X_S \times_S \bar{S}_0$ and $V_{\bar{S}_0} = V_S \times_S \bar{S}_0$. Note that $T_{\bar{S}_0} \to \bar{S}_0$ and $V_{\bar{S}_0} \to \bar{S}_0$ extend $T_{S_0} \to S_0$ and $V_{S_0} \to S_0$ respectively.

Let $W_{\bar{S}_0}$ be the normalization of $V_{\bar{S}_0}$ in $k(W_{S_0})$. Note that $W_{S_0} \to V_{S_0} \to T_{S_0}$ are finite morphism of normal varieties, so it is the restriction of $W_{\bar{S}_0} \to V_{\bar{S}_0} \to T_{\bar{S}_0}$. We summarize the setup in the following diagram.

$$\begin{tikzcd}
 & W \arrow[r] \arrow[d] & W_{S_0} \arrow[r] \arrow[d] & W_{\bar{S}_0} \arrow[d] \\
V_S \arrow[d] & V_T \arrow[l] \arrow[r] \arrow[d] & V_{S_0} \arrow[r] \arrow[d] & V_{\bar{S}_0} \arrow[d] \\
X_S \arrow[d] & T \arrow[l] \arrow[r] \arrow[d] & T_{S_0} \arrow[r] \arrow[d] & T_{\bar{S}_0} \arrow[d] \\
S & \hat{S}_{s'} \arrow[l] \arrow[r] & S_0 \arrow[r] & \bar{S}_0 \arrow[lll, bend left]
\end{tikzcd}$$

Let $\tau_S \in \bar{S}_0$ be such that $\bar{\pi}(\tau_S) = s$. Let $\bar{A}_0$ be a curve in $\bar{S}_0$ passing through τ_S such that $A_0 = \bar{A}_0 \cap S_0$ is non empty. Note that if $\tau_S \in S_0$ then the result follows from the assumptions on S_0.

Replacing $\bar{A}_0$ by an open neighbourhood of τ_S, we may assume that the fiber at all points of $\bar{A}_0$ of the morphism $V_{\bar{A}_0} \to \bar{A}_0$ are smooth irreducible curves. Let A and $\bar{A}$ be the normalization of A_0 and $\bar{A}_0$ respectively. Let $\tau_A \in \bar{A}$ be a point lying above $\tau_S \in \bar{A}_0$.

Let B_1 be the finitely many points in $\bar{A} \setminus A$. Let $b_i : \bar{A} \to V_{\bar{A}}$ be the section obtained by the pullback of $p_i : S \to V_S$ along the composition of the morphisms $\bar{A} \to \bar{A}_0 \to \bar{S}_0 \to S$. Recall that the sections p_i's correspond to the deformation of marked points on X (see Proposition 11). Let $B_2 = \cup_{i=1}^r \mathrm{im}(b_i)$ and B_V be the union of fibers $\cup_{\zeta \in B_1} V_\zeta$. Note that $W_{\bar{A}} \to V_{\bar{A}}$ is an H-cover étale away from $B_2 \cup B_V$. Since H is a prime-to-p group, the least common multiple m of the ramification indices at the generic points of B_V is coprime to p. Let $A' \to \bar{A}$ be a cyclic branched cover totally ramified at the points in B_1 with ramification indices m. Let $W_{A'} \to V_{A'}$ be the pull back of $W_{\bar{A}} \to V_{\bar{A}}$. Then applying Abhyankar's lemma we conclude that $W_{A'} \to V_{A'}$ is étale away from $B_2 \cup B_V$ and it is unramified at the generic points of B_V. Since $W_{A'} \to V_{A'}$ is a finite morphism of normal varieties, the purity of Branch locus implies that the morphism is étale away from B_2.

Let $\tau_{A'} \in A'$ be a point lying above τ_A. The fiber over $\tau_{A'}$ of the covering $V_{A'} \to T_{A'}$ is $V_s \to T_s$. But this is same as $V_X \to X$. Since $A' \to \bar{A}$ is proper and all the fibers of $V_{\bar{A}_0} \to \bar{A}_0$ are smooth irreducible curves, same is true for the fibers of $V_{A'} \to A'$.

Let $b_i' : A' \to V_{A'}$ be the sections obtained by the pullback of b_i along $A' \to A$. After shrinking A' to an open neighborhood of $\tau_{A'}$ if necessary, we may assume that $\mathrm{im}(b_i)$ and $\mathrm{im}(b_j)$ are disjoint for $i \neq j$. In particular, the branch locus of the cover $W_{A'} \to V_{A'}$ is smooth. Moreover, being a prime-to-p cover, it is étale locally a Kummer cover. Hence the fiber over every point $\tau \in A'$ of the cover $W_{A'} \to V_{A'}$ is a cover of smooth curves. Since the fibers of $W_{A'} \to A'$ are connected for all but finitely many points of A', Zariski's connectedness theorem tells us that every fiber of $W_{A'} \to A'$ must be connected. In particular, $W_{\tau_{A'}} \to T_{\tau_{A'}}$ is a Γ cover of smooth connected curves. This cover dominates $V_{\tau_{A'}} \to T_{\tau_{A'}}$ which is same as $V_X \to X$. $\square$

5. Degenerations of covers and solving embedding problems

In this section we use Proposition 11 and 12 to solve certain embedding problems. This is used to obtain more examples of effective subgroups of the fundamental group of a curve for a given embedding problem. The method below combines the technique of "adding branch points" as in [8] and "increasing the genus" as in [9, 10, 2].

Theorem 5.1. — *Let C be a smooth affine curve over k and X be the smooth completion of C. Let g be the genus of X and $r = \#(X \setminus C)$. Let $\mathcal{E}(5.1)$ denote the embedding problem*

(5.1)
$$\begin{tikzcd} & & & \pi_1(C) \arrow[dl, dashed] \arrow[d, "\alpha"] & \\ 1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r, "\phi"] & G \arrow[r] \arrow[d] & 1 \\ & & & 1. & \end{tikzcd}$$

Let $\Phi : V_X \to X$ be the G-cover of X étale over C corresponding to α. Let $B_X = X - C$ and $r = \#(B_X)$ and consider the set B_X to be r marked points on X. Suppose Φ has a degeneration $\Phi' : V'_X \to X'$ with associated data $(S; \Phi_S : V_S \to X_S; p_1, \ldots, p_r; s; s')$ and let $X_1, \ldots, X_m$ be the trivial components of X'. Let r_i be the number of smooth marked points of X' lying on X_i for $i = 1, \ldots, m$. Suppose there exists a group homomorphism

$$\theta : \pi_1(X_1 \setminus \{r_1 \text{ points}\}) \times \ldots \times \pi_1(X_m \setminus \{r_m \text{ points}\}) \to H/p(H)$$

such that the image of θ is a relative generating set for $H/p(H)$ in $\Gamma/p(H)$. Then there exists a Γ-cover $W_X \to X$ which corresponds to a proper solution to $\mathcal{E}(5.1)$ (i.e. $W_X \to X$ dominates $\Phi : V_X \to X$ and is étale over C).

Proof. — Let $\bar{\Gamma} = \Gamma/p(H)$, $\bar{H} = H/p(H)$, and $H_i = \theta(\pi_1(X_i \setminus r_i \text{ points}))$ for $i = 1, \ldots m$. Consider the embedding problem $\mathcal{E}(5.2)$

(5.2)
$$\begin{tikzcd} & & & \pi_1(C) \arrow[dl, dashed] \arrow[d, "\alpha"] & \\ 1 \arrow[r] & \bar{H} \arrow[r] & \bar{\Gamma} \arrow[r, "\phi"] & G \arrow[r] \arrow[d] & 1 \\ & & & 1. & \end{tikzcd}$$

This has prime to p kernel $\bar{H}$. Since the image of θ is a relative generating set for $\bar{H}$ in $\bar{\Gamma}$, the subgroups $H_1, \ldots H_m$ and G together generate $\bar{\Gamma}$. Moreover, by definition of H_i, there exists an H_i-cover of X_i étale away from r_i points for $1 \leqslant i \leqslant m$. So the hypotheses of Proposition 11 and 12 for the embedding problem $\mathcal{E}(5.2)$ are satisfied. By the conclusion of Proposition 12 there exists a $\bar{\Gamma}$-cover $W_s \to X$ dominating the G-cover $\Phi : V_X \to X$ which is étale over C. This is a solution to $\mathcal{E}(5.2)$.

Since $p(H)$ is a quasi-p group, by [6, Corollary 4.6] or [12, Theorem B], the following embedding problem has a solution.

$$\begin{tikzcd}
 & & & \pi_1(C) \arrow[dl, dashed] \arrow[d] & \\
1 \arrow[r] & p(H) \arrow[r] & \Gamma \arrow[r, "\phi"] & \bar{\Gamma} \arrow[r] \arrow[d] & 1 \\
 & & & 1 &
\end{tikzcd}$$

This provides the required Γ-cover. $\square$

As a consequence of Theorem 5.1, we obtain the following:

Corollary 13. — *Let C be a smooth affine curve over k and X be the smooth completion of C. Let g be the genus of X and $r = \#(X \setminus C)$. Let $\mathcal{E}(5.1)$ denote the embedding problem above for $\pi_1(C)$ and let μ be the relative rank of $H/p(H)$ in $\Gamma/p(H)$. Let $\Phi : V_X \to X$ be the G-cover of X étale over C corresponding to α. Suppose Φ has a degeneration $\Phi' : V'_X \to X'$ with a trivial component X_1. Let r_1 be the number of marked points of X' lying on X_1, $g(X_1)$ be the genus of X_1 and $n_{X_1} = 2g(X_1) + r_1 - 1$. If one of the following holds:*

(1) *$r_1 \geqslant 1$ and $n_{X_1} \geqslant \mu$*
(2) *$r_1 = 0$ and $g(X_1) \geqslant \mu$*

then there exists a Γ-cover of X dominating Φ which is étale over C.

Proof. — Let $u_1, \ldots, u_\mu$ be the relative generators of H/pH in Γ/pH and H' be the subgroup of H/pH generated by $u_1, \ldots, u_\mu$. Note that H' is a prime-to-p group generated by μ elements, so in both scenario (1) and (2) by [1] there exists an epimorphism from $\pi_1(X_1 \setminus \{\text{marked points}\})$ to H'. Hence the corollary follows from Theorem 5.1. $\square$

The above result restated in the terminology of effective subgroups becomes the following statement.

Corollary 14. — *Let C, X, g, r and μ be as in Corollary 13. Let $\Pi \trianglelefteq \pi_1(C)$ be of finite index such that the embedding problem $\mathcal{E}$(5.1) restricts to Π. Let Z be the cover of X étale over C with $\pi_1(Z) = \Pi$ and $\Phi : V_X \times_X Z \to Z$ be the induced G-cover. Suppose Φ has a degeneration with a trivial component X_1 such that one of the following holds:*

(1) $r_1 \geqslant 1$ *and* $n_{X_1} \geqslant \mu$

(2) $r_1 = 0$ *and* $g(X_1) \geqslant \mu$

then Π is an effective subgroup for the embedding problem $\mathcal{E}$(5.1).

Proof. — Note that Φ is a G-cover of Z étale over the preimage of C and it has degeneration with the same properties as in the hypothesis of the above corollary. So using that corollary, we obtain a Γ-cover of Z dominating Φ which is étale over the preimage of C. This Γ-cover provides a solution to the embedding problem $\mathcal{E}$(5.1) restricted to Π. Hence Π is an effective subgroup. □

6. The case of affine line

Let C be the affine line $\mathbb{A}^1_x = \mathrm{Spec}(k[x])$ and $\mathcal{E} = (\alpha : \pi_1(C) \to G, \phi : \Gamma \to G)$ be an embedding problem for $\pi_1(C)$ with $H = \ker \phi$. In [2, Theorem 1.3], it was shown that there are infinitely many index p effective subgroups of $\pi_1(\mathbb{A}^1_x)$ for the embedding problem $\mathcal{E}$.

In [8, Theorem 5] (see Proposition 2), it was shown that if the étale cover $D \to \mathbb{A}^1_x$ is such that the number of points above $x = \infty$ in the smooth completion of D is large enough then $\pi_1(D)$ is an effective subgroup of $\pi_1(C)$ as long as the embedding problem $\mathcal{E}$ restricts to $\pi_1(D)$.

In this section we will demonstrate some sufficient conditions on a subgroup of $\pi_1(\mathbb{A}^1_x)$ to be effective depending only on rank of H and the cover corresponding to α. In fact, we will show that in the collection of all p-cyclic étale covers of high enough genus of $\mathbb{A}^1_x$, every member $D \to \mathbb{A}^1_x$ leads to an effective subgroup $\pi_1(D) \trianglelefteq \pi_1(\mathbb{A}^1_x)$.

Assume Π is an index p normal subgroup of $\pi_1(C)$. Let $D \to C$ be the cover corresponding to Π, i.e., D be the normalization of C in $(K^{un})^{\Pi}$ where K^{un} is the compositum of the function fields of all étale covers of C (in a fixed separable closure of $k(C)$). Since D is an étale p-cyclic cover of the affine line, by Artin-Schreier theory it is given by the equation $z^p - z - f(x)$ for some non-constant polynomial $f(x) \in k[x]$. Let r be the degree of $f(x)$. By changing $f(x)$ if necessary we may assume r is prime to p. Let $Z \to X$ be the corresponding morphism between their smooth completions. The genus of Z is $g_Z = (p-1)(r-1)/2$ and Z is totally ramified at infinity. So $n_D = 2g_Z + r_D - 1 = 2g_Z = (p-1)(r-1)$.

Let $\mathcal{E} = (\alpha : \pi_1(C) \to G, \phi : \Gamma \to G)$ be an embedding problem for $\pi_1(C)$ with $H = \ker\phi$. As observed in the introduction if n_D is less than the rank of $\Gamma/p(\Gamma)$ then Π can not be effective. But $n_D \geqslant \operatorname{rank}(\Gamma/p(\Gamma))$ is certainly not a sufficient condition for $\Pi = \pi_1(D)$ to be effective, as indicated by the following example.

Example. — *Let $C = \mathbb{A}^1$, Γ a quasi-p group, $G = \mathbb{Z}/p\mathbb{Z}$ and H a nontrivial prime-to-p group. Note that* $\operatorname{rank}(\Gamma/p(\Gamma)) = 0$. *Suppose the map $\alpha : \pi_1(C) \to G$ be induced by a p-cyclic étale cover $V_C \to C$ where V_C is also isomorphic to $\mathbb{A}^1$. Let $D \to C$ be any p-cyclic étale cover linearly disjoint from $V_C \to C$ and $U = D \times_C V_C$, then the embedding problem $\mathcal{E}$ restricts to $\pi_1(D) \trianglelefteq \pi_1(C)$ but if $n_U < \operatorname{rank}(H/p(H))$ then the embedding problem $\mathcal{E}$ restricted to $\pi_1(D)$ has no solution. This is because the existence of a solution to the embedding problem implies that $H/p(H)$ is a quotient of $\pi_1(U)$. But this is impossible if $n_U < \operatorname{rank}(H/p(H))$.*

Though we shall see that if $C = \mathbb{A}^1$, $D \to C$ is p-cyclic étale and $n_D \geqslant 2\operatorname{rank}(H/p(H))$ then $\pi_1(D)$ is indeed an effective subgroup for the embedding problem $\mathcal{E}$ in many cases.

Proposition 15. — *Let $\mathcal{E}(6.1)$ be the embedding problem*

$$\begin{tikzcd}
 & & & \pi_1(\mathbb{A}^1) \arrow[dl, dashed] \arrow[d, "\alpha"] & \\
1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r, "\phi"] & G \arrow[r] \arrow[d] & 1 \\
 & & & 1 &
\end{tikzcd} \tag{6.1}$$

Let $V \to \mathbb{A}^1_x$ be the G-Galois cover corresponding to α and $V_X \to \mathbb{P}^1_x = X$ be the morphism corresponding to the smooth completion. Let g be such that there is a homomorphism θ from the surface group Π_g to $H/p(H)$ with the property that $\operatorname{im}(\theta)$ *is a relative generating set for $H/p(H)$ in $\Gamma/p(H)$. Let $D \to \mathbb{A}^1_x$ be an étale p-cyclic cover such that the genus of the smooth completion Z of D is at least g. Let V_Z be the normalization of $V_X \times_X Z$. Suppose that the genus of the normalization of $Z' \times_X V_X$ is same as $g(V_Z)$ for all but finitely p-cyclic covers $Z' \to X$ branched only at $x = \infty$ with the genus of Z' same as $g(Z)$. Then $\pi_1(D)$ is an effective subgroup for the embedding problem $\mathcal{E}(6.1)$.*

Proof. — For convenience we first sketch out the idea behind the proof before proceeding with the formal proof. In view of Theorem 5.1, the idea is to construct a degeneration of the cover $V_Z \to Z$ to $V_{s'} \to T_{s'}$ (in the notation below) such that $T_{s'}$ is the union of the line X and a curve Y of genus at least g. The preimage of X is $V_{s'}$ is V_X. And the copy of Y in $T_{s'}$ is the trivial component (i.e. preimage of Y in $V_{s'}$ is $|G|$ copies of Y). For this one starts with a projective line degenerating to two lines intersecting at a point. This along with a family which parametrizes all p-cyclic covers of a line of genus $g(Z)$, we cook up a family of cover $V_T \to T$ indexed by S such that the degenerate fiber over a point $s' \in S$ is as desired. Now the last hypothesis of the proposition together with some explicit calculation is used to show that for some $s \in S$, $V_{T,s} \to T_s$ is the same cover as $V_Z \to Z$.

Now we begin the formal proof. Since $D \to \mathbb{A}^1_x$ is p-cyclic étale cover, it is given by the equation $Z^p - Z - (a_r x^r + a_{r-1}x^{r-1} + \ldots + a_1 x)$ where r is coprime to p, $a_r \neq 0$ and $a_{ip} = 0$ for $0 \leqslant i \leqslant [r/p]$ ([13]). Let $A = k[t_1, \ldots, t_r]$ where $t_{ip} = 0$ for $1 \leqslant i \leqslant [r/p]$ and t_j are indeterminates if p does not divide j. Let $S = \mathrm{Spec}(A)$ and $S^o = S \backslash \{t_r = 0\}$. Let $X_S = X \times_k S$ and Y_S be the normal cover of $\mathbb{P}^1_y \times_k S$ given by $z^p - z - f(y^{-1})$ where

$$f(w) = w^r + t_{r-1}w^{r-1} + \ldots t_1 w^1 \tag{6.2}$$

Note that the genus of the normalization of the fiber of $Y_S \to S$ for any point of S is constant and Y_S is normal. Hence every fiber of Y_S over a closed point of S is a smooth curve (Lemma 10). Also note that the cover $Y_S \to \mathbb{P}^1_y \times_k S$ is branched only at $y = 0$ in $\mathbb{P}^1_y \times S$ because the discriminant is -1 on $\mathbb{A}^1_{y^{-1}} \times S$. Let $Y \to \mathbb{P}^1_y$ be the fiber of $Y_S \to \mathbb{P}^1_y \times_k S$ at the point $(t_1 = 0, \ldots, t_r = 0) \in S$ and note that $y = 0$ defines a unique point τ in Y.

Let F be the locus of $t_r - xy = 0$ in $\mathbb{P}^1_x \times_k \mathbb{P}^1_y \times_k S$ and $Y_F = Y_S \times_{\mathbb{P}^1_y \times_k S} F$. Let $X_F = X_S \times_{\mathbb{P}^1_x \times_k S} F$ and T be the fiber product $X_F \times_F Y_F$. Note that $X_F = F$ and $T = Y_F$. By definition, $\Psi_X : V_X \to X$ is a G-cover of X étale over $\mathbb{A}^1_x$.

$$\begin{tikzcd}
 & & & T = Y_F \arrow[dl] \arrow[dr] & \\
 & & X_F = F \arrow[dl] \arrow[d] \arrow[dr] & & Y_S \arrow[dl] \\
V_X \arrow[d] & X_S = \mathbb{P}^1_x \times S \arrow[dl] & \mathbb{P}^1_x \times \mathbb{P}^1_y \times S \arrow[l] \arrow[r] & \mathbb{P}^1_y \times S & \\
X = \mathbb{P}^1_x & & & &
\end{tikzcd}$$

The morphism $T \to X_F$ is a family of covers parametrized by S. Let $s' \in S$ be the point $(t_1 = 0, \ldots, t_r = 0)$ and s be any point in S^o. Note that the morphism $F \to \mathbb{P}^1_y \times S$ is an isomorphism away from $t_r = 0$. So the fiber $T_s = Y_{S,s}$ is smooth. Let $T_{s'} \to X_{s'}$ be the fiber of $T \to X_F$ over s'. Then $X_{s'}$ is the union of $X = \mathbb{P}^1_x$ and $\mathbb{P}^1_y$ intersecting transversally at $(x = 0, y = 0)$ and $T_{s'}$ is the union $X = \mathbb{P}^1_x$ and Y intersecting transversally at the point $(x = 0, \tau)$. The fiber over s, $T_s \to X_s$ is a p-cyclic cover of smooth curves. Since at $s \in S^o$, $t_r \neq 0$, the projection map $X_F \to X$ restricted to X_s is an isomorphism. Hence $X_s = \mathbb{P}^1_x = X$. Moreover, if s is the point $(t_1 = b_1, \ldots, t_r = b_r)$, then the cover T_s is locally given by the equation

$$Z^p - Z - (\frac{1}{b_r^r}x^r + \frac{b_{r-1}}{b_r^{r-1}}x^{r-1} + \ldots + \frac{b_1}{b_r}x)$$

because $y^{-1} = xt_r{}^{-1}$ on T.

Let $\Phi_T : V_T \to T$ be the normalized pullback of $V_X \to X$ to T along the morphism $T \to X_F \to X_S \to X$. So $V_T \to T$ is a G-cover. Let

$$S^1 = \{s \in S^o : \text{ the genus of the normalization of } V_X \times_X T_s = g(V_Z)\}.$$

By hypothesis, S^1 is an open dense subset of S^o. The normalization of the fiber $V_{T,s}$ of V_T at $s \in S^1$ is the normalization of $V_X \times_X T_s$. The genus of the normalization of $V_{T,s}$ for all $s \in S^1$ is constant. Moreover V_T is normal, hence $V_{T,s}$ is smooth for all $s \in S^1$ (Lemma 10). Hence Φ_s is a cover of smooth irreducible curves dominating V_X for all $s \in S^1$. Let the fiber of Φ_T at s' be denoted by the morphism $V_{s'} \to T_{s'}$. Since $V_X \to X$ is étale at $x = 0$ and $T_{s'}$ is the union of Y and X intersecting only at $x = 0$ in X and τ in Y, the fiber $V_{s'}$ is the union of V_X and $|G|$ copies of Y which intersect in V_X at the $|G|$ preimages of $x = 0$ and in Y at $y = 0$. In particular for any $s \in S^1$, $\Phi_{s'}$ is a degeneration of Φ_s and the irreducible component Y of $T_{s'}$ is a trivial component of $\Phi_{s'}$.

Finally choosing b_i, and hence $s \in S^o$, appropriately we may assume $T_s \to \mathbb{P}^1_x$ is the same cover as $Z \to \mathbb{P}^1_x$. More precisely, let $b_r = (a_r)^{-1/r}$, $b_i = a_i b_r^i$ for $1 \leqslant i \leqslant r-1$. Then the local equation of T_s is same as that of D. Hence $k(T_s) = k(D)$ and the genus $g(T_s) = g(Y)$ is at least g. Also note that $s \in S^1$.

Note that by [1] there exists an epimorphism from $\pi_1(Y)$ to the prime to p part of Π_g. Composing this with θ and noting that $H/p(H)$ is a prime to p group, we obtain a homomorphism $\tilde{\theta} : \pi_1(Y) \to H/p(H)$ such that $\mathrm{im}(\tilde{\theta})$ is a relative generating set for $H/p(H)$ in $\Gamma/p(H)$. So applying Theorem 5.1, we obtain a Γ-cover of T_s which dominates the G-cover $V_X \times_X T_s \to T_s$. Hence $\pi_1(D)$ is an effective subgroup of $\pi_1(\mathbb{A}^1_x)$ for the embedding problem $\mathcal{E}(6.1)$.

□

For a Galois cover $U \to V$ of smooth connected k-curves, by the upper jumps at a branch point $v \in V$ we mean the upper jumps of the ramification filtration of the local field extension $\hat{\mathcal{O}}_{U,u}/\hat{\mathcal{O}}_{V,v}$ for any point $u \in U$ lying above v. Note that the set upper jumps does not depend upon the choice of $u \in U$ lying above $v \in V$ since $U \to V$ is a Galois cover.

Corollary 16. — *Let $\mathcal{E}(6.1)$ be the embedding problem in Proposition 15, $V_X \to \mathbb{P}^1_x$ be the G-Galois cover corresponding to α and g be as in Proposition 15. Let $D \to \mathbb{A}^1_x$ be an étale p-cyclic cover such that the genus of the smooth completion Z of D is at least g and the upper jump of the cover $Z \to \mathbb{P}^1_x$ at $x = \infty$ is different from all the upper jumps of $V_X \to \mathbb{P}^1_x$ at $x = \infty$. Then $\pi_1(D)$ is an effective subgroup for the embedding problem $\mathcal{E}(6.1)$.*

Proof. — Note that $Z \to \mathbb{P}^1_x$ being a p-cyclic cover is totally ramified at $x = \infty$. Let $\tau \in Z$ be the point lying above $x = \infty$. Let $\beta_1, \ldots, \beta_l \in V_X$ be the points lying above $x = \infty$ under the morphism $V_X \to \mathbb{P}^1_x$. Let $I_j \leqslant G$ be the inertia group of $V_X \to \mathbb{P}^1_x$ at β_j for $1 \leqslant j \leqslant l$. Since $V_X \to \mathbb{P}^1_x$ is a Galois cover all the I_j's are conjugates of each other. Also we know that the degree of the morphism $V_X \to \mathbb{P}^1_x$ is $|G|$ and the ramification index at β_j is $|I_j| = e$ (say). So $|G| = el$.

Let $\hat{R}$ be the completion of the stalk of V_X at β_1, K the fraction field of $\hat{R}$, $\hat{S}$ be the completion of the stalk of Z at τ and L the fraction field of $\hat{S}$. Then $\mathrm{Gal}(K/k((x^{-1}))) = I_1$, $\mathrm{Gal}(L/k((x^{-1}))) = \mathbb{Z}/p\mathbb{Z}$. Since the upper jumps of the two local extensions are distinct, K and L are linearly disjoint over $k((x^{-1}))$. Hence $k(V_X)$ and $k(Z)$ are linearly disjoint over

$k(x)$. So $V_X \times_{\mathbb{P}^1_x} Z$ is an irreducible curve with the function field being the compositum $k(V_X)k(Z)$. Let V_Z be the normalization of $V_X \times_{\mathbb{P}^1_x} Z$. Then $V_Z \to \mathbb{P}^1_x$ is a $G \times \mathbb{Z}/p\mathbb{Z}$-Galois cover branched only at ∞. At a point in V_Z lying above $(\beta_1, \tau) \in V_X \times_{\mathbb{P}^1_x} Z$, the ramification index of the cover $V_Z \to \mathbb{P}^1_x$, which can be computed by passing to the completion of stalks at β_1 of V_X and τ of Z, comes out to be the degree of the field extension $[KL : k((x^{-1}))]$. But this degree equals pe, since K and L are linearly disjoint, $[K : k((x^{-1}))] = e$ and $[L : k((x^{-1}))] = p$. Also the degree of the morphism $V_Z \to \mathbb{P}^1_x$ is $p|G| = pel$. Since $V_Z \to \mathbb{P}^1_x$ is a Galois cover, there are exactly l points in V_Z over $x = \infty$ one each lying above $(\beta_j, \tau) \in V_X \times_{\mathbb{P}^1_x} Z$.

Let $u_1, \ldots, u_a$ be the upper jumps of the ramification filtration on the inertia group I_1 of the cover $V_X \to \mathbb{P}^1_x$ and r be the upper jump of the inertia group $\mathbb{Z}/p\mathbb{Z}$ at τ of the cover $Z \to \mathbb{P}^1_x$. Since r is different from $u_1, \ldots, u_a$ by [11, Corollary 2.5] the ramification filtration is completely determined by the ramification filtration on I_1 and r. Let $Z' \to \mathbb{P}^1_x$ be another p-cyclic cover branched only at $x = \infty$ and the genus of Z' is $g(Z)$. Then the upper jump at $x = \infty$ of $Z' \to \mathbb{P}^1_x$ is also r (since $g(Z) = (p-1)(r-1)/2$ depends only on r). Let $V_{Z'}$ be the normalization of $V_X \times_{\mathbb{P}^1_x} Z'$. We observe that like $V_Z \to \mathbb{P}^1_x$, the cover $V_Z' \to \mathbb{P}^1_x$ is branched only at $x = \infty$, like in V_Z, there are exactly l points in $V_{Z'}$ lying above $x = \infty$ and the ramification filtration at these l points are same as the ramification filtration on the l points in V_Z lying above $x = \infty$. Since the degree and the ramification behaviour of $V_Z \to \mathbb{P}^1_x$ and $V_{Z'} \to \mathbb{P}^1_x$ are same, by Riemann-Hurwitz formula and Hilbert's different formula, the genus $g(V_Z) = g(V_{Z'})$. The result now follows from Proposition 15. $\square$

Corollary 17. — *Let $\mathcal{E}(6.1)$ be the embedding problem and $V \to \mathbb{A}^1_x$ be the G-Galois cover corresponding to α as in Proposition 15. Let $Z \to \mathbb{P}^1_x$ be a p-cyclic cover branched only at $x = \infty$ such that $g = g(Z)$ is at least the relative rank of $H/p(H)$ in $\Gamma/p(H)$ and the upper jump of the cover $Z \to \mathbb{P}^1_x$ at $x = \infty$ is different from all the upper jumps of $V_X \to \mathbb{P}^1_x$ at $x = \infty$. Let $D \subset Z$ be the complement of points lying above $x = \infty$. Then $\pi_1(D)$ is an effective subgroup for the embedding problem $\mathcal{E}(6.1)$.*

Proof. — Let $\{a_1, \ldots, a_g\} \subset H/p(H)$ be a relative generating set for $H/p(H)$ in $\Gamma/p(H)$. Note that Π_g is the quotient of the free group on $2g$ generators $A_1, \ldots, A_g$, $B_1, \ldots, B_g$ by the subgroup generated by $[A_1, B_1] \cdot [A_2, B_2] \cdots [A_g, B_g]$. So there exists a homomorphism from Π_g to H/pH which takes $A_i \to a_i$ and B_i to identity. Hence there exists $\theta : \Pi_g \to H/p(H)$ such that image of θ is a relative generating set for $H/p(H)$ in $\Gamma/p(H)$. The result now follows from the above corollary. $\square$

Remark 18. — *Recall that an étale p-cyclic cover $D \to \mathbb{A}^1_x$ is given by the polynomial equation $z^p - z = f(x)$ where $f(x)$ is polynomial of degree r coprime to p. For such a cover the upper jump of the inertia group $\mathbb{Z}/p\mathbb{Z}$ at $x = \infty$ is r and genus of the smooth completion Z of D is $(p-1)(r-1)/2$. So given an étale cover $V \to \mathbb{A}^1_x$, the hypothesis on the cover $Z \to \mathbb{P}^1_x$ of the above two corollaries will hold for all but finitely many values of r.*

7. Existence of degenerations

We will give a few more examples below where degenerations exist.

Let X be a smooth irreducible projective curve over k with a nonempty set of marked points. Let $C = X \setminus \{\text{the marked points}\}$ and $\Theta : X \to \mathbb{P}^1_x$ be a finite surjective generically separable morphism such that $\Theta^{-1}(x = \infty)$ is any given nonempty subset of $X \setminus C$ and Θ is étale over $x = 0$. Such a Θ exists by Lemma 6. Let m be the degree of the morphism Θ and $\{r_1, \ldots, r_m\} = \Theta^{-1}(x = 0)$. Let $\Phi : V_X \to X$ be a G-cover étale over C. For a smooth k-variety B, let $\Psi_B : \mathcal{Y} \to \mathbb{P}^1_y \times B$ be a family of smooth covers of $\mathbb{P}^1_y$ ramified only at $y = 0$ and let m' be the number of points in each fiber $\Psi_b : \mathcal{Y}_b \to \mathbb{P}^1_y$ lying above $y = 0$, for all $b \in B$. Let $S = \mathbb{A}^1_t \times B$ and let F be the closure of the zero locus of $t - xy$ in $\mathbb{P}^1_x \times \mathbb{P}^1_y \times S$. Let $X_F = X \times_{\mathbb{P}^1_x} F$, $V_{X_F} = V_X \times_X X_F = V_X \times_{\mathbb{P}^1_x} F$, $\mathcal{Y}_S = \mathcal{Y} \times \mathbb{A}^1_t = \mathcal{Y} \times_{\mathbb{P}^1_y \times B} (\mathbb{P}^1_y \times S)$, $\mathcal{Y}_F = \mathcal{Y}_S \times_{\mathbb{P}^1_y \times S} F = \mathcal{Y} \times_{\mathbb{P}^1_y \times B} F$, T be the normalization of $X_F \times_F \mathcal{Y}_F$ and V_T be the normalization of $V_{X_F} \times_{X_F} T$ which is same as the normalization of $V_{X_F} \times_F \mathcal{Y}_F$. Observe that the normalized base change of $\Phi : V_X \to X$ via the morphism $T \to X_F \to X$, is a G-cover of S-curves $\Phi_T : V_T \to T$.

$$\begin{tikzcd}
 & & & V_T \arrow[dll] \arrow[d, "\Phi_T"'] & & & & \\
 & V_{X_F} \arrow[dl] \arrow[d] & & T \arrow[dll] \arrow[drr] & & & & \\
V_X \arrow[d, "\Phi"'] & X_F \arrow[dl] \arrow[drr] & & & & \mathcal{Y}_F \arrow[dll] \arrow[dr] & & \\
X \arrow[dr] & & & F \subset \mathbb{P}^1_x \times \mathbb{P}^1_y \times S \arrow[dll] \arrow[drr] & & & \mathcal{Y}_S \arrow[r] \arrow[dl] & \mathcal{Y} \arrow[dl, "\Psi_B"] \\
 & \mathbb{P}^1_x & & & & \mathbb{P}^1_y \times S \arrow[r] & \mathbb{P}^1_y \times B &
\end{tikzcd}$$

Let $\Phi' : V' \to T'$ be the G-cover of B-curves obtained by looking at the fiber of $\Phi_T : V_T \to T$ over $t = 0$. For $b \in B$, let $\Phi'_b : V_{0,b} \to T_{0,b}$ denote the fiber of Φ' at b.

Proposition 19. — *Let the setup be as above. For a closed point $s = (a, b) \in S = \mathbb{A}^1_t \times B$, let Φ_s be the fiber of Φ_T at s. The set*

$$U = \{s \in S : \Phi_s \text{ is a } G\text{-cover of smooth irreducible curves}\}$$

is a nonempty open subset of S. Moreover for $s = (a, b) \in U$, Φ'_b is a degeneration of Φ_s and the fiber $T_{0,b}$ of T at $(0, b)$ consists of m copies of $\mathcal{Y}_b$ which are trivial components of Φ'_b. Also the cover $T_s \to X$ is étale over C.

Proof. — This result is a generalization of a part of Proposition 15 in which X is allowed to be any smooth projective curve instead of the affine line. But we restrict to only finding a degeneration. Again the starting point is to consider the family of p-cyclic covers $\mathcal{Y}_F \to F$ indexed by B and combine it with $V_X \to X$ to obtain $V_T \to T$ as above. There are m copies of $\mathcal{Y}_b$ in $T_{0,b}$ and they are trivial component follows in a similar fashion by using base change. The essential part is to show that U is nonempty. This reduces to showing that for any fixed $b \in B$, the cover $V_{T,\eta_b} \to T_{\eta_b}$ is a cover of smooth curves where η_b is the generic point of $\mathbb{A}^1_t \times b$. This is achieved by showing linear disjointness of two field extensions of $k(t, x)$ whose compositum is the function field of $k(T_{\eta_b})$.

Now we begin the formal proof. The fiber $F_{0,b}$ of $F \to S$ at $(0, b) \in S$ for any $b \in B$ is $\mathbb{P}^1_x$ and $\mathbb{P}^1_y$ intersecting transversally at $x = y = 0$. Let

$\Psi_b^{-1}(y=0) = \{s_1, \dots, s_{m'}\}$. The fiber $T_{0,b}$ of $T \to S$ at $(0,b)$ consists of m copies of $\mathcal{Y}_b$ and m' copies of X where each copy of X intersect each copy of $\mathcal{Y}_b$ at exactly one point. This can be seen as follows. Let $\bar{T} = X_F \times_F \mathcal{Y}_F$ and $T \to \bar{T}$ be the normalization morphism.

$$
\begin{aligned}
\bar{T}_{0,b} &= \bar{T} \times_S (0,b) \\
&= (X_F \times_F \mathcal{Y}_F) \times_S (0,b) \\
&= (X \times_{\mathbb{P}^1_x} F) \times_F (F \times_{\mathbb{P}^1_y \times S} \mathcal{Y}_S) \times_S (0,b) \\
&= (X \times_{\mathbb{P}^1_x} F) \times_F (F_{0,b} \times_{\mathbb{P}^1_y \times (0,b)} \mathcal{Y}_b) \\
&= X \times_{\mathbb{P}^1_x} (F \times_F ((\mathbb{P}^1_x \cup \mathbb{P}^1_y \text{ intersecting transversally at } x=y=0) \times_{\mathbb{P}^1_y} \mathcal{Y}_b)) \\
&= X \times_{\mathbb{P}^1_x} (\mathcal{Y}_b \cup m' \text{ copies of } \mathbb{P}^1_x \text{ where } j^{\text{th}} \text{ copy of } \mathbb{P}^1_x \text{ intersect } \mathcal{Y}_b \text{ at exactly} \\
&\quad \text{one point } (x=0, s_j) \in \mathbb{P}^1_x \times \mathcal{Y}_b, 1 \leqslant j \leqslant m') \\
&= m \text{ copies of } \mathcal{Y}_b \text{ and } m' \text{ copies of } X \text{ where the } j^{\text{th}} \text{ copy of } X \text{ intersect the} \\
&\quad i^{\text{th}} \text{ copy of } \mathcal{Y}_b \text{ at exactly one point } (r_i, s_j) \in X \times \mathcal{Y}_b, 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m'.
\end{aligned}
$$

Note that $\mathcal{O}_F = (\mathcal{O}_{\mathbb{P}^1_x} \otimes_k \mathcal{O}_{\mathbb{P}^1_y} \otimes_k k[t] \otimes_k \mathcal{O}_B)/(t - xy)$. So $\mathcal{O}_{\bar{T}} = (\mathcal{O}_X \otimes_k \mathcal{O}_{\mathcal{Y}} \otimes_k k[t])/(t - xy)$ since $\mathcal{O}_X$ is a flat $\mathcal{O}_{\mathbb{P}^1_x}$-algebra and $\mathcal{O}_{\mathcal{Y}}$ is a flat $\mathcal{O}_{\mathbb{P}^1_y} \otimes_k \mathcal{O}_B$-algebra. So $\bar{T}$ is regular at any closed point in the $t=0$ locus. Hence the normalization morphism $T \to \bar{T}$ is an isomorphism at these points. So $T_{0,b} = \bar{T}_{0,b}$.

Note that Φ_T is the normalized base change of $V_X \to X$ and the fiber of $T \to S$ over $(0,b)$ consist of components isomorphic to X and $\mathcal{Y}_b$. So for the G-cover $\Phi'_b : V_{0,b} \to T_{0,b}$ each copy of $\mathcal{Y}_b$ is a trivial component of Φ'_b.

Let $\eta = S \times_{\mathbb{A}^1_t} \operatorname{Spec}(k(t)) = B \times \operatorname{Spec}(k(t))$. For any closed point $b \in B$, let $\eta_b = b \times \operatorname{Spec}(k(t))$ denote the corresponding closed point of η.

Claim. — *Over the point η_b of S, $V_{T,\eta_b} \to T_{\eta_b}$ is a G-cover of irreducible η_b-curves.*

Proof. — Note that

$$
\begin{aligned}
T_{\eta_b} &= T \times_S \eta_b = (\mathcal{Y}_F \times_F X_F) \times_S \eta_b \\
&= (\mathcal{Y}_b \otimes_k k(t)) \times_{F \times_S \eta_b} (X \otimes_k k(t))
\end{aligned}
$$

So show that T_{η_b} is irreducible, it is enough to show that the function fields of the covers $\mathcal{Y}_b \otimes_k k(t)$ and $X \otimes_k k(t)$ of $F \times_S \eta_b$ are linearly disjoint over the function field of $F \times_S \eta_b$. Also note that the composition of morphisms $\mathcal{Y}_b \otimes_k k(t) \to F \times_S \eta_b \xrightarrow{\sim} \mathbb{P}^1_y \otimes_k k(t)$ is the extension of base field of $\mathcal{Y}_b \to \mathbb{P}^1_y$ to $k(t)$ and similarly $X \otimes_k k(t) \to F \times_S \eta_b \xrightarrow{\sim} \mathbb{P}^1_x \otimes_k k(t)$ is the extension of base field of $X \to \mathbb{P}^1_x$ to $k(t)$. Moreover the composition of isomorphisms $\mathbb{P}^1_y \otimes_k k(t) \to F \times_S \eta_b \to \mathbb{P}^1_x \otimes_k k(t)$ which we will call Θ is given by $y \mapsto t/x$.

Let $\alpha \in k(\mathcal{Y}_b)$ be such that $k(\mathcal{Y}_b) = k(y)[\alpha]$ and $f(y, Z)$ be the minimal polynomial of α in $k(y)[Z]$. We have a cover $\mathcal{Y}_b \otimes_k k(t) \to \mathbb{P}^1_x \otimes_k k(t)$ via the isomorphism Θ. Consider the resulting field extension $L_2 = k(t)(\mathcal{Y}_b)$ of $k(t, x)$. We also have a field extension $L_1 = k(t)(X)$ of $k(t, x)$ obtained from the morphism $X \to \mathbb{P}^1_x$ base changed to $k(t)$. To see that T_{η_b} is irreducible, it is enough to show that L_1 and L_2 are linearly disjoint over $k(t, x)$. Note that $L_2 \cong k(t, y)[\alpha]$, hence viewing L_2 as an extension $k(t, x)$, we get that $L_2 = k(t, x)[\alpha']$ where α' is a root of the irreducible polynomial $f(t/x, Z)$ in $k(t, x)[Z]$. We observe that L_1 and L_2 are linearly disjoint over $k(t, x)$ iff $[L_1 L_2 : L_1] = [L_2 : k(t, x)] = \deg_Z(f(t/x, Z))$. But $L_1 L_2 = L_1[\alpha']$, so it is enough to show $f(t/x, Z)$ is irreducible in $k(t)(X)[Z]$. Let $f(t/x, Z) = Z^n + a_{n-1}(t/x)Z^{n-1} + \ldots + a_0(t/x)$ and nonzero $\gamma \in k$ be such that $x = \gamma$ is not a pole of $a_0(t/x), \ldots, a_{n-1}(t/x)$. Let $\tilde{\gamma} \in X$ be a closed point lying above the closed $x = \gamma$ of $\mathbb{P}^1_x$. Then at the point $\tilde{\gamma}$ the polynomial $f(t/x, Z)$ reduces to $f(t/\gamma, Z) \in k(t)[Z]$. Since $f(y, Z)$ is irreducible in $k(y)[Z]$, $f(t/\gamma, Z)$ is irreducible in $k(t)[Z]$. Hence $f(t/x, Z)$ is also irreducible in $k(t)(X)[Z]$.

The proof of the irreducibility of V_{T,η_b} is also similar and can be obtained by replacing X by V_X in the above argument. □

From the claim it follows that for any $b \in B$ there exists a nonempty open subset U_b of $\mathbb{A}^1_t$ such that the fiber of Φ_T over $(a, b) \in S$ for any closed point a of U_b is a G-cover of irreducible curves. Note that T and V_T are normal, hence so is T_η and $V_{T,\eta}$. So most of the fibers of Φ_T are covers of smooth irreducible curves. Hence U is a nonempty open set.

Finally there is a morphism $T_\eta \to X_\eta = X \times_k \eta$ coming from the morphism $T \to X_F$ which is étale away from $y = 0$. But the rational function y on X_η is same as t/x. So $T_\eta \to X_\eta$ is étale away from points lying over $x = \infty$. In other words, $T_\eta \to X_\eta$ is étale over $C \times_k \eta$.

So for any closed point (a, b) of $U \subset S = \mathbb{A}^1_t \times B$, Φ'_b is a degeneration of the G-cover $\Phi_{a,b} : V_{a,b} \to T_{a,b}$ and $T_{a,b} \to X$ is a smooth irreducible cover étale over C (of the same degree as $\mathcal{Y}_b \to \mathbb{P}^1_y$). □

Corollary 20. — *Let $\mathcal{E}(7.1)$ be the embedding problem*

(7.1)
$$\begin{tikzcd} & & & \pi_1(C) \arrow[dl, dashed] \arrow[d, "\alpha"] & \\ 1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r, "\phi"] & G \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

Let X be the smooth completion of C and $\Phi : V_X \to X$ be a G-cover étale over C corresponding to α. Let the notation and hypothesis be as in Proposition 19. Let $s = (a, b) \in U \subset S$ be a fixed point, $D \subset T_s$ be the preimage of C under the morphism $T_s \to X$ and g be the genus of $\mathcal{Y}_b$. Then $\pi_1(D)$ is an effective subgroup of $\pi_1(C)$ for the given embedding problem if there exists a homomorphism $\theta : \Pi_g^m \to H/p(H)$ with $\mathrm{im}(\theta)$ a relative generating subset of $H/p(H)$ in $\Gamma/p(H)$. Here Π_g is the surface group of genus g.

Proof. — By the above proposition Φ_s has a degeneration to Φ'_b with m copies of $\mathcal{Y}_b$ as trivial components. Also prime to p part of $\pi_1(\mathcal{Y}_b)$ is the prime to p part of the profinite completion of Π_g. Hence the corollary follows from Theorem 5.1. □

Corollary 21. — *Let Π be an index p-normal subgroup of $\pi_1(C)$ and $D \to C$ be the corresponding étale cover. Consider the embedding problem $\mathcal{E}(7.1)$ in the above corollary and let $V \to C$ be the G-Galois cover corresponding to α. Let X, V_X and Z be the smooth completion of C, V and D respectively. Suppose there exist a separable cover $\theta : X \to \mathbb{P}^1_x$ étale over $x = 0$ with $\theta^{-1}(x = \infty) \cap C$ empty and a p-cyclic cover $Y \to \mathbb{P}^1_x$ branched only at $x = \infty$ such that the normalization of the cover $X \times_{\mathbb{P}^1_x} Y \to X$ is same as the cover $Z \to X$. Also assume that the genus g_Y is at least the relative rank of $H/p(H)$ in $\Gamma/p(H)$ and the upper jump of $Y \to \mathbb{P}^1_x$ is different from all the upper jumps of $V_X \to \mathbb{P}^1_x$ at all the points of V_X lying above $x = \infty$. Then Π is an effective subgroup of $\pi_1(C)$ for the embedding problem* (7.1).

Proof. — Since $Y \to \mathbb{P}^1_x$ is a p-cyclic cover, it is given by an Artin-Schreier polynomial $z^p - z - f(x)$ where $f(x)$ is a polynomial of degree r for some r coprime to p. Let $f(x) = a_r x^r + a_{r-1}x^{r-1} + \ldots + a_0$ with $a_r \neq 0$. Let $B = \mathbb{A}^r$ and $\mathcal{Y} \to B \times \mathbb{P}^1_y$ be the cover given by $z^p - z - (y^{-r} + b_{r-1}y^{-r+1} + \ldots + b_0)$ where b_i's are coordinates of B. Note that this is a family of p-cyclic covers of $\mathbb{P}^1_y$ branched only at $y = 0$.

Let $S = B \times \mathbb{A}^1_t$, F, T, $\Phi_T : V_T \to T$, etc. be defined as in the setup before Proposition 19. Let $U = \{s \in S : \Phi_s \text{ is a cover of smooth curves }\}$ also be as in Proposition 19.

Note that Φ_s is the fiber of $\Phi_T : V_T \to T$ where V_T and T are the normalization of $V_{X_F} \times_F \mathcal{Y}_F$ and $X_F \times_F \mathcal{Y}_F$ respectively. So for $s = (t, b) \in U$ with $b = (\beta_{r-1}, \ldots, \beta_0)$, T is the normalization of $X \times_{\mathbb{P}^1_x} \mathcal{Y}_s$ where $\mathcal{Y}_s \to \mathbb{P}^1_x$ is given by

$$z^p - z - (t^{-r}x^r + \beta_{r-1}t^{-r+1}x^{r-1} \ldots + \beta_0).$$

By the hypothesis on the upper jumps of $V_X \to \mathbb{P}^1_x$ and the upper jump of $Y \to \mathbb{P}^1_x$ which is same as the upper jump of $\mathcal{Y}_s \to \mathbb{P}^1_x$, we obtain that for every point $s = (t, b) \in S$ with $t \neq 0$ the normalization of T_s and $V_{T,s}$ have constant genus (i.e. independent of s). Hence by Lemma 10 $U = S \backslash \{t = 0\}$. So by Corollary 20 we conclude that $\pi_1(T_s \setminus \{\text{points lying above } x = \infty\})$ is an effective subgroup of $\pi_1(C)$ for all $s = (t, b) \in S$ with $t \neq 0$. By an appropriate choice of t and b one can arrange that the cover $T_s \to X$ is same as $Z \to X$. Hence $\pi_1(D)$ is an effective subgroup of $\pi_1(C)$. $\square$

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