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BEILINSON-HODGE CYCLES ON SEMIABELIAN VARIETIES

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Given a smooth not necessarily proper complex variety U, Beilinson [B] conjectured that all Hodge cycles in $H^*(U,\mathbb{Q})$ come from motivic cohomology, or more precisely that the so called regulator map

$$reg : CH^i(U,j)\otimes\mathbb{Q} \to Hom_{MHS}(\mathbb{Q}(-i), H^{2i-j}(U,\mathbb{Q}))$$

from Bloch's higher Chow group [Bl] is surjective. This is a very natural and appealing statement which includes the usual Hodge conjecture. Unfortunately, it has turned out that it is not true in this generallity, c.f. [J, 9.11], [KL]. There is presumably a restricted range of (i,j) for which this conjecture is viable. For instance the line $j = 0$, which corresponds to the usual Hodge conjecture, should lie in this set. Work of Asakura and Saito [AS] suggests that the conjecture should also hold when $i = j$. Following these authors, we refer to this special case as the Beilinson-Hodge conjecture.

Our goal here is to prove the Beilinson-Hodge conjecture when U is either a semiabelian variety or a product of smooth curves. The method is based on the study of invariants under the Mumford-Tate group.

1. Reduction lemma

We recall [Bl, L] that given a variety U, Bloch has defined a bigraded abelian group $\bigoplus CH^i(U,j)$. The elements are represented by certain codimension i algebraic cycles on $U\times\mathbb{A}^j$. There are products

$$CH^i(U,j)\times CH^p(U,q)\to CH^{i+p}(U,j+q)$$

when U is smooth. A cycle $Z\subset U\times\mathbb{A}^j$, representing an element of $CH^i(U,j)$, has a fundamental class in

$$H^{2i}(U\times\mathbb{A}^j, U\times\partial\mathbb{A}^j)(i)\cong H^{2i-j}(U)(i)$$

where $\partial\mathbb{A}^j$ is a union of the hyperplanes corresponding to the faces of $\mathbb{A}^j$ when viewed as an algebraic simplex. This extends to a homomorphism

$$reg : CH^i(U,j)\to Hom_{MHS}(\mathbb{Z}(-i), H^{2i-j}(U,\mathbb{Z}))$$

This description was indicated in [Bl]. Other explicit constructions of this map can be found in [KLM], and [AS, §1] for the subgroup of decomposable cycles. From these formulas, it is clear that the map respects products, and the special case

$$reg : CH^1(U,1) = \mathcal{O}(U)^*\to \mathrm{Hom}_{MHS}(\mathbb{Z}(-1), H^1(U,\mathbb{Z}))\subset H^1(U,\mathbb{Z}(1))$$

is just the composition of the inclusion $\mathcal{O}(U)^*\subset\mathcal{O}^{an}(U)^*$ with the connecting map associated to the exponential sequence.

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It is convenient to define the space of Beilinson-Hodge cycles

$$BH^q(U) = Hom_{MHS}(\mathbb{Q}(-q), H^q(U, \mathbb{Q}))$$

Then the Beilinson-Hodge conjecture asserts that $CH^q(U, q)$ surjects onto $BH^q(U)$. Note that the conjecture is only interesting for open varieties, because it is vacuously true if the variety is proper, since $BH^* = 0$ in this case by [D2]. The first nontrivial case of the conjecture, when $q = 1$, turns out to be easy to understand and prove, even integrally. It is not unreasonable to attribute this to Abel, since it is closely related to his classical theorem.

Theorem 1.1 (Abel)**.** *For any smooth variety U, the map*

$$reg : \mathcal{O}(U)^* \to \mathrm{Hom}_{MHS}(\mathbb{Z}(-1), H^1(U, \mathbb{Z}))$$

is surjective

Proof. Choose a smooth compactification X such that $D = X - U$ has normal crossings. Let $d\mathcal{O}_U^{an}$ denote the image of $d : \mathcal{O}_U^{an} \to \Omega_U^{an1}$ in the category of sheaves. The group $H^1(U, \mathbb{Z}(1))$ is torsion free by the universal coefficient theorem, so it can be viewed as a subgroup of $H^1(U, \mathbb{C})$. An element in $H^1(U, \mathbb{Z}(1))$ is in $BH^1(U)$ if and only if it lies in $F^1H^1(U) = \ker[H^1(U, \mathbb{C}) \to H^1(X, \mathcal{O}_X)]$. Chasing the following commutative diagram, with exact rows,

$$\begin{tikzcd}
H^0(U, \mathcal{O}_U^{an*}) \arrow[r, "\delta"] \arrow[d, "d\log"] & H^1(U, \mathbb{Z}(1)) \arrow[r] \arrow[d] & H^1(U, \mathcal{O}_U^{an}) \arrow[d, equal] \\
H^0(U, d\mathcal{O}_U^{an}) \arrow[r] & H^1(U, \mathbb{C}) \arrow[r] & H^1(U, \mathcal{O}_U^{an}) \\
H^0(X, \Omega_X^1(\log D)) \arrow[r] \arrow[u] & H^1(U, \mathbb{C}) \arrow[r] \arrow[u, equal] & H^1(X, \mathcal{O}_X) \arrow[u]
\end{tikzcd}$$

shows that the set of these classes coincides with $\{\delta(f) \mid d\log(f) \in H^0(\Omega_X(\log D))\}$. The condition $d\log(f) \in H^0(\Omega_X(\log D))$ can be seen to force f to have singularities of finite order along D. Thus

$$BH^1(U) \cap H^1(X, \mathbb{Z}) = \delta(\mathcal{O}(U)^*).$$

□

Lemma 1.1. *If the products $BH^1(U) \times \ldots \times BH^1(U) \to BH^q(U)$ are surjective for all q, then the Beilinson-Hodge conjecture holds for U.*

Proof. This follows from the following commutative diagram and theorem 1.1

$$\begin{tikzcd}
CH^1(U,1) \times \ldots \times CH^1(U,1) \arrow[r] \arrow[d] & CH^q(U,q) \arrow[d] \\
BH^1(U) \times \ldots \times BH^1(U) \arrow[r] & BH^q(U)
\end{tikzcd}$$

□

2. Mumford-Tate groups

The category of rational mixed Hodge structures form a neutral Tannakian category over $\mathbb{Q}$ [DMOS, chap II]. Let $\langle H\rangle$ denote the Tannakian category generated by a mixed Hodge structure H. This is the full subcategory consisting of all subquotients of tensor powers $T^{m,n}H = H^{\otimes m} \otimes (H^*)^{\otimes n}$. This construction extends to any set of Hodge structures. The Mumford-Tate group $MT(H)$ is the group of tensor automorphisms of the forgetful functor from $\langle H\rangle$ to $\mathbb{Q}$-vector spaces. By Tannaka duality $\langle H\rangle$ is equivalent to the category of representations of this group. When H is a pure Hodge structure, $MT(H)$ can be defined in a more elementary fashion as the smallest $\mathbb{Q}$-algebraic group whose real points contains the image of the torus defining the Hodge structure. We define two auxillary groups. The extended Mumford-Tate group $EMT(H)$ is $MT(\langle H, \mathbb{Q}(1)\rangle)$, and it surjects onto $MT(H)$. (Some authors consider $EMT(H)$ to be the Mumford-Tate group). The special Mumford-Tate group $SMT(H) = \ker[EMT(H) \to \mathbb{G}_m]$ with respect to the map that is induced by the inclusion $\langle \mathbb{Q}(1)\rangle \subset \langle H, \mathbb{Q}(1)\rangle$.

Theorem 2.1.

(1) *If* $\mathbb{Q}(1)$ *(respectively* $\mathbb{Q}(m)$ *with* $m \neq 0$*) lies in* $\langle H\rangle$*, then* $MT(H)$ *is isomorphic (respectively isogenous) to* $EMT(H)$*. Otherwise* $EMT(H) \cong MT(H) \times \mathbb{G}_m$*.*

(2) $MT(H) \subset GL(H)$ *is the largest subgroup leaving every rational element of type* $(0,0)$ *in* $T^{m,n}H$ *invariant for all* m, n*.* $SMT(H)$ *leaves rational elements of type* (q,q) *in* $T^{m,n}H$ *invariant for all* m, n, q*.*

(3) *If* H *is pure and polarizable, then* $MT(H)$ *is connected and reductive.*

(4) *Let* $H^{split} = \bigoplus_k Gr_k^W H$*, then* $MT(H)$ *is a semidirect product of* $MT(H^{split})$ *with a unipotent group.*

Proof. For the first statement, see [Mi, pp 466-467]. The next two properties are standard and proved in [DMOS, chap I], although [An, §2] would be a more concise reference. The last part is essentially given in [An]. We indicate the proof for completeness. Let P be the group linear automorphisms of H preserving the flag $W_\bullet$. The unipotent radical $UP \subset P$ is the subgroup which acts trivially on Gr_k^W. We have inclusion of tensor categories

$$\iota : \langle H^{split}\rangle \to \langle H\rangle$$

with a right inverse $H' \mapsto (H')^{split}$. Therefore we get a split surjection of Tannaka duals $\iota^* : MT(H) \to MT(H^{split})$. The kernel ι^* lies in UP, and is therefore unipotent. □

Corollary 2.2. $MT(H^{split})$ *is the quotient of* $MT(H)$ *by its unipotent radical.*

Let us turn to the case where U is either a semiabelian variety or a smooth curve. Set $MT(U) = MT(H^1(U)) = EMT(H^1(U))$, where the last equality follows from the theorem. Also let $SMT(U) = SMT(H^1(U))$.

Let $H = H^1(U)$ and let $W = W_1 H = H^1(X)$. Choose a complementary subspace V to W in H. We also know that $MT(U)$ preserves the weight filtration on $H^1(U)$ ([An, Lemma 2c]). Hence Φ the kernel of $MT(H) \to MT(H^{split})$ and the unipotent radical of $MT(U)$ is a subspace of $\mathrm{Hom}_{\mathbb{Q}}(V, W)$.

Corollary 2.3. *As a subgroup of $GL(H) = GL(V \oplus W)$*

$$SMT(U) = \{\begin{pmatrix} I & 0 \\ f & S \end{pmatrix} \mid S \in SMT(W) \text{ and } f \in \Phi\}.$$

3. Main theorem

Let H be the first cohomology of a semiabelian variety or a smooth affine curve. We want to refine the description of $SMT(H)$ given by corollary 2.3. We define three subspaces $V_i \subset H$. Let $V_3 = W_1 H$, let $V_1 \subseteq H^{SMT(H)}$ be a complement to V_3 in $W_1 H + H^{SMT(H)}$, and finally choose V_2 to be a complement to $V_1 + V_3$ in H. Thus we have a decomposition

$$H = V_1 \oplus V_2 \oplus V_3 \tag{1}$$

with respect to which $SMT(H)$ becomes a subgroup of the following matrix group:

$$\{\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & f & S \end{pmatrix} \mid S \in SMT(V_3) \text{ and } f \in Hom(V_2, V_3)\}.$$

The unipotent radical $U(SMT(H))$ lies in the subgroup

$$\{\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & f & I \end{pmatrix} \mid S \in SMT(V_3) \text{ and } f \in Hom(V_2, V_3)\}. \tag{2}$$

Lemma 3.1. *For any nonzero $u \in V_2$, we can find a $g \in U(SMT(H))$ such that $gu \neq u$, or equivalently such that $f(u) \neq 0$ with respect to the matrix (2).*

Proof. Given a nonzero $u \in V_2$, we have $g_1 u \neq u$ for some $g_1 \in SMT(H)$. Writing

$$g_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & f & S \end{pmatrix}$$

we see that $f(u) \neq 0$. Set

$$g_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & S^{-1} \end{pmatrix}$$

This lies in $SMT(H)$, since the map $SMT(H) \to SMT(H)/U(SMT(H))$ splits. Then $g = g_1 g_2$ has the desired property. □

Let

$$BH^q(H) = \mathrm{Hom}(\mathbb{Q}(-q), H^{\otimes q})$$

for H as above.

Theorem 3.1. *The product maps $BH^1(H) \times \ldots \times BH^1(H) \to BH^q(H)$ are surjective for all q*

Proof. To simplify book keeping, we will usually write tuples $(j_1, \dots j_n)$ as strings $j_1 \dots j_n$. Juxtaposition is used to denote concatenation of strings, with exponents used for repetition. For example, $1^2\, 2\, 3^0 = 1\,1\,2$.

(1) leads to a decomposition

$$H^{\otimes n} = \bigoplus_{j_1,\dots,j_n} V(j_1 \dots j_n), \tag{3}$$

where

$$V(j_1 \dots j_n) = V_{j_1} \otimes \dots \otimes V_{j_n}$$

Let $\tau \in BH^n(H)$ i.e. suppose that it is a Beilinson-Hodge cycle. Our goal is to show that $\tau \in BH^1(H)^{\otimes n}$. Let us decompose

$$\tau = \sum \tau_{j_1 \dots j_n}$$

with respect to (3). It suffices to show that $\tau \in V_1^{\otimes n}$, since $V_1 \subseteq BH^1(H)$. After replacing τ by $\tau - \tau_{1^n}$, we will show τ equals 0.

We next argue that any component $\tau' = \tau_{j_1 j_2 \dots j_n}$ with all of the $j_i \in \{1, 2\}$ must be zero. Assume that $\tau' \neq 0$, then we will derive a contradiction. Let

$$\tau_{j_1 j_2 \dots j_n} = x_1 \otimes x_2 \otimes \dots \otimes x_n$$

with $x_i \in V_{j_i}$. From the previous paragraph, $j_1 \dots j_n = 1^{n_1} 2^{n_2}\, 1^{n_3} \dots$ must have at least one 2. Since $u = x_{n_1+1} \in V_2 - \{0\}$, we can choose a $g \in U(SMT(H))$ so that $f(u) \neq 0$, with f as in (2). Then $g\tau' - \tau'$ will have a nonzero component in $V(1^{n_1}\, 3\, 2^{n_2-1} 1^{n_3} \dots)$. We must have $g\tau - \tau = 0$, since τ is invariant under $SMT(H)$ by theorem 2.1. Thus τ must have another term τ'' whose image under $g - I$ has a nonzero component of type $1^{n_1}\, 3\, 2^{n_2-1} \dots$. The only possible candidate is $\tau'' = \tau_{1^{n_1} 3 2^{n_2-1} \dots}$. However, after expanding this as a product of x_i's as above, we can see that $(g - I)\tau''$ has no nonzero components of the required type. For example, $(g - I)\tau'' = 0$ if the second 2 is absent from $j_1 j_2 \dots j_n$, $(g - I)\tau''$ is sum of types $1^{n_1} 3^2 1^{n_3}, 1^{n_1} 321^{n_3}$ and $1^{n_1} 231^{n_3}$ if $j_1 j_2 \dots j_n = 1^{n_1} 2^2 1^{n_3}$ and so on. Therefore $\tau' = 0$ as claimed.

To conclude, we note that the projection of a nonzero Beilinson-Hodge cycle to $(Gr_2^W H)^{\otimes n}$ must be nonzero. We deduce from the previous paragraph that for every component of τ, must have at least one $j_i = 3$. This implies that τ projects to zero in $(Gr_2^W H)^{\otimes n}$. Therefore it must already be zero. □

Corollary 3.2. *The Beilinson-Hodge conjecture holds for a product of smooth curves.*

Proof. Let $U = \prod U_i$, where U_i are smooth curves. Let $H = H^1(U)$. Then by Künneth's formula and the theorem, the conditions of lemma 1.1 hold. □

Corollary 3.3. *The Beilinson-Hodge conjecture holds for a semiabelian variety.*

Proof. Let U be a semiabelian variety. Let $H = H^1(U)$. By the theorem, we have that $BH^n(H) = BH^1(H)^{\otimes n}$. Now observe that $H^*(U) = \wedge^* H$ which is a direct summand of the tensor algebra. So the BH cycles on $H^n(U)$ are given by products of BH-cycles on H. □

The referee pointed out the following interesting corollary which can be proved along the same lines as the first corollary.

Corollary 3.4. *Let $U = \prod U_i$ be a product of n smooth curves with smooth projective completions X_i. Then $BH^n(U) \neq 0$ if and only if there exists torsion cycles in $J(X_i)$ with nonempty support on $X_i - U_i$ for each i.*

Proof. Using the theorem, this can be reduced to the case of $n = 1$. By theorem 1.1, a nonzero element of $BH^1(U_1)$ lifts to an element $f \in \mathcal{O}(U_1)^* \otimes \mathbb{Q}$, which in turn defines a divisor $(f) \in Div(U_1) \otimes \mathbb{Q}$ with nonempty support in $X_1 - U_1$. Conversely, any such $\mathbb{Q}$-divisor determines a nonzero element of $BH^1(U_1)$ □

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