WOLD DECOMPOSITION FOR DOUBLY COMMUTING ISOMETRIES

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Abstract. In this paper, we obtain a complete description of the class of n-tuples ($n \geq 2$) of doubly commuting isometries. In particular, we present a several variables analogue of the Wold decomposition for isometries on Hilbert spaces. Our main result is a generalization of M. Slocinski's Wold-type decomposition of a pair of doubly commuting isometries.

1. Introduction

Let V be an isometry on a Hilbert space $\mathcal{H}$, that is, $V^*V = I_{\mathcal{H}}$. A closed subspace $\mathcal{W} \subseteq \mathcal{H}$ is said to be *wandering subspace* for V if $V^k\mathcal{W} \perp V^l\mathcal{W}$ for all $k, l \in \mathbb{N}$ with $k \neq l$, or equivalently, if $V^m\mathcal{W} \perp \mathcal{W}$ for all $m \geq 1$. An isometry V on $\mathcal{H}$ is said to be a *unilateral shift* or *shift* if

$$\mathcal{H} = \bigoplus_{m \geq 0} V^m \mathcal{W},$$

for some wandering subspace $\mathcal{W}$ for V.

For a shift V on $\mathcal{H}$ with a wandering subspace $\mathcal{W}$ we have

$$\mathcal{H} \ominus V\mathcal{H} = \bigoplus_{m \geq 0} V^m\mathcal{W} \ominus V(\bigoplus_{m \geq 0} V^m\mathcal{W}) = \bigoplus_{m \geq 0} V^m\mathcal{W} \ominus \bigoplus_{m \geq 1} V^m\mathcal{W} = \mathcal{W}.$$

In other words, the wandering subspace of a shift is unique and is given by $\mathcal{W} = \mathcal{H} \ominus V\mathcal{H}$. The dimension of the wandering subspace of a shift is called the *multiplicity* of the shift.

The classical Wold decomposition theorem ([19], see also page 3 in [11]) states that every isometry on a Hilbert space is either a shift, or a unitary, or a direct sum of shift and unitary (see Theorem 2.1). The Wold decomposition theorem, or results analogous of the Wold decomposition theorem, plays an important role in many areas of operator algebras and operator theory, including the invariant subspace problem for Hilbert spaces of holomorphic functions (cf. [7], [8], [10], [14], [15]).

The natural question then becomes: Let $n \geq 2$ and $V = (V_1, \ldots, V_n)$ be an n-tuple of commuting isometries on a Hilbert space $\mathcal{H}$. Does there exists a Wold-type decomposition for V? What happens if we have a family of isometries?

Several interesting results have been obtained in many directions. For instance, in [17] Suciu developed a structure theory for semigroup of isometries (see also [4], [6], [9]). However, to get a more precise result it will presumably always be necessary to make additional assumptions on the family of operators. Before proceeding, we recall the definition of the doubly commuting isometries.

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Let $V = (V_1, \ldots, V_n)$ be an n-tuple ($n \geq 2$) of commuting isometries on $\mathcal{H}$. Then V is said to *doubly commute* if

$$V_i V_j^* = V_j^* V_i,$$

for all $1 \leq i < j \leq n$.

The simplest example of an n-tuple of doubly commuting isometries is the tuple of multiplication operators $(M_{z_1}, \ldots, M_{z_n})$ by the coordinate functions on the Hardy space $H^2(\mathbb{D}^n)$ over the polydisc $\mathbb{D}^n$ ($n \geq 2$).

In [16], M. Slocinski obtained an analogous result of Wold decomposition theorem for a pair of doubly commuting isometries.

THEOREM 1.1. *(M. Slocinski) Let $V = (V_1, V_2)$ be a pair of doubly commuting isometries on a Hilbert space $\mathcal{H}$. Then there exists a unique decomposition*

$$\mathcal{H} = \mathcal{H}_{ss} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{uu},$$

where $\mathcal{H}_{ij}$ are joint V-reducing subspace of $\mathcal{H}$ for all $i, j = s, u$. Moreover, V_1 on $\mathcal{H}_{i,j}$ is a shift if $i = s$ and unitary if $i = u$ and that V_2 is a shift if $j = s$ and unitary if $j = u$.

We refer to [9] for a new proof of Slocinski's result (see also [3], [7], [8]).

Slocinski's Wold-type decomposition does not hold for general tuples of commuting isometries (cf. Example 1 in [16]). However, if V is an n-tuple of commuting isometries such that

$$\dim \ker \left(\prod_{i=1}^{n} V_i^* \right) < \infty,$$

then V admits a Wold-type decomposition (see Theorem 2.4 in [2]).

In this paper, we obtain a Wold-type decomposition for tuples of doubly commuting isometries. We extend the ideas of M. Slocinski on the Wold-type decomposition for a pair of isometries to the multivariable case ($n \geq 2$). Our approach is simple and based on the classical Wold decomposition for a single isometry. Moreover, our method yields a new proof of Slocinski's result for the base case $n = 2$. In addition, we obtain an explicit description of the closed subspaces in the orthogonal decomposition of the Hilbert space (see the equality (3.2)).

The paper is organized as follows. In Section 2, we set up notations and definitions and establish some preliminary results. In Section 3, we prove our main result and some of its consequences.

Note added in proof: After this work was completed, we became aware that generalization of Slocinski's results to n-tuples of doubly commuting isometries has been obtained independently by Timotin [18], Gaspar and Suciu [5] and in the context of C^*-correspondence, by Skalski and Zacharias [14]. However, our main theorem, Theorem 3.1, is stronger in the sense that the orthogonal decomposition in (3.1) works for any $m \in \{2, \ldots, n\}$ (with $2 < n$). Also our approach yields an explicit representation of the closed subspaces in the orthogonal decomposition (see the equality (3.2)) and a new and simpler proof of earlier generalizations of Slocinski's result. It is a question of general interest whether, the present method, with explicit description of the closed subspaces (3.2) in the Wold decomposition (3.1) yields more structure theory results in the study of C^*-correspondences.

2. Preparatory results

In this section we recall the Wold decomposition theorem and present some elementary facts concerning doubly commuting isometries.

We begin with the Wold decomposition of an isometry.

Theorem 2.1. *(H. Wold) Let V be an isometry on $\mathcal{H}$. Then $\mathcal{H}$ admits a unique decomposition $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_u$, where $\mathcal{H}_s$ and $\mathcal{H}_u$ are V-reducing subspaces of $\mathcal{H}$ and $V|_{\mathcal{H}_s}$ is a shift and $V|_{\mathcal{H}_u}$ is unitary. Moreover,*

$$\mathcal{H}_s = \bigoplus_{m=0}^{\infty} V^m \mathcal{W} \qquad \text{and} \qquad \mathcal{H}_u = \bigcap_{m=0}^{\infty} V^m \mathcal{H},$$

where $\mathcal{W} = ran(I - VV^*) = kerV^*$ is the wandering subspace for V.

Proof. Let $\mathcal{W} = \text{ran}(I - VV^*)$ be the wandering subspace for V and

$$\mathcal{H}_s := \bigoplus_{m=0}^{\infty} V^m \mathcal{W}.$$

Consequently, $\mathcal{H}_s$ is a V-reducing subspace of $\mathcal{H}$ and that $V|_{\mathcal{H}_s}$ is an isometry. Furthermore,

$$\mathcal{H}_u := \mathcal{H}_s^{\perp} = \Big(\bigoplus_{m=0}^{\infty} V^m \mathcal{W}\Big)^{\perp} = \bigcap_{m=0}^{\infty} (V^m \mathcal{W})^{\perp}.$$

We observe now that $I - VV^*$ is an orthogonal projection, hence $V^l(I - VV^*)V^{*l}$ is also an orthogonal projection and

$$V^l(I - VV^*)V^{*l} = (V^l(I - VV^*))(V^l(I - VV^*))^*,$$

for all $l \geq 0$. Thus we obtain

$$\text{ran}V^l(I - VV^*) = \text{ran}\big((V^l(I - VV^*))(V^l(I - VV^*))^*\big) = \text{ran}V^l(I - VV^*)V^{*l},$$

and hence

$$\begin{aligned}(V^l\mathcal{W})^{\perp} &= (V^l\text{ran}(I - VV^*))^{\perp} = (\text{ran}V^l(I - VV^*))^{\perp} \\ &= (\text{ran}V^l(I - VV^*)V^{*l})^{\perp} = \text{ran}(I - V^l(I - VV^*)V^{*l}) \\ &= \text{ran}[(I - V^lV^{*l}) \oplus V^{l+1}V^{*\,l+1}] = \text{ran}(I - V^lV^{*l}) \oplus \text{ran}V^{l+1} \\ &= (V^l\mathcal{H})^{\perp} \oplus V^{l+1}\mathcal{H} = \text{ker}V^{*l} \oplus V^{l+1}\mathcal{H},\end{aligned}$$

for all $l \geq 0$. Consequently, we have

$$\mathcal{H}_u = \bigcap_{m=0}^{\infty} \big(\text{ker}V^{*m} \oplus V^{m+1}\mathcal{H}\big) = \bigcap_{m=0}^{\infty} V^m\mathcal{H}.$$

Uniqueness of the decomposition readily follows from the uniqueness of the wandering subspace $\mathcal{W}$ for V. This completes the proof. ■

We now introduce some notation which will remain fixed for the rest of the paper. Given an integer $1 \leq m \leq n$, we denote the set $\{1, \ldots, m\}$ by I_m. In particular, $I_n = \{1, \ldots, n\}$.

Define $\mathcal{W}_i := \text{ran}(I - V_iV_i^*)$ for each $1 \leq i \leq n$ and

$$\mathcal{W}_A := \text{ran}(\prod_{i\in A}(I - V_iV_i^*)),$$

where A is a non-empty subset of I_m and $1 \leq m \leq n$.

By doubly commutativity of V it follows that

$$(I - V_iV_i^*)(I - V_jV_j^*) = (I - V_jV_j^*)(I - V_iV_i^*),$$

for all $i \neq j$. In particular,

$$\{(I - V_iV_i^*)\}_{i=1}^n$$

is a family of commuting orthogonal projections on $\mathcal{H}$. Therefore, for all non-empty subsets A of I_m $(1 \leq m \leq n)$ we have

$$\mathcal{W}_A = \text{ran}(\prod_{i\in A}(I - V_iV_i^*)) = \bigcap_{i\in A}\text{ran}(I - V_iV_i^*)) = \bigcap_{i\in A}\mathcal{W}_i. \tag{2.1}$$

The following simple result plays a basic role in describing the class of n-tuples of doubly commuting isometries.

Proposition 2.2. *Let $V = (V_1, \ldots, V_n)$ be an n-tuple ($n \geq 2$) of doubly commuting isometries on $\mathcal{H}$ and A be a non-empty subset of I_m for $1 \leq m \leq n$. Then $\mathcal{W}_A$ is a V_j-reducing subspace of $\mathcal{H}$ for all $j \in I_n \setminus A$.*

Proof. By doubly commutativity of V we have

$$V_j(I - V_iV_i^*) = (I - V_iV_i^*)V_j,$$

for all $i \neq j$, and thus

$$V_j(\prod_{i\in A}(I - V_iV_i^*)) = (\prod_{i\in A}(I - V_iV_i^*))V_j,$$

for all $j \in I_n \setminus A$, that is,

$$V_jP_{\mathcal{W}_A} = P_{\mathcal{W}_A}V_j,$$

where $P_{\mathcal{W}_A}$ is the orthogonal projection of $\mathcal{W}_A$ onto $\mathcal{H}$. This completes the proof. ∎

To complete this section we will use the preceding proposition to obtain the generalized wandering subspaces for n-tuple of doubly commuting isometries.

Corollary 2.3. *Let $V = (V_1, \ldots, V_n)$ be an n-tuple ($n \geq 2$) of doubly commuting isometries on $\mathcal{H}$ and $m \leq n$. Then for each non-empty subset A of I_m and $j \in I_n \setminus A$,*

$$\mathcal{W}_A \ominus V_j\mathcal{W}_A = ran(\prod_{i\in A}(I - V_iV_i^*)(I - V_jV_j^*)) = \Big(\bigcap_{i\in A}\mathcal{W}_i\Big) \cap \mathcal{W}_j.$$

Proof. Doubly commutativity of V implies that

$$\prod_{i\in A}(I - V_iV_i^*)(I - V_jV_j^*) = \prod_{i\in A}(I - V_iV_i^*) - V_j(\prod_{i\in A}(I - V_iV_i^*))V_j^*.$$

By Proposition 2.2 we have $V_j\mathcal{W}_A \subseteq \mathcal{W}_A$ for all $j \notin A$. Moreover

$$V_j\mathcal{W}_A = \operatorname{ran}[V_j \prod_{i\in A}(I - V_iV_i^*)V_j^*],$$

and hence

$$\mathcal{W}_A \ominus V_j\mathcal{W}_A = \operatorname{ran}(\prod_{i\in A}(I - V_iV_i^*) - V_j(\prod_{i\in A}(I - V_iV_i^*))V_j^*) = \operatorname{ran}(\prod_{i\in A}(I - V_iV_i^*)(I - V_jV_j^*)),$$

for all $j \notin A$. The second equality follows from (2.1). This completes the proof. ∎

3. The Main Theorem

In this section we will prove the main result of this paper.

Before proceeding, we shall adopt the following set of notations. Let $(T_1, \ldots, T_n)$ be an n-tuple of commuting operators on a Hilbert space $\mathcal{H}$ and $1 \leq m \leq n$. Let $A = \{i_1, \ldots, i_l\} \subseteq I_m$ and $1 \leq l \leq m$. We denote by T_A the $|A|$-tuple of commuting operators $(T_{i_1}, \ldots, T_{i_l})$ and $\mathbb{N}^A := \{\boldsymbol{k} = (k_{i_1}, \ldots, k_{i_l}) : k_{i_j} \in \mathbb{N}, 1 \leq j \leq l\}$. We also denote $T_{i_1}^{k_{i_1}} \cdots T_{i_l}^{k_{i_l}}$ by $T_A^{\boldsymbol{k}}$ for all $\boldsymbol{k} \in \mathbb{N}^A$.

Theorem 3.1. *Let $V = (V_1, \ldots, V_n)$ be an n-tuple ($n \geq 2$) of doubly commuting isometries on $\mathcal{H}$ and $m \in \{2, \ldots, n\}$. Then there exists 2^m joint $(V_1, \ldots, V_m)$-reducing subspaces $\{\mathcal{H}_A : A \subseteq I_m\}$ (counting the trivial subspace $\{0\}$) such that*

$$\mathcal{H} = \bigoplus_{A\subseteq I_m} \mathcal{H}_A, \tag{3.1}$$

where for each $A \subseteq I_m$,

$$\mathcal{H}_A = \bigoplus_{\boldsymbol{k}\in\mathbb{N}^A} V_A^{\boldsymbol{k}}\Big(\bigcap_{\boldsymbol{j}\in\mathbb{N}^{I_m\setminus A}} V_{I_m\setminus A}^{\boldsymbol{j}}\mathcal{W}_A\Big). \tag{3.2}$$

In particular, there exists 2^n orthogonal joint V-reducing subspaces $\{\mathcal{H}_A : A \subseteq I_n\}$ such that

$$\mathcal{H} = \sum_{A\subseteq I_n} \oplus\mathcal{H}_A,$$

and for each $A \subseteq I_n$ and $\mathcal{H}_A \neq \{0\}$, $V_i|_{\mathcal{H}_A}$ is a shift if $i \in A$ and unitary if $i \in I_n \setminus A$ for all $i = 1, \ldots, n$. Moreover, the above decomposition is unique, in the sense that

$$\mathcal{H}_A = \bigoplus_{\boldsymbol{k}\in\mathbb{N}^A} V_A^{\boldsymbol{k}}\Big(\bigcap_{\boldsymbol{j}\in\mathbb{N}^{I_n\setminus A}} V_{I_n\setminus A}^{\boldsymbol{j}}\mathcal{W}_A\Big),$$

for all $A \subseteq I_n$.

Proof. We shall prove the statement using mathematical induction.
For $m = 2$: By applying the Wold decomposition theorem, Theorem 2.1, to the isometry V_1 on $\mathcal{H}$ we have

$$\mathcal{H} = \bigoplus_{k_1\in\mathbb{N}} V_1^{k_1}\mathcal{W}_1 \bigoplus \Big(\bigcap_{k_1\in\mathbb{N}} V_1^{k_1}\mathcal{H}.\Big)$$

As $\mathcal{W}_1$ is a V_2-reducing subspace, it follows from the Wold decomposition theorem for the isometry $V_2|_{\mathcal{W}_1} \in \mathcal{L}(\mathcal{W}_1)$ that

$$\mathcal{W}_1 = \bigoplus_{k_2 \in \mathbb{N}} V_2^{k_2} (\mathcal{W}_1 \ominus V_2 \mathcal{W}_1) \bigoplus \Big(\bigcap_{k_2 \in \mathbb{N}} V_2^{k_2} \mathcal{W}_1 \Big) = \bigoplus_{k_2 \in \mathbb{N}} V_2^{k_2} (\mathcal{W}_1 \cap \mathcal{W}_2) \Big(\bigoplus \bigcap_{k_2 \in \mathbb{N}} V_2^{k_2} \mathcal{W}_1 \Big),$$

where the second equality follows from Corollary 2.3. Consequently,

$$\begin{aligned}
\mathcal{H} &= \bigoplus_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{W}_1 \bigoplus \Big(\bigcap_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{H} \Big) \\
&= \bigoplus_{k_1 \in \mathbb{N}} V_1^{k_1} \Big(\bigoplus_{k_2 \in \mathbb{N}} V_2^{k_2} \Big(\mathcal{W}_1 \cap \mathcal{W}_2 \Big) \bigoplus \bigcap_{k_2 \in \mathbb{N}} V_2^{k_2} \mathcal{W}_1 \Big) \bigoplus \bigcap_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{H} \\
&= \bigoplus_{k_1, k_2 \in \mathbb{N}} V_1^{k_1} V_2^{k_2} \Big(\mathcal{W}_1 \cap \mathcal{W}_2 \Big) \bigoplus_{k_1 \in \mathbb{N}} V_1^{k_1} \Big(\bigcap_{k_2 \in \mathbb{N}} V_2^{k_2} \mathcal{W}_1 \Big) \bigoplus \bigcap_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{H}.
\end{aligned}$$

Furthermore, the Wold decomposition of the isometry V_2 on $\mathcal{H}$

$$\mathcal{H} = \bigoplus_{k_2 \in \mathbb{N}} V_2^{k_2} \mathcal{W}_2 \bigoplus \bigcap_{k_2 \in \mathbb{N}} V_2^{k_2} \mathcal{H},$$

yields

$$V_1^{k_1} \mathcal{H} = \bigoplus_{k_2 \in \mathbb{N}} V_2^{k_2} V_1^{k_1} \mathcal{W}_2 \bigoplus \bigcap_{k_2 \in \mathbb{N}} V_1^{k_1} V_2^{k_2} \mathcal{H},$$

for all $k_1 \in \mathbb{N}$. From this we infer that

$$\bigoplus_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{H} = \bigoplus_{k_2 \in \mathbb{N}} V_2^{k_2} \Big(\bigcap_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{W}_2 \Big) \bigoplus \bigcap_{k_1, k_2 \in \mathbb{N}} V_1^{k_1} V_2^{k_2} \mathcal{H}.$$

Therefore,

$$\mathcal{H} = \bigoplus_{\boldsymbol{k} \in \mathbb{N}^2} V^{\boldsymbol{k}} \Big(\mathcal{W}_1 \cap \mathcal{W}_2 \Big) \bigoplus_{k_1 \in \mathbb{N}} V_1^{k_1} \Big(\bigcap_{k_2 \in \mathbb{N}} V_2^{k_2} \mathcal{W}_1 \Big) \bigoplus_{k_2 \in \mathbb{N}} V_2^{k_2} \Big(\bigcap_{k_1 \in \mathbb{N}} V_1^{k_1} \mathcal{W}_2 \Big) \bigoplus \bigcap_{\boldsymbol{k} \in \mathbb{N}^2} V^{\boldsymbol{k}} \mathcal{H}, \tag{3.3}$$

that is,

$$\mathcal{H} = \bigoplus_{A \subseteq I_2} \mathcal{H}_A,$$

where $\mathcal{H}_A$ is as in (3.2) for each $A \subseteq I_2$.

For $m+1 \leq n$: Now let for $m < n$, we have $\mathcal{H} = \bigoplus_{A \subseteq I_m} \mathcal{H}_A$, where for each non-empty subset A of I_m

$$\mathcal{H}_A = \bigoplus_{\boldsymbol{k} \in \mathbb{N}^A} V_A^{\boldsymbol{k}} \Big(\bigcap_{\boldsymbol{j} \in \mathbb{N}^{I_m \setminus A}} V_{I_m \setminus A}^{\boldsymbol{j}} \mathcal{W}_A \Big),$$

and for $A = \phi \subseteq I_m$,

$$\mathcal{H}_A = \bigcap_{\boldsymbol{k} \in \mathbb{N}^m} V_{I_m}^{\boldsymbol{k}} \mathcal{H}.$$

We claim that

$$\mathcal{H} = \sum_{A \subseteq I_{m+1}} \oplus \mathcal{H}_A.$$

Since $\mathcal{W}_A$ is V_{m+1}-reducing subspace for all none-empty $A \subseteq I_m$, we have

$$\begin{aligned}\mathcal{W}_A &= \bigoplus_{k_{m+1}\in\mathbb{N}} V_{m+1}^{k_{m+1}}(\mathcal{W}_A \ominus V_{m+1}\mathcal{W}_A) \bigoplus \Big(\bigcap_{k_{m+1}\in\mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{W}_A \Big) \\ &= \bigoplus_{k_{m+1}\in\mathbb{N}} V_{m+1}^{k_{m+1}}(\bigcap_{i\in A} \mathcal{W}_i \bigcap \mathcal{W}_{m+1}) \bigoplus \Big(\bigcap_{k_{m+1}\in\mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{W}_A \Big),\end{aligned}$$

and hence it follows that

$$\begin{aligned}\mathcal{H}_A &= \bigoplus_{\boldsymbol{k}\in\mathbb{N}^A} V_A^{\boldsymbol{k}} \Big(\bigcap_{\boldsymbol{j}\in\mathbb{N}^{I_m\setminus A}} V_{I_m\setminus A}^{\boldsymbol{j}} \mathcal{W}_A \Big) \\ &= \bigoplus_{\boldsymbol{k}\in\mathbb{N}^A} V_A^{\boldsymbol{k}} \Big(\bigcap_{\boldsymbol{j}\in\mathbb{N}^{I_m\setminus A}} V_{I_m\setminus A}^{\boldsymbol{j}} \Big(\bigoplus_{k_{m+1}\in\mathbb{N}} V_{m+1}^{k_{m+1}}(\bigcap_{i\in A} \mathcal{W}_i \cap \mathcal{W}_{m+1}) \bigoplus \Big(\bigcap_{k_{m+1}\in\mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{W}_A \Big)\Big)\Big) \\ &= \Big[\bigoplus_{\substack{\boldsymbol{k}\in\mathbb{N}^A \\ k_{m+1}\in\mathbb{N}}} V_A^{\boldsymbol{k}} V_{m+1}^{k_{m+1}} \Big(\bigcap_{\boldsymbol{j}\in\mathbb{N}^{I_m\setminus A}} V_{I_m\setminus A}^{\boldsymbol{j}} (\bigcap_{i\in A\cup\{m+1\}} \mathcal{W}_i) \Big) \Big] \bigoplus \Big[\bigoplus_{\boldsymbol{k}\in\mathbb{N}^A} V_A^{\boldsymbol{k}} \bigcap_{k_{m+1}\in\mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{W}_A \Big]\end{aligned}$$

Applying again the Wold decomposition to the isometry V_{m+1} on $\mathcal{H}$, we have

$$\mathcal{H} = \bigoplus_{k_{m+1}\in\mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{W}_{m+1} \bigoplus \bigcap_{k_{m+1}\in\mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{H},$$

and hence for $A = \phi \subseteq I_m$,

$$\begin{aligned}\mathcal{H}_A &= \bigcap_{\boldsymbol{k}\in\mathbb{N}^m} V_1^{k_1} \cdots V_m^{k_m} \mathcal{H} \\ &= \bigcap_{\boldsymbol{k}\in\mathbb{N}^m} V_1^{k_1} \cdots V_m^{k_m} (\bigoplus_{k_{m+1}\in\mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{W}_{m+1} \bigoplus \bigcap_{k_{m+1}\in\mathbb{N}} V_{m+1}^{k_{m+1}} \mathcal{H}) \\ &= \Big[\bigoplus_{k_{m+1}\in\mathbb{N}} V_{m+1}^{k_{m+1}} (\bigcap_{\boldsymbol{k}\in\mathbb{N}^m} V_1^{k_1} \cdots V_m^{k_m} \mathcal{W}_{m+1}) \Big] \bigoplus \Big[\bigcap_{\boldsymbol{k}\in\mathbb{N}^{A\cup\{m+1\}}} V_1^{k_1} \cdots V_m^{k_m} V_{m+1}^{k_{m+1}} \mathcal{H} \Big].\end{aligned}$$

Consequently,

$$\mathcal{H} = \sum_{A\subseteq I_{m+1}} \oplus\, \mathcal{H}_A.$$

It follows immediately from the above orthogonal decomposition of $\mathcal{H}$ that $V_i|_{\mathcal{H}_A}$ is a shift for all $i \in A$ and unitary for all $i \in I_n \setminus A$.
The uniqueness part follows from the uniqueness of the classical Wold decomposition of isometries and the canonical construction of the present orthogonal decomposition. This completes the proof. ■

Note that if $n = 2$, then (3.3) yields a new proof of Slocinski's Wold-type decomposition of a pair of doubly commuting isometries.

The following corollary is an n-variables analogue of the wandering subspace representations of pure isometries (that is, shift operators) on Hilbert spaces.

COROLLARY 3.2. *Let $V = (V_1, \ldots, V_n)$ be an n-tuple ($n \geq 2$) of doubly commuting shift operators on $\mathcal{H}$. Then*

$$\mathcal{W} = \bigcap_{i=1}^{n} ran(I - V_i V_i^*),$$

is a wandering subspace for V and

$$\mathcal{H} = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \oplus V^{\boldsymbol{k}} \mathcal{W}.$$

Proof. Let us note first that the given condition is equivalent to

$$\bigcap_{k \in \mathbb{N}} V_i^k \mathcal{H} = \{0\},$$

for all $1 \leq i \leq n$. Then the result readily follows from the proof of Theorem 3.1. ∎

Recall that a pair of n-tuples $V = (V_1, \ldots, V_n)$ and $W = (W_1, \ldots, W_n)$ on $\mathcal{H}$ and $\mathcal{K}$, respectively, are said to be unitarily equivalent if there exists a unitary map $U : \mathcal{H} \to \mathcal{K}$ such that $UV_i = W_i U$ for all $1 \leq i \leq n$.

The following corollary is a generalization of Theorem 1 in [16].

THEOREM 3.3. *Let $V = (V_1, \ldots, V_n)$ be an n-tuple ($n \geq 2$) of commuting isometries on $\mathcal{H}$. Then the following conditions are equivalent:*

(i) There exists a wandering subspace $\mathcal{W}$ for V such that

$$\mathcal{H} = \bigoplus_{\boldsymbol{k} \in \mathbb{N}^n} V^{\boldsymbol{k}} \mathcal{W}.$$

(ii) V_m is a shift for all $m = 1, \ldots, n$ and V is doubly commuting tuple.

(iii) There exists $m \in \{1, \ldots, n\}$ such that V_m is a shift and the wandering subspace for V_m is given by

$$\mathcal{W}_m = \bigoplus_{\boldsymbol{k} \in \mathbb{N}^n, k_m = 0} V^{\boldsymbol{k}} (\bigcap_{i=0}^{n} \mathcal{W}_i).$$

(iv) $\mathcal{W} := \bigcap_{i=1}^{n} \mathcal{W}_i$ is a wandering subspace for V and that $\mathcal{H} = \bigoplus_{\boldsymbol{k} \in \mathbb{N}^n} V^{\boldsymbol{k}} \mathcal{W}$.

(v) V is unitarily equivalent to $M_z = (M_{z_1}, \ldots, M_{z_n})$ on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ for some Hilbert space $\mathcal{E}$ with $dim \mathcal{E} = dim \mathcal{W}$.

Proof. (i) implies (ii) : That V_m is a shift, for all $1 \leq m \leq n$, follows from the fact that

$$\mathcal{H} = \bigoplus_{k \in \mathbb{N}} V_m^k \Big(\bigoplus_{\boldsymbol{k} \in \mathbb{N}^n, k_m = 0} V^{\boldsymbol{k}} \mathcal{W} \Big).$$

Now let $h \in \mathcal{H}$ and that

$$h = \sum_{i=0}^{\infty} V_m^i f_i. \qquad (f_i \in \bigoplus_{\boldsymbol{k} \in \mathbb{N}^n, k_m = 0} V^{\boldsymbol{k}} \mathcal{W})$$

It follows that for all $l \neq m$,

$$V_l V_m^* h = V_l(\sum_{i=1}^{\infty} V_m^{i-1} f_i) = \sum_{i=1}^{\infty} V_m^{i-1}(V_l f_i) = V_m^*(\sum_{i=0}^{\infty} V_m^i (V_l f_i)) = V_m^* V_l(\sum_{i=0}^{\infty} V_m^i f_i),$$

that is, V is doubly commuting.

(ii) implies (iii): By Corollary 3.1 we have

$$\mathcal{H} = \bigoplus_{\boldsymbol{k} \in \mathbb{N}^n} V^{\boldsymbol{k}}(\bigcap_{i=1}^{n} \mathcal{W}_i) = \bigoplus_{k \in \mathbb{N}} V_m^k \Big(\bigoplus_{\boldsymbol{k} \in \mathbb{N}^n,\, k_m = 0} V^{\boldsymbol{k}}(\bigcap_{i=1}^{n} \mathcal{W}_i)\Big),$$

and hence (iii) follows.

(iii) implies (iv): Since V_m is a shift with the wandering subspace

$$\mathcal{W}_m = \bigoplus_{\boldsymbol{k} \in \mathbb{N}^n,\, k_m = 0} V^{\boldsymbol{k}}(\bigcap_{i=0}^{n} \mathcal{W}_i),$$

we infer that

$$\mathcal{H} = \bigoplus_{k \in \mathbb{N}} V_m^k \mathcal{W}_m = \bigoplus_{k \in \mathbb{N}} V_m^k \Big(\bigoplus_{\boldsymbol{k} \in \mathbb{N}^n,\, k_m = 0} V^{\boldsymbol{k}}(\bigcap_{i=0}^{n} \mathcal{W}_i)\Big) = \bigoplus_{k \in \mathbb{N}} V_m^k (\bigcap_{i=1}^{n} \mathcal{W}_i),$$

and hence (iv) follows.

(iv) implies (v): Let $\mathcal{E} = \bigcap_{i=1}^{n} \mathcal{W}_i$. Define the unitary operator

$$U : \mathcal{H} = \bigoplus_{\boldsymbol{k} \in \mathbb{N}^n} V^{\boldsymbol{k}}(\bigcap_{i=1}^{n} \mathcal{W}_i) \longrightarrow H^2_{\mathcal{E}}(\mathbb{D}^n) = \bigoplus_{\boldsymbol{k} \in \mathbb{N}^n} z^{\boldsymbol{k}} \mathcal{E},$$

by $U(V^{\boldsymbol{k}} \eta) = z^{\boldsymbol{k}} \eta$ for all $\eta \in \mathcal{E}$ and $\boldsymbol{k} \in \mathbb{N}^n$. It is obvious that $UV_i = M_{z_i} U$ for all $i = 1, \ldots, n$.

That (v) implies (i) is trivial.

This concludes the proof of the theorem. ∎

As we indicated earlier, the Wold decomposition theorem may be regarded as a very powerful tool from which a number of classical results may be deduced. In [13] we will use techniques developed in this paper to obtain a complete classification of doubly commuting invariant subspaces of the Hardy space over the polydisc.

It is worth mentioning that in [12], Popovici obtained a Wold type decomposition for general isometries. See [1] (Proposition 2.8 and Corollary 2.9) and [4] for more results along this line.

Finally, we point out that all results of this paper are also valid for a family (that is, not necessarily finite tuple) of doubly commuting isometries.

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