Submodules in polydomains and noncommutative varieties

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Abstract. Tensor product of Fock spaces is analogous to the Hardy space over the unit polydisc. This plays an important role in the development of noncommutative operator theory and function theory in the sense of noncommutative polydomains and noncommutative varieties. In this paper we study joint invariant subspaces of tensor product of full Fock spaces and noncommutative varieties. We also obtain, in particular, by using techniques of noncommutative varieties, a classification of joint invariant subspaces of n-fold tensor products of Drury-Arveson spaces.

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1. Introduction

In this paper we study certain invariant subspaces (that is, submodules) of tuples of operators (that is, Hilbert modules over unital and associative free algebras generated by noncommuting variables) in the setting of noncommutative operator theory and noncommutative varieties. The noncommutative operator theory was introduced in the middle eighties by Frazho [10, 11] and Bunce [5] (see, however, Taylor [27]). However, in the late eighties, a more systematic formalism of noncommutative operator theory began with the work of Popescu on isometric dilations and analytic models of infinite sequences of noncommuting operators [16, 17]. Popescu's noncommutative operator theory has a wide range of applications in different context and research areas as, free analysis and matrix convex sets [4], Hilbert C^*-modules [13], moment problem and Cuntz algebras [9] and operator algebras [7, 8] and multivariable operator theory and function theory [1], just to name a few. On the other hand, the theory of noncommutative varieties, again introduced by

Popescu [20], establishes a fundamental connection between noncommutative and commutative operator theory and function theory in several complex variables.

In the setting of noncommutative operator theory, noncommutative polydomains and noncommutative varieties, introduced by Popescu in [21], are analogue of polydisc in $\mathbb{C}^n$. Popescu's theory of polydomains can be seen as an attempt to unify the function theory and multivariable operator theory (both commutative and noncommutative) on the open unit ball and polydisc like domains in $\mathbb{C}^n$.

The goal of the present paper is to examine a general technique for characterizing joint invariant subspaces of the noncommutative Hardy space on noncommutative polydomains and noncommutative varieties.

We emphasize that the notion of Fock space (we also call Fock module) that plays the central role in noncommutative operator theory and used in the free analytic models also plays significant role in noncommutative polydomains. Here one actually needs to deal with the tensor products of Fock spaces. From this point of view, in this paper, we characterize invariant subspaces of tensor products of Fock spaces. In order to be more specific, here we introduce the notions of Fock module, Fock $\boldsymbol{n}$-modules and multi-analytic maps, the most necessary technical background for the study of noncommutative multivariable operator theory, and refer the reader to Section 2 for a more detailed discussion.

Throughout this article, n and k will denote natural numbers. We also denote by $\boldsymbol{n} = (n_1, \ldots, n_k)$ a k-tuple of natural numbers. Consider the n-dimensional Hilbert space $\mathbb{C}^n$ with the standard orthonormal basis $\{e_1, \ldots, e_n\}$. The *Fock module* F^2_n is defined by

$$F^2_n = \bigoplus_{m \in \mathbb{Z}_+} (\mathbb{C}^n)^{\otimes m},$$

where $(\mathbb{C}^n)^{\otimes 0} = \mathbb{C}$ and $(\mathbb{C}^n)^{\otimes m}$ is the m-fold Hilbert space tensor product of $\mathbb{C}^n$. Define the *left creation operators* $S_1, \ldots, S_n$ on F^2_n by $S_i f := e_i \otimes f$, $f \in F^2_n$. It is easy to see that $S_i^* S_j = \delta_{i,j} I_{F^2_n}$ for all $i, j = 1, \ldots, n$, that is, $(S_1, \ldots, S_n)$ is an n tuple noncommuting isometries with orthogonal ranges. Similarly, we define the *right creation operators* $(R_1, \ldots, R_n)$ by setting $R_i f = f \otimes e_i$, $f \in F^2_n$. The *Fock* $\boldsymbol{n}$*-module* $F^2_{\boldsymbol{n}}$ is defined by

$$F^2_{\boldsymbol{n}} = F^2_{n_1} \otimes \cdots \otimes F^2_{n_k}.$$

Now for each $i \in \{1, \ldots, k\}$, we denote the n_i-tuple of creation operators on $F^2_{n_i}$ (instead of $(S_1, \ldots, S_{n_i})$) by

$$S_{n_i} = (S_{i1}, \ldots, S_{in_i}).$$

Then, for each $j \in \{1, \ldots, n_i\}$ we define the operator $\boldsymbol{S}_{ij}$ acting on the Fock $\boldsymbol{n}$-module $F^2_{\boldsymbol{n}}$ by setting

$$\boldsymbol{S}_{ij} := I_{F^2_{n_1}} \otimes \cdots \otimes I_{F^2_{n_{i-1}}} \otimes S_{ij} \otimes I_{F^2_{n_{i+1}}} \otimes \cdots I_{F^2_{n_k}}.$$

It is now evident that $\boldsymbol{S}_{ij}\boldsymbol{S}_{pq} = \boldsymbol{S}_{pq}\boldsymbol{S}_{ij}$ and $\boldsymbol{S}_{ij}^*\boldsymbol{S}_{pq} = \boldsymbol{S}_{pq}\boldsymbol{S}_{ij}^*$ for all $1 \leq i < p \leq k$, $j = 1, \ldots, n_i$, and $q = 1, \ldots, n_p$. In other words, for each $i = 1, \ldots, k$,

$$\boldsymbol{S}_{n_i} := (\boldsymbol{S}_{i1}, \ldots, \boldsymbol{S}_{in_i}),$$

is an n_i-tuple of noncommuting isometries with orthogonal ranges acting on the Fock $\boldsymbol{n}$-module $F^2_{\boldsymbol{n}}$. We set the k-tuple of tuples of noncommuting isometries $\boldsymbol{S}$ as

$$\boldsymbol{S} = (\boldsymbol{S}_{n_1}, \ldots, \boldsymbol{S}_{n_k}).$$

Finally, for Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, an operator $\Theta : F^2_n \otimes \mathcal{E} \to F^2_n \otimes \mathcal{E}_*$ is called *multi-analytic* if

$$\Theta(S_i \otimes I_{\mathcal{E}}) = (S_i \otimes I_{\mathcal{E}_*})\Theta \qquad (i = 1, \ldots, n).$$

The set of all such multi-analytic operators will be denoted by $R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$.

In this paper we classify joint invariant subspaces of $\boldsymbol{S}$. We also aim to illustrate our ideas in the setting of noncommutative varieties (see Section 5). At the present stage, it is worthwhile to point out that a certain classes of joint invariant subspaces of $\boldsymbol{S}$ as well as shifts on noncommutative varieties has been considered by Popescu (for instance, see [22, Corollary 5.3], and the recent one [24, Theorem 5.2]). These are either analog of doubly commuting invariant subspaces or Brehmer type invariant subspaces. In the present setting, our classification results hold for general joint invariant subspaces of $\boldsymbol{S}$ and shifts on noncommutative varieties

We summarize the main contribution of this paper as follows: Section 2 contains preliminary notions, and some basic observations. In section 3 we lay the foundation for the main body of this paper, and prove a key result concerning representations of commutant of pure isometric tuples. An n tuple of operators $V = (V_1, \ldots, V_n)$ acting on some Hilbert space $\mathcal{H}$ is called *a pure isometric tuple* if $V_i^* V_j = \delta_{ij} I_{\mathcal{H}}$ for all $i, j \in \{1, \ldots, n\}$ and

$$\text{SOT-}\lim_{m \to \infty} \sum_{\substack{|\alpha| = m \\ \alpha \in F_n^+}} V^\alpha V^{*\alpha} = 0,$$

where F_n^+ denotes the unital free semigroup on n generators $g_1, \ldots, g_n$ and the identity e, $X^\alpha = X_{i_1} \cdots X_{i_m}$ and $|\alpha| = m$ for all $\alpha = g_{i_1} \cdots g_{i_m} \in F_n^+$. In this case (see Lemma 3.1), there exists a canonical unitary map $L_V : \mathcal{H} \to F^2_n \otimes \mathcal{E}$, where $\mathcal{E} = \cap_{i=1}^n \ker V_i^*$, such that

$$L_V V_i = (S_i \otimes I_{\mathcal{E}}) \qquad (i = 1, \ldots, n).$$

The unitary L_V is essentially the dilation map (which we call canonical module unitary operator) and related to the noncommutative Poisson transforms of Popescu [22]. In Theorem 3.2, we use this approach (which is not very different from the earlier approach of Popescu) to reprove Popescu's noncommutative version of the classical Beurling-Lax-Halmos theorem [17]: A closed subspace $\mathcal{M} \subseteq F^2_n \otimes \mathcal{E}_*$ is invariant under $S \otimes I_{\mathcal{E}_*} = (S_1 \otimes I_{\mathcal{E}_*}, \ldots, S_n \otimes I_{\mathcal{E}_*})$ if and only if there exists an inner multi-analytic operator $\Theta : F_n^2 \otimes \mathcal{E} \to F_n^2 \otimes \mathcal{E}_*$ such that $\mathcal{M} = \Theta(F_n^2 \otimes \mathcal{E})$, where

$$\mathcal{E} = \mathcal{M} \ominus \sum_{i=1}^{n} (S_i \otimes I_{\mathcal{E}_*})\mathcal{M}.$$

Up until this point, the results of Section 3 are all due to Popescu. However, the present unification (or rearrangement) is slightly different, which is also essential for the main body of this paper. The only new result of Section 3 is the explicit representations of commutants of pure isometric tuples (see Theorem 3.3): $C \in \{V_1, \dots, V_n\}'$ if and only if there exists a multi-analytic operator $\Phi \in R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E})$ such that $L_V C L_V^* = \Phi$ and the Fourier coefficients of Φ are given by

$$\varphi_{\alpha^t} = P_{\mathcal{E}} V^{\alpha *} C|_{\mathcal{E}} \qquad (\alpha \in F_n^+).$$

This key result plays an important role in what follows. Here R_n^∞ (respectively F_n^∞) is the *noncommutative analytic Toeplitz algebra*, the weakly closed algebra generated by the right (respectively left) creation operators and the identity operator $\{R_i : i = 1, \dots, n\} \cup \{I_{F_n^2}\}$ (respectively $\{S_i : i = 1, \dots, n\} \cup \{I_{F_n^2}\}$).

Now we turn to Section 4. We continue to assume that $\mathcal{M} \subseteq F_n^2 \otimes \mathcal{E}_*$ is a closed subspace. Let $T \in \mathcal{B}(F_n^2 \otimes \mathcal{E}_*)$ and suppose

$$T(S_i \otimes I_{\mathcal{E}_*}) = (S_i \otimes I_{\mathcal{E}_*})T \qquad (i = 1, \dots, n).$$

We know from Popescu's noncommutative version of the classical Beurling-Lax-Halmos theorem (as also stated above) that $\mathcal{M}$ is invariant under $S \otimes I_{\mathcal{E}_*}$ if and only if $\mathcal{M} = \Theta(F_n^2 \otimes \mathcal{E})$ for some inner multi-analytic operator $\Theta : F_n^2 \otimes \mathcal{E} \to F_n^2 \otimes \mathcal{E}_*$. In Theorem 4.1, we prove that $T\mathcal{M} \subseteq \mathcal{M}$ if and only if there exists a multi-analytic operator $\Phi \in R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E})$ with (explicit) Fourier coefficients

$$\varphi_{\alpha^t} = P_{\mathcal{E}}(S^{\alpha *} \otimes I_{\mathcal{E}_*})T|_{\mathcal{E}} \qquad (\alpha \in F_n^+),$$

such that

$$T\Theta = \Theta\Phi.$$

In other words, $T\mathcal{M} \subseteq \mathcal{M}$ if and only if Popescu's inner multi-analytic operator Θ satisfies the above operator equation for some explicit Φ. While this is the only new component of Theorem 4.1, the proof of this also requires proving Popescu's noncommutative version of the Beurling-Lax-Halmos theorem in our present terminology. In other words, the present proof of Popescu's result (which is a little different than that of Popescu) in our terminology is inescapable for the purpose of the new part of Theorem 4.1. It is also worthwhile to note that the entire proof of Theorem 4.1 follows the ground rules laid down in all the previous results.

Putting all the pieces together from the above, in Corollary 4.3, we characterize invariant subspaces of Fock $\boldsymbol{n}$-module as follows: Let $\mathcal{M}$ be a closed subspace of the Fock $\boldsymbol{n}$-module $F^2_{\boldsymbol{n}}$. Let

$$\mathcal{E} = \mathcal{M} \ominus \sum_{j=1}^{n_1} \boldsymbol{S}_{1j}\mathcal{M},$$

and $\mathcal{E}_{\boldsymbol{n}} = F^2_{n_2} \otimes \cdots \otimes F^2_{n_k}$. Then $X\mathcal{M} \subseteq \mathcal{M}$ for all $X \in \boldsymbol{S}_{n_i}$, $i = 1, \ldots, k$, if and only if there exist an inner multi-analytic operator $\Theta \in R^\infty_{n_1}\overline{\otimes}\mathcal{B}(\mathcal{E}, \mathcal{E}_{\boldsymbol{n}})$ and multi-analytic operators $\Phi_{ij} \in R^\infty_{n_1}\overline{\otimes}\mathcal{B}(\mathcal{E})$, for all $i = 2, \ldots, k$ and $j = 1, \ldots, n_i$, such that

$$\mathcal{M} = \Theta(F^2_{n_1} \otimes \mathcal{E}) \quad \text{and} \quad \boldsymbol{S}_{ij}\Theta = \Theta\Phi_{ij}.$$

In this case, the multi-analytic operators $\Phi_{ij} \in R^\infty_{n_1}\overline{\otimes}\mathcal{B}(\mathcal{E})$, $i = 2, \ldots, k$ and $j = 1, \ldots, n_i$, are uniquely determined by $\mathcal{M}$ (which essentially follows from Corollary 4.3 and Remark 4.4). Moreover, for each $i = 2, \ldots, k$, the n_i-tuple $\boldsymbol{\Phi}_{n_i} = (\Phi_{i1}, \ldots, \Phi_{in_i})$ on $F^2_{n_1} \otimes \mathcal{E}$ is a pure isometric tuple (see Remark 4.4), whereas the restriction tuple

$$(\boldsymbol{S}_{n_1}|_{\mathcal{M}}, \boldsymbol{S}_{n_2}|_{\mathcal{M}}, \ldots, \boldsymbol{S}_{n_k}|_{\mathcal{M}}) \text{ on } \mathcal{M},$$

and the pure isometric tuple

$$(\boldsymbol{S}_{n_1} \otimes I_{\mathcal{E}}, \boldsymbol{\Phi}_{n_2}, \ldots, \boldsymbol{\Phi}_{n_k}) \text{ on } F^2_{n_1} \otimes \mathcal{E},$$

are jointly unitarily equivalent. Finally, in Theorem 4.5, we prove that the above tuple $(\boldsymbol{\Phi}_{n_2}, \ldots, \boldsymbol{\Phi}_{n_k})$ on $F^2_{n_1} \otimes \mathcal{E}$ is unique in an appropriate sense.

In Section 5, we pass from the Fock $\boldsymbol{n}$-module to constrained Fock $\boldsymbol{n}$-modules and prove analogous results for invariant subspaces of constrained tuples. Suppose J be a weak operator topology closed two-sided proper ideal in F^∞_n. Define $N_J := F^2_n \ominus JF^2_n$. Then N_J is a joint invariant subspace of $(S_1^*, \ldots, S_n^*)$. Following [20], define the *constrained left creation operators* as

$$B_i := P_{N_J} S_i|_{N_J} \qquad (i = 1, \ldots, n).$$

Then $B = (B_1, \ldots, B_n)$ is an n-tuple of constrained left creation operator on N_J. In Theorem 5.2, we prove the following analog of Theorem 4.1 in the setting of noncommutative varieties: Let $\mathcal{E}_*$ be a Hilbert space, $T \in \mathcal{B}(N_J \otimes \mathcal{E}_*)$, $\mathcal{M} \subseteq N_J \otimes \mathcal{E}_*$ be a closed subspace, and let $T(B_i \otimes I_{\mathcal{E}_*}) = (B_i \otimes I_{\mathcal{E}_*})T$ for all $i = 1, \ldots n$. Then

$$T\mathcal{M} \subseteq \mathcal{M} \text{ and } (B_i \otimes I_{\mathcal{E}_*})\mathcal{M} \subseteq \mathcal{M} \qquad (i = 1, \ldots, n),$$

if and only if there exist a closed subspace $\mathcal{E}$ of $N_J \otimes \mathcal{E}_*$, a constrained multi-analytic partial isometry

$$\Theta(W_1, \ldots, W_n) \in \mathscr{W}(W_1, \ldots, W_n)\overline{\otimes}\mathcal{B}(\mathcal{E}, \mathcal{E}_*),$$

and a constrained multi-analytic operator

$$\Phi(W_1, \ldots, W_n) \in \mathscr{W}(W_1, \ldots, W_n)\overline{\otimes}\mathcal{B}(\mathcal{E}),$$

such that

$$\mathcal{M} = \Theta(W_1, \ldots, W_n)(N_J \otimes \mathcal{E}),$$

and

$$T\Theta(W_1, \ldots, W_n) = \Theta(W_1, \ldots, W_n)\Phi(W_1, \ldots, W_n).$$

Just as in the case of Theorem 4.1, the above result, without the T part, is Popescu's version of constrained Beurling, Lax and Halmos theorem [20, Theorem 1.2]. However, this time, our proof also brings out additional geometric flavor to Popescu's constrained Beurling, Lax and Halmos theorem. For instance, we prove that the initial coefficient space of $\Theta(W_1, \ldots, W_n)$ is contained in $N_J \otimes \mathcal{E}_*$, that is

$$\mathcal{E} \subseteq N_J \otimes \mathcal{E}_*.$$

This inclusion appears to be a new addition to Popescu's original result. This extra piece of information (along with other techniques invoked in the proof of Theorem 5.2) appears to be more fruitful in concrete situations. However, to keep the flow of the relevant results of this section, we move the proof of the above inclusion in Section 6 (see Lemma 6.2).
The final result of Section 5 is a classification of invariant subspaces of Fock $\boldsymbol{n}$-modules: Given $\boldsymbol{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ and weak operator topology closed two-sided proper ideal J_i in $F_{n_i}^\infty$, $i = 1, \ldots, k$, the *constrained Fock $\boldsymbol{n}$-module* $N_{\boldsymbol{J_n}}$ is defined by

$$N_{\boldsymbol{J_n}} := N_{J_1} \otimes \cdots \otimes N_{J_k}.$$

Set $\mathcal{E}_{\boldsymbol{n}} = N_{J_2} \otimes \cdots \otimes N_{J_k}$, and define $\boldsymbol{B}_{n_i} = (\boldsymbol{B}_{i1}, \ldots, \boldsymbol{B}_{in_i})$, the n_i-tuple on $N_{\boldsymbol{J_n}}$, where $\boldsymbol{B}_{ij} = P_{N_{\boldsymbol{J_n}}} \boldsymbol{S}_{ij}|_{N_{\boldsymbol{J_n}}}$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$. Now suppose $\mathcal{M} \subseteq N_{\boldsymbol{J_n}}$ be a closed subspace. Then $X\mathcal{M} \subseteq \mathcal{M}$ for all $X \in \boldsymbol{B}_{n_i}$, $i = 1, \ldots, k$, if and only if there exist a Hilbert space $\mathcal{E}_*$, a constrained multi-analytic partial isometry $\Theta \in \mathscr{W}(\mathbf{W}_{11}, \ldots, \mathbf{W}_{1n_1})\overline{\otimes}\mathcal{B}(\mathcal{E}_*, \mathcal{E}_{\boldsymbol{n}})$, and a constrained multi-analytic operator

$$\Phi_{ij} \in \mathscr{W}(\mathbf{W}_{11}, \ldots, \mathbf{W}_{1n_1})\overline{\otimes}\mathcal{B}(\mathcal{E}),$$

such that $\mathcal{M} = \Theta(N_{J_1} \otimes \mathcal{E}_*)$, and

$$\mathbf{B}_{ij}\Theta = \Theta\Phi_{ij},$$

for all $i = 2, \ldots, k$ and $j = 1, \ldots, n_i$ (see Section 5 for more details).

In Section 6, we examine the structure of invariant subspaces of *Drury-Arveson $\boldsymbol{n}$-module*

$$H^2_{\boldsymbol{n}} = H^2_{n_1} \otimes \cdots \otimes H^2_{n_k}.$$

In this particular situation, our results are more definite compared to that of Section 5 (see for instance, Corollaries 6.1, 6.3, and 6.4). It is worthwhile to note that the problem of describing invariant subspaces of constrained tuples in the setting of noncommutative varieties is somewhat more challenging (essentially due to the non-uniqueness issue of the commutant lifting theorem, see Section 6).

Section 7 present an example of dimension inequality of fibres of the noncommutative Beurling, Lax and Halmos theorem which also, in particular, show that certain natural generalizations of the classical results are not possible in noncommutative operator theory.

Part of the present investigation may be regarded as a generalization of some of the results concerning invariant subspaces of the Hardy module $H^2(\mathbb{D}^n)$ over unit polydisc $\mathbb{D}^n$ [12]. However, we wish to point out that even in the particular case of $H^2(\mathbb{D}^n)$, our results are slightly different and somewhat more convenient from that of [12] (see the final paragraph in Section 4). Also, in what follows, we will use the standard terminology of Hilbert modules. In particular, this setting is more convenient and economic to deal with the techniques involved in the results presented here.

2. Preliminaries and basic observations

Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, the set of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ will be denoted by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. If $\mathcal{H}_1 = \mathcal{H}_2$, then we shall write $\mathcal{B}(\mathcal{H}_1)$ for $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$. Given an n-tuple $X = (X_1, \ldots, X_n)$ on $\mathcal{H}$, we define a completely positive map $Q_X : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ (see [15]) by

$$Q_X(A) := \sum_{i=1}^{n} X_i A X_i^* \qquad (A \in \mathcal{B}(\mathcal{H})).$$

We say that X is a *row contraction* if $(I_{\mathcal{B}(\mathcal{H})} - Q_X)(I_{\mathcal{H}}) \geq 0$, or equivalently

$$\sum_{i=1}^{n} X_i X_i^* \leq I_{\mathcal{H}}.$$

Therefore, if we denote

$$\mathcal{B}(\mathcal{H})^n = \{X = (X_1, \ldots, X_n) : X_1, \ldots, X_n \in \mathcal{B}(\mathcal{H})\},$$

then, the set of all row contractions on $\mathcal{H}$, given by

$$\mathfrak{B}^{(n)}(\mathcal{H}) := \{X \in \mathcal{B}(\mathcal{H})^n : (I_{\mathcal{B}(\mathcal{H})} - Q_X)(I_{\mathcal{H}}) \geq 0\},$$

is the noncommutative unit ball in $\mathcal{B}(\mathcal{H})^n$. Also it is easy to see that if $X \in \mathfrak{B}^{(n)}(\mathcal{H})$, then

$$I_{\mathcal{H}} \geq Q_X(I_{\mathcal{H}}) \geq Q_X^2(I_{\mathcal{H}}) \geq \ldots \geq 0,$$

which allows one to define a self-adjoint and contraction operator Q_X^{∞} in $\mathcal{B}(\mathcal{H})$ as

$$Q_X^{\infty} := \mathrm{SOT} - \lim_{l \to \infty} Q_X^l(I_{\mathcal{H}}).$$

Note also that

$$Q_X^m(I_{\mathcal{H}}) = \sum_{\substack{|\alpha|=m \\ \alpha \in F_n^+}} X^{\alpha} X^{*\alpha}.$$

The following is now immediate:

Lemma 2.1. *$Q_X^{\infty} = 0$ if and only if SOT-* $\lim_{m \to \infty} \sum_{\substack{|\alpha|=m \\ \alpha \in F_n^+}} X^{\alpha} X^{*\alpha} = 0.$

The tuples of left and right creation operators are closely related to each other. In order to see this, we define

$$e_{\alpha} = \begin{cases} e_{i_1} \otimes \cdots \otimes e_{i_m} & \text{if } \alpha = g_{i_1} \cdots g_{i_m} \\ \quad 1 & \text{if } \alpha = e, \end{cases}$$

for all $\alpha \in F_n^+$. Next we define the *flip operator* $U_t : F_n^2 \to F_n^2$ by

$$U_t(e_\alpha) := e_{\alpha^t},$$

where $\alpha^t := g_{i_m} \cdots g_{i_1}$ and $\alpha = g_{i_1} \cdots g_{i_m} \in F_n^+$. Clearly, U_t is unitary, $U_t^2 = I_{F_n^2}$ and

$$U_t(f \otimes g) = U_t f \otimes U_t g,$$

for all $f, g \in F_n^2$. Moreover, for $\alpha \in F_n^+$, since $R^\alpha f = f \otimes e_{\alpha^t}$, $f \in F_n^2$, it follows that

$$R^\alpha = U_t S^\alpha U_t.$$

Let $\mathbb{C}\langle Z_1, \dots, Z_n\rangle$ denote the unital and associative free algebra generated by n noncommutative variables $Z_1, \dots, Z_n$ over $\mathbb{C}$. Then

$$\mathbb{C}\langle Z_1, \dots, Z_n\rangle = \bigoplus_{\alpha \in F_n^+} \mathbb{C} Z^\alpha,$$

where $Z^\alpha = Z_{i_1} \cdots Z_{i_m}$ for each $\alpha = g_{i_1} \dots g_{i_m} \in F_n^+$. Now let $\{X_1, \dots, X_n\}$ be (not necessarily commuting) bounded linear operators on a Hilbert space $\mathcal{H}$. We realize $\mathcal{H}$ as a $\mathbb{C}\langle Z_1, \dots, Z_n\rangle$*-Hilbert module* as follows:

$$\mathbb{C}\langle Z_1, \dots, Z_n\rangle \times \mathcal{H} \to \mathcal{H},$$

with

$$(p(Z_1, \dots, Z_n), f) \mapsto p(Z_1, \dots, Z_n) \cdot f := p(X_1, \dots, X_n) f,$$

for $p(Z_1, \dots, Z_n)$ in $\mathbb{C}\langle Z_1, \dots, Z_n\rangle$ and $f \in \mathcal{H}$. We say that $\mathcal{H}$ is a *(left) Hilbert module* corresponding to $X = (X_1, \dots, X_n) \in \mathcal{B}(\mathcal{H})^n$. Often we will simply say that $\mathcal{H}$ is a $\mathbb{C}\langle Z_1, \dots, Z_n\rangle$-Hilbert module (or simply a Hilbert module when no confusion can result) if X is clear from the context. Given $\mathbb{C}\langle Z_1, \dots, Z_n\rangle$-Hilbert modules $\mathcal{H}$ and $\mathcal{K}$ corresponding to $X \in \mathcal{B}(\mathcal{H})^n$ and $Y \in \mathcal{B}(\mathcal{K})^n$, respectively, a bounded linear operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is said to be a module map if $AX_i = Y_i A$, $i = 1, \dots, n$. We say that $\mathcal{H}$ is a *row-contractive Hilbert Module* if $(X_1, \dots, X_n) \in \mathfrak{B}^{(n)}(\mathcal{H})$. In addition, if $Q_X^\infty = 0$, then we say that the Hilbert module $\mathcal{H}$ is *pure.*

Let $\mathcal{H}$ be a row-contractive $\mathbb{C}\langle Z_1, \dots, Z_n\rangle$-Hilbert module, and let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. We say that $\mathcal{M}$ is a *submodule* of $\mathcal{H}$ if $X_i \mathcal{M} \subseteq \mathcal{M}$ for all $i = 1, \dots, n$. In this case, we also treat $\mathcal{M}$ as a $\mathbb{C}\langle Z_1, \dots, Z_n\rangle$-Hilbert module corresponding to the n-tuple

$$X|_{\mathcal{M}} = (X_1|_{\mathcal{M}}, \dots, X_n|_{\mathcal{M}}).$$

We record for clarity and future use that the pure property of Hilbert modules carry over to submodules (see the first part of the proof of [26, Theorem 3.2]):

Lemma 2.2. *Any submodule of a pure and row-contractive* $\mathbb{C}\langle Z_1, \dots, Z_n\rangle$*-Hilbert module is pure and row-contractive* $\mathbb{C}\langle Z_1, \dots, Z_n\rangle$*-Hilbert module.*

Proof. Let $\mathcal{H}$ be a pure and row-contractive $\mathbb{C}\langle Z_1, \dots, Z_n\rangle$-Hilbert module, and let $\mathcal{M}$ be a submodule of $\mathcal{H}$. For $h_1, \dots, h_n \in \mathcal{M}$, we have

$$\| \sum_{i=1}^n X_i|_{\mathcal{M}} h_i \|^2 = \| \sum_{i=1}^n X_i h_i \|^2 \le \sum_{i=1}^n \|h_i\|^2,$$

and hence $I_{\mathcal{M}} - Q_{X|_{\mathcal{M}}}(I_{\mathcal{M}}) \geq 0$ or, equivalently $X|_{\mathcal{M}} \in \mathfrak{B}^{(n)}(\mathcal{M})$. Now, it can be checked easily, using $X_i|_{\mathcal{M}}(X_j|_{\mathcal{M}})^* = (X_i P_{\mathcal{M}} X_j^*)|_{\mathcal{M}}$ for all $i, j = 1, \ldots, n$, that

$$Q^m_{X|_{\mathcal{M}}}(I_{\mathcal{M}}) = \Big(\sum_{\substack{|\alpha|=m \\ \alpha \in F_n^+}} X^\alpha P_{\mathcal{M}} X^{*\alpha} \Big)|_{\mathcal{M}},$$

for all $m \geq 0$, and hence $Q^\infty_{X|_{\mathcal{M}}} = 0$. □

The quintessential example of pure and row-contractive Hilbert modules over the noncommutative algebra $\mathbb{C}\langle Z_1, \ldots, Z_n\rangle$ is the Fock module F_n^2 corresponding to $(S_1, \ldots, S_n)$. If $n = 1$, then the full Fock module F_1^2 can be identified with the *Hardy* module $H^2(\mathbb{D})$ over the unit disk and both F_1^∞, R_1^∞ coincide with $H^\infty(\mathbb{D})$. Also note that the $\mathbb{C}\langle Z_1, \ldots, Z_n\rangle$-Hilbert module corresponding to the right creation operators $(R_1, \ldots, R_n)$ is isometrically isomorphic, via the flip operator, to the Fock module.

Given a Hilbert space $\mathcal{E}$, the $\mathcal{E}$-valued Fock module is the $\mathbb{C}\langle Z_1, \ldots, Z_n\rangle$-Hilbert module $F_n^2 \otimes \mathcal{E}$ corresponding to the tuple

$$S \otimes I_{\mathcal{E}} = (S_1 \otimes I_{\mathcal{E}}, \ldots, S_n \otimes I_{\mathcal{E}}) \in \mathfrak{B}^{(n)}(F_n^2 \otimes \mathcal{E}).$$

When the Hilbert space $\mathcal{E}$ is clear from the context, we also write S instead of $S \otimes I_{\mathcal{E}}$. It is well known that

$$\mathbb{C} \otimes \mathcal{E} = \bigcap_{i=1}^n \ker(S_i \otimes I_{\mathcal{E}})^*,$$

and

$$I_{F_n^2 \otimes \mathcal{E}} - \sum_{i=1}^n (S_i \otimes I_{\mathcal{E}})(S_i \otimes I_{\mathcal{E}})^* = P_{\mathbb{C}} \otimes I_{\mathcal{E}},$$

where $P_{\mathbb{C}}$ denotes the orthogonal projection of F_n^2 onto the vacuum space $\mathbb{C} \subseteq F_n^2$. We say that a bounded linear operator $\Theta : F_n^2 \otimes \mathcal{E} \to F_n^2 \otimes \mathcal{E}_*$ is *multi-analytic* if Θ is a module map, that is, $\Theta(S_i \otimes I_{\mathcal{E}}) = (S_i \otimes I_{\mathcal{E}_*})\Theta$ for all $i = 1, \ldots, n$. It is well known (see for example[19, Theorem 1.1]) that the set of module maps coincides with the weakly closure, denoted by $R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$, of by the spatial tensor product R_n^∞ with $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$. Here, by an abuse of notation, the functional calculus (Proposition4.2 [18]) of $\Theta(R_1, \ldots, R_n)$ is given by

$$\Theta(R_1, \ldots, R_n) = \text{SOT-} \lim_{r \to 1^-} \sum_{m=0}^\infty \sum_{|\alpha|=m} r^{|\alpha|} R^\alpha \otimes \theta_\alpha. \tag{2.1}$$

Moreover, each *Fourier coefficient* $\theta_\alpha \in \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$, $\alpha \in F_n^+$, is uniquely determined by Θ as follows:

$$\langle \theta_{\alpha^t} \eta, \zeta \rangle = \langle \Theta(1 \otimes \eta), e_\alpha \otimes \zeta \rangle, \tag{2.2}$$

for all $\eta \in \mathcal{E}$ and $\zeta \in \mathcal{E}_*$. Elements of $R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$ will be denoted by $\Theta(R_1, \ldots, R_n)$, or simply by Θ if $(R_1, \ldots, R_n)$ is clear from the context.

A bounded linear operator $M \in \mathcal{B}(F^2_n \otimes \mathcal{E}, F^2_n \otimes \mathcal{E}_*)$ is said to be *multi-coanalytic* if

$$M(S_i^* \otimes I_{\mathcal{E}}) = (S_i^* \otimes I_{\mathcal{E}_*})M,$$

for all $i = 1, \ldots, n$. If $\Theta \in \mathcal{B}(F^2_n \otimes \mathcal{E}, F^2_n \otimes \mathcal{E}_*)$ is both multi-analytic and multi-coanalytic, then by (2.2), we have that

$$\langle \theta_{\alpha^t} \eta, \zeta \rangle = \langle (S^{\alpha *} \otimes I_{\mathcal{E}})(1 \otimes \eta), \Theta^*(1 \otimes \zeta) \rangle = 0,$$

for all $\alpha \in F_n^+ \setminus \{e\}$, $\eta \in \mathcal{E}$ and $\zeta \in \mathcal{E}_*$. On the other hand, if $\Theta \in \mathcal{B}(F^2_n \otimes \mathcal{E}, F^2_n \otimes \mathcal{E}_*)$ and $\theta_\alpha = 0$ for all $F_n^+ \setminus \{e\}$, then one can easily check that Θ is multi-coanalytic. Thus, we have proved the following lemma:

Lemma 2.3. *A module map $\Theta \in \mathcal{B}(F^2_n \otimes \mathcal{E}, F^2_n \otimes \mathcal{E}_*)$ is multi-coanalytic if and only if the associated Fourier coefficients $\theta_\alpha = 0$ for all $\alpha \in F_n^+ \setminus \{e\}$.*

This also proves the following (well known) observation: A closed subspace $\mathcal{M} \subseteq F^2_n \otimes \mathcal{E}$ is joint reducing for $S \otimes I_{\mathcal{E}}$ if and only if there exists a closed subspace $\mathcal{K} \subseteq \mathcal{E}$ such that $\mathcal{M} = F^2_n \otimes \mathcal{K}$. To prove the non-trivial implication, let $\mathcal{M}$ is reducing for $S \otimes I_{\mathcal{E}}$. Then the orthogonal projection $P_{\mathcal{M}}$ onto $\mathcal{M}$ is a module map. Since $P_{\mathcal{M}}$ is self-adjoint it is also a multi-coanlaytic operator, and hence, by the above lemma, $P_{\mathcal{M}}$ must be constant. Finally, since $P_{\mathcal{M}}$ is positive and idempotent, it follows that $P_{\mathcal{M}} = I_{F^2_n} \otimes P_{\mathcal{K}}$ for some $\mathcal{K} \subseteq \mathcal{E}$.

3. Beurling, Lax and Halmos theorem and commutants

The classical Beurling, Lax and Halmos theorem [14, page 198, Theorem 3.3] deals with a complete classification of invariant subspaces of vector-valued Hardy spaces over the open unit disc $\mathbb{D}$. To be more specific, let $\mathcal{E}_*$ be a Hilbert space, and let $\mathcal{S}$ be a closed subspace of $H^2_{\mathcal{E}_*}(\mathbb{D})$. Then the Beurling, Lax and Halmos theorem says that $\mathcal{S}$ is M_z-invariant if and only if there exist a Hilbert space $\mathcal{E}$ and an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ such that

$$\mathcal{S} = \Theta H^2_{\mathcal{E}}(\mathbb{D}).$$

In particular, the shift M_z on the Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$ and the restriction operator $M_z|_{\mathcal{S}}$ on $\mathcal{S}$ are unitarily equivalent. This has been generalized for Fock module by Popescu [17]. Here, however, we give a slightly direct (or geometric) approach to reprove Popescu's result. This will be useful in characterizing submodules of Fock $\boldsymbol{n}$-modules. Along the way we will also parametrize commutants of tuples of noncommutative pure isometries.

Let $V = (V_1, \ldots, V_n) \in \mathfrak{B}^n(\mathcal{H})$ be a tuple of isometries with orthogonal ranges. We call such a tuple a *pure isometric tuple* if $\mathcal{Q}_V^\infty = 0$. The following lemma follows from Popescu's noncommutative Wold decomposition theorem [16, 23]. However, the explicit description of L_V in our version below will be useful in analysing commutants of pure isometric tuples and all the remaining results of this paper.

Lemma 3.1. *Let $\mathcal{H}$ be a $\mathbb{C}\langle Z_1, \dots, Z_n\rangle$-Hilbert module corresponding to a pure isometric tuple $V = (V_1, \dots, V_n)$. Let*

$$\mathcal{E} := \mathcal{H} \ominus \sum_{i=1}^{n} V_i \mathcal{H} = \bigcap_{i=1}^{n} \ker V_i^*.$$

Then

$$SOT - \sum_{\alpha \in \mathbb{F}_n^+} V^\alpha P_{\mathcal{E}} V^{\alpha *} = I_{\mathcal{H}},$$

and

$$\mathcal{H} = \bigoplus_{\alpha \in F_n^+} V^\alpha \mathcal{E}.$$

Moreover, the map $L_V : \mathcal{H} \to F_n^2 \otimes \mathcal{E}$ defined by

$$L_V(V^\alpha \eta) := e_\alpha \otimes \eta = S^\alpha (1 \otimes \eta) \qquad (\alpha \in F_n^+, \eta \in \mathcal{E}), \tag{3.1}$$

is a unitary module map and

$$L_V f = \sum_{\alpha \in F_n^+} e_\alpha \otimes (P_{\mathcal{E}} V^{*\alpha} f) \qquad (f \in \mathcal{H}).$$

Proof. We first note that $P_{\mathcal{E}} = I_{\mathcal{H}} - \sum_{i=1}^{n} V_i V_i^*$. Hence

$$V^\alpha P_{\mathcal{E}} V^{\alpha *} = V^\alpha (I_H - \sum_{i=1}^{n} V_i V_i^*) V^{\alpha *} = V^\alpha V^{\alpha *} - \sum_{i=1}^{n} V^\alpha V_i V_i^* V^{\alpha *},$$

for all $\alpha \in F_n^+$. For each $k \geq 1$, we have

$$\sum_{|\alpha| \leq k} V^\alpha P_{\mathcal{E}} V^{\alpha *} = \sum_{l=0}^{k} \sum_{|\alpha| = l} (V^\alpha V^{\alpha *} - \sum_{|\alpha| = l+1} V^\alpha V^{\alpha *}) = I_{\mathcal{H}} - \sum_{|\alpha| = k+1} V^\alpha V^{\alpha *},$$

and hence the first equality follows from Lemma 2.1. It is now easy to prove the second equality: Observe that $\bigoplus_{\alpha \in F_n^+} V^\alpha \mathcal{E}$ is a joint reducing subspace of V. If $f \in \mathcal{H}$ and $f \perp V^\alpha \mathcal{E}$ for all $\alpha \in F_n^+$, then $V^{\alpha *} f \perp \mathcal{E}$ and hence $P_{\mathcal{E}} V^{*\alpha} f = 0$. The first equality then implies that $f = 0$, which proves the validity of the second equality. The fact that L_V, as defined in (3.1), is a unitary module map follows readily from the grading of $\mathcal{H}$ in the second equality. The final equality follows from the first and the definition of L_V in (3.1). ☐

Given a pure isometric tuple $V = (V_1, \dots, V_n) \in \mathfrak{B}^n(\mathcal{H})$, the unitary module map L_V constructed in the above lemma is called the *canonical module unitary operator* corresponding to V.

The following general fact will be useful: Given a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, the inclusion map $\iota_{\mathcal{S}} : \mathcal{S} \hookrightarrow \mathcal{H}$ satisfies the following properties:

$$\iota_{\mathcal{S}}^* \iota_{\mathcal{S}} = I_{\mathcal{S}} \quad \text{and} \quad \iota_{\mathcal{S}} \iota_{\mathcal{S}}^* = P_{\mathcal{S}}.$$

Now, let $\mathcal{E}_*$ be a Hilbert space, and let $\mathcal{M} \subseteq F_n^2 \otimes \mathcal{E}_*$ be a submodule of $F_n^2 \otimes \mathcal{E}_*$. Applying Lemma 3.1, by virtue of Lemma 2.2, to

$$(S \otimes I_{\mathcal{E}_*})|_{\mathcal{M}} = ((S_i \otimes I_{\mathcal{E}_*})|_{\mathcal{M}}, \ldots, (S_n \otimes I_{\mathcal{E}_*})|_{\mathcal{M}}),$$

we obtain the canonical module unitary operator corresponding to $(S \otimes I_{\mathcal{E}_*})|_{\mathcal{M}}$ as $L_{S|_{\mathcal{M}}} : \mathcal{M} \to F_n^2 \otimes \mathcal{E}$, where

$$\mathcal{E} = \bigcap_{i=1}^{n} \ker \Big((S_i \otimes I_{\mathcal{E}_*})|_{\mathcal{M}} \Big)^* \subseteq \mathcal{M}. \tag{3.2}$$

Then $\iota_{\mathcal{M}} L^*_{(S \otimes I_{\mathcal{E}_*})|_{\mathcal{M}}} : F_n^2 \otimes \mathcal{E} \to F_n^2 \otimes \mathcal{E}_*$ is an isometric module map and

$$\mathcal{M} = \operatorname{ran}(\iota_{\mathcal{M}} L^*_{(S \otimes I_{\mathcal{E}_*})|_{\mathcal{M}}}).$$

If we define

$$\Theta = \iota_{\mathcal{M}} L^*_{(S \otimes I_{\mathcal{E}_*})|_{\mathcal{M}}},$$

then $\Theta \in R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$ is an *inner* (that is, isometric) multi-analytic operator, and moreover

$$\mathcal{M} = \Theta(F_n^2 \otimes \mathcal{E}).$$

We now proceed to the uniqueness part. Let $\tilde{\mathcal{E}}$ be a Hilbert space, and let $\tilde{\Theta}$ be an inner multi-analytic operator in $R_n^\infty \overline{\otimes} \mathcal{B}(\tilde{\mathcal{E}}, \mathcal{E}_*)$ such that $\mathcal{M} = \tilde{\Theta}(F_n^2 \otimes \tilde{\mathcal{E}})$. Then for

$$\Theta^* \tilde{\Theta} : F_n^2 \otimes \tilde{\mathcal{E}} \to F_n^2 \otimes \mathcal{E},$$

we have

$$\Theta^* \tilde{\Theta} S_i \otimes I_{\tilde{\mathcal{E}}} = \Theta^* S_i \otimes I_{\mathcal{E}_*} \tilde{\Theta} = \Theta^* S_i \otimes I_{\mathcal{E}_*} \Theta \Theta^* \tilde{\Theta} = S_i \otimes I_{\mathcal{E}} \Theta^* \Psi = S_i \otimes I_{\mathcal{E}} \Theta^* \tilde{\Theta},$$

and hence

$$(S_i \otimes I_{\mathcal{E}}) \Theta^* \tilde{\Theta} = \Theta^* \tilde{\Theta} (S_i \otimes I_{\mathcal{E}}),$$

for all $i = 1, \ldots, n$. Therefore $\Theta^* \tilde{\Theta}$ is a module map. On the other hand, since

$$\Theta(F_n^2 \otimes \mathcal{E}) = \tilde{\Theta}(F_n^2 \otimes \tilde{\mathcal{E}}),$$

for $h \in F_n^2 \otimes \mathcal{E}$, there exists $\tilde{h} \in F_n^2 \otimes \tilde{\mathcal{E}}$ such that $\Theta h = \tilde{\Theta} \tilde{h}$. Then we have

$$(S_i \otimes I_{\tilde{\mathcal{E}}}) \tilde{\Theta}^* \Theta h = (S_i \otimes I_{\tilde{\mathcal{E}}}) \tilde{h} = \tilde{\Theta}^* (S_i \otimes I_{\mathcal{E}_*}) \tilde{\Theta} \tilde{h} = \tilde{\Theta}^* (S_i \otimes I_{\mathcal{E}_*}) \Theta h,$$

that is, $(S_i \otimes I_{\tilde{\mathcal{E}}}) \tilde{\Theta}^* \Theta h = \tilde{\Theta}^* \Theta (S_i \otimes I_{\mathcal{E}}) h$ for all $i = 1, \ldots, n$, and hence that $\Theta^* \tilde{\Theta}$ is multi-coanalytic. Lemma 2.3 then implies that $\Theta^* \tilde{\Theta}$ is a constant map, that is, $\Theta^* \tilde{\Theta} = I_{F_n^2} \otimes \tau$ for some $\tau \in \mathcal{B}(\tilde{\mathcal{E}}, \mathcal{E})$. Thus

$$\tilde{\Theta} = \Theta(I_{F_n^2} \otimes \tau),$$

as $\Theta \Theta^* = P_{\operatorname{ran} \tilde{\Theta}}$. That τ is a unitary follows from the fact that both Θ and $\tilde{\Theta}$ are isometries and $\Theta(F_n^2 \otimes \mathcal{E}) = \tilde{\Theta}(F_n^2 \otimes \tilde{\mathcal{E}})$.

Thus we have proved Popescu's noncommutative version of the classical Beurling-Lax-Halmos theorem (see [2], [17] and [6] for the original versions).

Theorem 3.2. *Let $\mathcal{E}_*$ be a Hilbert space and let $\mathcal{M}$ be a closed subspace of $F_n^2 \otimes \mathcal{E}_*$. Suppose*

$$\mathcal{E} = \mathcal{M} \ominus \sum_{i=1}^{n} (S_i \otimes I_{\mathcal{E}_*})\mathcal{M}.$$

Then the following are equivalent:

(i) $\mathcal{M}$ is a submodule of $F_n^2 \otimes \mathcal{E}_*$.

(ii) There exists an inner multi-analytic operator $\Theta : F_n^2 \otimes \mathcal{E} \to F_n^2 \otimes \mathcal{E}_*$ such that $\mathcal{M} = \Theta(F_n^2 \otimes \mathcal{E})$.

Moreover, in the latter case, if $\mathcal{M} = \tilde{\Theta}(F_n^2 \otimes \tilde{\mathcal{E}})$ for some Hilbert space $\tilde{\mathcal{E}}$ and inner multi-analytic operator $\tilde{\Theta} : F_n^2 \otimes \tilde{\mathcal{E}} \to F_n^2 \otimes \mathcal{E}_*$, then there exists a unitary $\tau \in \mathcal{B}(\tilde{\mathcal{E}}, \mathcal{E})$ such that

$$\tilde{\Theta} = \Theta(I_{F_n^2} \otimes \tau).$$

It is worthwhile to note that our approach and presentation is slightly different than that of Popescu. It is somewhat more convenient to work with the canonical coefficient space $\mathcal{E}$ as described in the above theorem. In particular, as we will see, the present approach will be useful in the study of submodules of Fock module and Fock $\boldsymbol{n}$-modules.

We now turn to the representations of a commutant of pure isometric tuples.

Theorem 3.3. *Let $\mathcal{H}$ be a $\mathbb{C}\langle Z_1, \ldots, Z_n\rangle$-Hilbert module corresponding to a pure isometric tuple $V = (V_1, \ldots, V_n)$. Let $L_V : \mathcal{H} \to F_n^2 \otimes \mathcal{E}$ be the canonical module unitary operator. Then $C \in \{V_1, \ldots, V_n\}'$ if and only if there exists a multi-analytic operator $\Phi \in R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E})$ such that $L_V C L_V^* = \Phi$ and the Fourier coefficients of Φ are given by*

$$\varphi_{\alpha^t} = P_{\mathcal{E}} V^{\alpha *} C|_{\mathcal{E}} \qquad (\alpha \in F_n^+).$$

Proof. Let $C \in \mathcal{B}(\mathcal{H})$ and let $\eta \in \mathcal{E}$. Then $CL_V^*(1 \otimes \eta) = C\eta$, as $L_V^*(1 \otimes \eta) = \eta$ by (3.1). By the definition of L_V, we have

$$L_V C L_V^*(1 \otimes \eta) = L_V C\eta = \sum_{\alpha \in F_n^+} e_\alpha \otimes (P_{\mathcal{E}} V^{\alpha *} C\eta).$$

Clearly, if $C \in \{V_1, \ldots, V_n\}'$, then

$$(L_V C L_V^*)(S_i \otimes I_{\mathcal{E}}) = (S_i \otimes I_{\mathcal{E}})(L_V C L_V^*),$$

for all $i = 1, \ldots, n$, and hence $L_V C L_V^*$ is a multi-analytic operator. Let

$$L_V C L_V^* = \Phi \in R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E}).$$

Then by (2.1) we have

$$\Phi(R_1, \ldots, R_n) = \text{SOT} - \lim_{r \to 1^-} \sum_{m=0}^{\infty} \sum_{|\alpha|=m} r^{|\alpha|} R^\alpha \otimes \varphi_\alpha,$$

where $\varphi_\alpha \in \mathcal{B}(\mathcal{E})$ for all $\alpha \in F_n^+$. Finally, if $\eta, \zeta \in \mathcal{E}$, then by (2.2) we have

$$\begin{aligned}\langle \varphi_{\alpha^t}\eta, \zeta\rangle &= \langle L_V C L_V^*(1 \otimes \eta), e_\alpha \otimes \zeta\rangle \\ &= \langle \sum_{\beta \in F_n^+} e_\beta \otimes (P_\mathcal{E} V^{\beta *} C\eta), e_\alpha \otimes \zeta\rangle \\ &= \langle P_\mathcal{E} V^{\alpha *} C\eta, \zeta\rangle,\end{aligned}$$

and hence $\varphi_{\alpha^t} = P_\mathcal{E} V^{\alpha *} C|_\mathcal{E}$ for all $\alpha \in F_n^+$. The converse is obvious. □

Therefore, the unique formal Fourier expansion of $\Phi \in R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E})$ in the above proposition is given by

$$\Phi(R_1, \dots, R_n) = \sum_{\alpha \in F_n^+} R^\alpha \otimes \Big(P_\mathcal{E} V^{\alpha^t *} C|_\mathcal{E}\Big).$$

Finally, a word of caution is necessary here: Since

$$\mathcal{E} = \bigcap_{i=1}^n \ker V_i^* \subseteq \mathcal{H},$$

the representation of φ_{α^t}, $\alpha \in F_n^+$, in Theorem 3.3 is well-defined.

4. Submodules of Fock $\boldsymbol{n}$-modules

This section deals with representations of submodules of the Fock $\boldsymbol{n}$-module $F^2_{\boldsymbol{n}}$. This problem originated from the natural question of whether Beurling, Lax and Halmos type inner function-based characterizations of invariant subspaces can be valid on Hardy space over the unit polydisc $\mathbb{D}^n$, $n > 1$. The answer is negative even for $n = 2$ (see Rudin [25, Theorems 4.1.1 and 4.4.2]). On the other hand, recently in [12], an abstract classification of invariant subspaces of the Hardy space over the unit polydisc has been proposed. Here we refine the method of [12] to handle the noncommutative tuples and carry out the classification of submodules of $F^2_{\boldsymbol{n}}$. We point out once again that noncommutative analog of doubly commuting and Brehmer type invariant subspaces have been considered by Popescu (for instance, see [22, Corollary 5.3]). Our classification results in this section hold for general submodules of $F^2_{\boldsymbol{n}}$.

Let $\mathcal{E}$ and $\mathcal{K}$ be Hilbert spaces and let $T \in \mathcal{B}(\mathcal{K})$. We treat $\mathcal{K}$ as a $\mathbb{C}[Z]$-Hilbert module corresponding to T. Now consider the $\mathcal{E}$-valued Fock module $F_n^2 \otimes \mathcal{E}$ as $\mathbb{C}\langle Z_1, \dots, Z_n\rangle$-Hilbert module corresponding to $S \otimes I_\mathcal{E} = (S_1 \otimes I_\mathcal{E}, \dots, S_n \otimes I_\mathcal{E})$. Consider the free algebra

$$\mathbb{C}\langle Z_1, \dots, Z_n\rangle \otimes_\mathbb{C} \mathbb{C}[Z] = \mathbb{C}\langle Z_1, \dots, Z_n, Z\rangle,$$

generated by the indeterminates $\{Z_1, \dots, Z_n, Z\}$. Note that

$$(Z \otimes 1)(1 \otimes Z_i) - (1 \otimes Z_i)(Z \otimes 1) = 0 \qquad (i = 1, \dots, n).$$

Now we treat $(F_n^2 \otimes \mathcal{K}) \otimes \mathcal{E}$ as a $\mathbb{C}\langle Z_1, \dots, Z_n, Z\rangle$-Hilbert module, where Z_i and Z corresponds to $(S_i \otimes I_\mathcal{K}) \otimes I_\mathcal{E}$ and $(I_{F_n^2} \otimes T) \otimes I_\mathcal{E}$, respectively, and

$1 \le i \le n$. Note that since $(S_i \otimes I_{\mathcal{K}}) \otimes I_{\mathcal{E}}$, $i = 1, \ldots, n$, commutes (and doubly commutes) with $(I_{F_n^2} \otimes T) \otimes I_{\mathcal{E}}$, the above identification is well defined.

On the other hand, let $\mathcal{H}$ be a $\mathbb{C}\langle Z_1, \ldots, Z_n\rangle$-Hilbert module corresponding to a pure isometric tuple $V \in \mathfrak{B}^{(n)}(\mathcal{H})$ and let $T \in \mathcal{B}(\mathcal{H})$. Suppose T commutes and also doubly commutes with V. Let $F_n^2 \otimes \mathcal{E}$ be the identification of $\mathcal{H}$ as in (3.1). It follows from Lemma 3.1 that the representation of T in $F_n^2 \otimes \mathcal{E}$ is a constant multi-analytic operator.

The above discussion is the underlying theme of this section, where we aim at characterizing joint invariant subspaces of Fock $\boldsymbol{n}$-module $F^2_{\boldsymbol{n}}$. Such a characterization is a consequence of the following key theorem.

Theorem 4.1. *Let $\mathcal{E}_*$ be a Hilbert space, $T \in \mathcal{B}(F_n^2 \otimes \mathcal{E}_*)$ and let $\mathcal{M}$ a closed subspace of $F_n^2 \otimes \mathcal{E}_*$. Let*

$$\mathcal{E} = \mathcal{M} \ominus \sum_{i=1}^{n} (S_i \otimes I_{\mathcal{E}_*})\mathcal{M}.$$

and suppose $T(S_i \otimes I_{\mathcal{E}_*}) = (S_i \otimes I_{\mathcal{E}_*})T$ for all $i = 1, \ldots, n$. Then the following statements are equivalent:

(a) *$\mathcal{M}$ is submodule of $F_n^2 \otimes \mathcal{E}_*$ and $T\mathcal{M} \subseteq \mathcal{M}$.*
(b) *There exist an inner multi-analytic operator $\Theta \in R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$ and a multi-analytic operator $\Phi \in R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E})$ such that*

$$\mathcal{M} = \Theta(F_n^2 \otimes \mathcal{E}),$$

and

$$T\Theta = \Theta\Phi.$$

Moreover, if either of the above conditions hold, then the Fourier coefficients of Φ are given by

$$\varphi_{\alpha^t} = P_{\mathcal{E}}(S^{\alpha *} \otimes I_{\mathcal{E}_*})T|_{\mathcal{E}} \qquad (\alpha \in F_n^+).$$

Proof. The implication $(b) \Rightarrow (a)$ follows from the well known Douglas' range-inclusion theorem. So we proceed to prove that $(a) \Rightarrow (b)$. Suppose $\mathcal{M}$ is a submodule of $F_n^2 \otimes \mathcal{E}_*$ and suppose that $\mathcal{M}$ is T-invariant. Define $\tilde{T} = T|_{\mathcal{M}}$ and

$$(S \otimes I_{\mathcal{E}_*})|_{\mathcal{M}} = ((S_1 \otimes I_{\mathcal{E}_*})|_{\mathcal{M}}, \ldots, (S_n \otimes I_{\mathcal{E}_*})|_{\mathcal{M}}).$$

Clearly $(S_i \otimes I_{\mathcal{E}_*})|_{\mathcal{M}}$ is an isometry and $[(S_i \otimes I_{\mathcal{E}_*})|_{\mathcal{M}}, \tilde{T}] = 0$ for all $i = 1, \ldots, n$. Hence, taking into account of Lemma 2.2, it follows that $(S \otimes I_{\mathcal{E}_*})|_{\mathcal{M}}$ is a pure isometric tuple. Now we are in the setting of the proof of Theorem 3.2. Therefore

$$\mathcal{M} = \Theta(F_n^2 \otimes \mathcal{E}),$$

where

$$\mathcal{E} = \bigcap_{i=1}^{n} \ker\left((S_i \otimes I_{\mathcal{E}_*})|_{\mathcal{M}}\right)^*,$$

and

$$\Theta = \iota_{\mathcal{M}} L^*_{(S \otimes I_{\mathcal{E}_*})|_{\mathcal{M}}} \in R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*).$$

Moreover, by Theorem 3.3, there exists $\Phi \in R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E})$ such that

$$\Phi = L_{(S\otimes I_{\mathcal{E}_*})|_{\mathcal{M}}} \tilde{T} L^*_{(S\otimes I_{\mathcal{E}_*})|_{\mathcal{M}}},$$

and the Fourier coefficients of Φ are given by

$$\varphi_{\alpha^t} = P_{\mathcal{E}}\Big((S\otimes I_{\mathcal{E}_*})|_{\mathcal{M}}\Big)^{\alpha *}\tilde{T}|_{\mathcal{E}} \qquad (\alpha \in F_n^+).$$

Now for each $\alpha \in F_n^+$ we have

$$P_{\mathcal{E}}\Big((S\otimes I_{\mathcal{E}_*})|_{\mathcal{M}}\Big)^{\alpha *}\tilde{T}|_{\mathcal{E}} = P_{\mathcal{E}}P_{\mathcal{M}}(S\otimes I_{\mathcal{E}_*})^{\alpha *}|_{\mathcal{M}}\tilde{T}|_{\mathcal{E}} = P_{\mathcal{E}}P_{\mathcal{M}}(S^{\alpha *}\otimes I_{\mathcal{E}_*})\tilde{T}|_{\mathcal{E}},$$

hence, by the fact that $\mathcal{E} \subseteq \mathcal{M}$, we have

$$P_{\mathcal{E}}\Big((S\otimes I_{\mathcal{E}_*})|_{\mathcal{M}}\Big)^{\alpha *}\tilde{T}|_{\mathcal{E}} = P_{\mathcal{E}}(S^{\alpha *}\otimes I_{\mathcal{E}_*})\tilde{T}|_{\mathcal{E}}.$$

Finally, from the definitions of Θ and Φ above and the fact that $\tilde{T} = T|_{\mathcal{M}} = \iota_{\mathcal{M}}^* T \iota_{\mathcal{M}}$, we conclude that

$$\Phi = L_{(S\otimes I_{\mathcal{E}_*})|_{\mathcal{M}}} \tilde{T} L^*_{(S\otimes I_{\mathcal{E}_*})|_{\mathcal{M}}} = L_{(S\otimes I_{\mathcal{E}_*})|_{\mathcal{M}}} \iota_{\mathcal{M}}^* T \iota_{\mathcal{M}} L^*_{(S\otimes I_{\mathcal{E}_*})|_{\mathcal{M}}} = \Theta^* T \Theta,$$

that is

$$\Phi = \Theta^* T \Theta, \tag{4.1}$$

and hence $\Theta\Phi = T\Theta$, as $\Theta\Theta^* = P_{\mathcal{M}}$ and $\operatorname{ran}(T\Theta) \subseteq \mathcal{M}$. This completes the proof of the theorem. □

Now, let $\mathcal{M}$ be a closed subspace of $F_n^2 \otimes \mathcal{E}_*$. Then $\mathcal{M}$ is submodule of $F_n^2 \otimes \mathcal{E}_*$ if and only if there exist a Hilbert space $\mathcal{E}$ and an inner multi-analytic operator $\Theta \in R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$ such that $\mathcal{M} = \Theta(F_n^2 \otimes \mathcal{E})$. This is Popescu's Beurling-Lax-Halmos theorem [16]. However, the above theorem says that $\mathcal{M}$ is also invariant under T if and only if T and Popescu's inner function Θ satisfies the operator equation $T\Theta = \Theta\Phi$ for some explicit Φ as in the statement above.

Remark 4.2. In the setting of Theorem 4.1, if, in addition, T is an isometry, then

$$\|\Phi h\| = \|\Theta \Phi h\| = \|T\Theta h\| = \|h\| \qquad (h \in F_n^2 \otimes \mathcal{E}),$$

and hence it follows that Φ is also an isometry. Suppose now that T is pure, that is, $T^{*m} \to 0$ as $m \to \infty$ in the strong operator topology. Then $\Phi^{*m} = \Theta^* T^{*m} \Theta$, $m \geq 1$, implies that Φ is also pure.

Now we proceed to submodules of the Fock $\boldsymbol{n}$-module $F^2_{\boldsymbol{n}}$. Recall that the Fock $\boldsymbol{n}$-module $F^2_{\boldsymbol{n}}$, for $\boldsymbol{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, is given by

$$F^2_{\boldsymbol{n}} = F^2_{n_1} \otimes \cdots \otimes F^2_{n_k}.$$

Clearly, if we denote by

$$\mathbb{C}\langle \boldsymbol{Z}\rangle_{\boldsymbol{n}} := \mathbb{C}\langle Z_1, \ldots, Z_{n_1}\rangle \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}\langle Z_1, \ldots, Z_{n_k}\rangle,$$

the tensor product of free algebras over $\mathbb{C}$, then $F^2_{\boldsymbol{n}}$ is naturally a $\mathbb{C}\langle \boldsymbol{Z}\rangle_{\boldsymbol{n}}$-Hilbert module corresponding to $\boldsymbol{S} = (\boldsymbol{S}_{n_1}, \ldots, \boldsymbol{S}_{n_k})$. From this point of view, a closed subspace $\mathcal{M} \subseteq F^2_{\boldsymbol{n}}$ is said to be a submodule if

$$X\mathcal{M} \subseteq \mathcal{M},$$

for all $X \in \boldsymbol{S}_{n_i}$, $i = 1, \ldots, k$. Now if we set

$$\mathcal{E}_{\boldsymbol{n}} = F^2_{n_2} \otimes \cdots \otimes F^2_{n_k},$$

then Theorem 4.1 (applied to $\mathcal{E}_{\boldsymbol{n}}$ in place of $\mathcal{E}_*$) directly leads to the following corollary concerning representations of submodules of $F^2_{\boldsymbol{n}}$:

Corollary 4.3. *Let $\mathcal{M}$ be a closed subspace of the Fock $\boldsymbol{n}$-module $F^2_{\boldsymbol{n}}$. Let*

$$\mathcal{E} = \mathcal{M} \ominus \sum_{j=1}^{n_1} \boldsymbol{S}_{1j}\mathcal{M},$$

and $\mathcal{E}_{\boldsymbol{n}} = F^2_{n_2} \otimes \cdots \otimes F^2_{n_k}$. Then $\mathcal{M}$ is a submodule of $F^2_{\boldsymbol{n}}$ if and only if there exist an inner multi-analytic operator $\Theta \in R^\infty_{n_1} \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_{\boldsymbol{n}})$ and multi-analytic operators $\Phi_{ij} \in R^\infty_{n_1} \overline{\otimes} \mathcal{B}(\mathcal{E})$, $i = 2, \ldots, k$, and $j = 1, \ldots, n_i$, such that

$$\mathcal{M} = \Theta(F^2_{n_1} \otimes \mathcal{E}),$$

and

$$\boldsymbol{S}_{ij}\Theta = \Theta\Phi_{ij}.$$

In this case, the Fourier coefficients of Φ_{ij} are given by

$$\varphi_{ij,\alpha^t} = P_{\mathcal{E}}(S^{\alpha*} \otimes I_{\mathcal{E}_{\boldsymbol{n}}})\boldsymbol{S}_{ij}|_{\mathcal{E}},$$

for all $\alpha \in F^+_{n_1}$, $i = 2, \ldots, k$, and $j = 1, \ldots, n_i$.

Remark 4.4. A few comments about the above classification result are in order.

1. In view of our notation $\boldsymbol{S}_{n_1} = (\boldsymbol{S}_{11}, \ldots, \boldsymbol{S}_{1n_1})$, the Fourier coefficients of Φ_{ij} can be further simplified to
$$\varphi_{ij,\alpha^t} = P_{\mathcal{E}}\boldsymbol{S}^{\alpha*}_{n_1}\boldsymbol{S}_{ij}|_{\mathcal{E}},$$
for all $\alpha \in F^+_{n_1}$ (see Theorem 4.1).
2. In view of Remark 4.2, it follows that Φ_{ij} is a pure isometry for each $i = 2, \ldots, k$, and $j = 1, \ldots, n_i$.
3. Fix $i \in \{2, \ldots, k\}$. Then by (4.1), it follows that
$$\Phi_{ij} = \Theta^*\boldsymbol{S}_{ij}\Theta,$$
for all $j = 1, \ldots, n_i$. Consequently, $\Phi^*_{ip}\Phi_{iq} = \delta_{pq}I_{F^2_{n_1}\otimes\mathcal{E}}$ for all $p, q = 1, \ldots, n_i$. Hence, the n_i-tuple $\boldsymbol{\Phi}_{n_i} = (\Phi_{i1}, \ldots, \Phi_{in_i})$ is a pure isometric tuple on $F^2_{n_1} \otimes \mathcal{E}$ for all $i = 2, \ldots, k$.
4. In the setting of Corollary 4.3, if $\mathcal{M}$ is a submodule of $F^2_{\boldsymbol{n}}$, then
$$(\boldsymbol{S}_{n_1}|_{\mathcal{M}}, \boldsymbol{S}_{n_2}|_{\mathcal{M}}, \ldots, \boldsymbol{S}_{n_k}|_{\mathcal{M}}) \text{ on } \mathcal{M},$$
and
$$(\boldsymbol{S}_{n_1} \otimes I_{\mathcal{E}}, \boldsymbol{\Phi}_{n_2}, \ldots, \boldsymbol{\Phi}_{n_k}) \text{ on } F^2_{n_1} \otimes \mathcal{E},$$
are unitarily equivalent.

We conclude this section with the uniqueness of the tuple $(\boldsymbol{\Phi}_{n_2}, \ldots, \boldsymbol{\Phi}_{n_k})$ on $F^2_{n_1} \otimes \mathcal{E}$:

Theorem 4.5. *In the setting of Corollary* 4.3*, let* $\mathcal{M} = \tilde{\Theta}(F^2_{n_1} \otimes \tilde{\mathcal{E}})$ *for some Hilbert space* $\tilde{\mathcal{E}}$ *and an inner multi-analytic map* $\tilde{\Theta} \in R^\infty_{n_1} \overline{\otimes} \mathcal{B}(\tilde{\mathcal{E}}, \mathcal{E}_{\boldsymbol{n}})$*, and let* $\boldsymbol{S}_{ij}\tilde{\Theta} = \tilde{\Theta}\tilde{\Phi}_{ij}$ *for some pure isometry* $\tilde{\Phi}_{ij} \in R^\infty_n \overline{\otimes} \mathcal{B}(\tilde{\mathcal{E}})$ *and* $i = 2, \dots, k$ *and* $j = 1, \dots, n_i$*. Then there exists a unitary* $\tau : \mathcal{E} \to \tilde{\mathcal{E}}$ *such that*

$$\tilde{\Theta}(I_{F^2_{n_1}} \otimes \tau) = \Theta,$$

and

$$(I_{F^2_{n_1}} \otimes \tau)\Phi_{ij} = \tilde{\Phi}_{ij}(I_{F^2_{n_1}} \otimes \tau),$$

for all $i = 2, \dots, k$*, and* $j = 1, \dots, n_i$*.*

Proof. By Theorem 3.2, there exists a unitary $\tau \in \mathcal{B}(\mathcal{E}, \tilde{\mathcal{E}})$ such that $\tilde{\Theta}(I_{F^2_{n_1}} \otimes \tau) = \Theta$. Moreover

$$\tilde{\Theta}(I_{F^2_{n_1}} \otimes \tau)\Phi_{ij} = \Theta\Phi_{ij} = \boldsymbol{S}_{ij}\Theta = \boldsymbol{S}_{ij}\tilde{\Theta}(I_{F^2_{n_1}} \otimes \tau),$$

that is, $\tilde{\Theta}(I_{F^2_{n_1}} \otimes \tau)\Phi_{ij} = \tilde{\Theta}\tilde{\Phi}_{ij}(I_{F^2_{n_1}} \otimes \tau)$. The result now follows from the fact that $\tilde{\Theta}$ is an isometry. ☐

In particular, if $(n_1, \dots, n_k) = (1, \dots, 1)$, then the Fock module $F^2_{\boldsymbol{n}}$ is the Hardy module $H^2(\mathbb{D}^k)$, and hence Corollary 4.3 (and the uniqueness theorem above) recovers [12, Theorem 3.2]. However, it is worth mentioning that the representation of submodules in Corollary 4.3 somewhat finer than [12, Theorem 3.2]. The major difference here is the coordinate free approach to submodules of $F^2_{\boldsymbol{n}}$ as in Theorem 4.1 (for instance, the formalism of κ_i in [12, Theorem 3.1] is not essential in the present consideration). This slightly different technical advantage may actually result in the further development of multivariable operator theory in noncommutative polydomains.

5. Noncommutative varieties and submodules

First, we briefly recall the necessary definitions and results about noncommutative varieties in $\mathcal{B}(\mathcal{H})^n$ from [20]. Given a weak operator topology closed two-sided proper ideal J of F^∞_n, the *non-commutative variety* $\mathcal{V}_J(\mathcal{H})$ corresponding to J is defined by

$$\mathcal{V}_J(\mathcal{H}) = \{(X_1, \dots, X_n) \in \mathfrak{B}^{(n)}(\mathcal{H}) : \varphi(X_1, \dots, X_n) = 0 \text{ for all } \varphi \in J\},$$

where $\varphi(X_1, \dots, X_n)$ is defined in the sense of F^∞_n non-commutative functional calculus for completely non-coisometric contractions [18].

Now let J be a weak operator topology closed two-sided proper ideal in F^∞_n. Define

$$M_J := \overline{JF^2_n} \quad \text{and} \quad N_J := F^2_n \ominus JF^2_n.$$

Since J is a two-sided weakly closed ideal, it follows that M_J is a submodule of F^2_n, and hence N_J is a quotient module of F^2_n [20, Lemma 1.1]. Moreover

$$M_J = \overline{\text{span}}\{S^\alpha\varphi(1) : \varphi \in J, \alpha \in F^+_n\} \quad \text{and} \quad N_J = \bigcap_{\varphi \in J} \ker \varphi^*.$$

Following [20], define the *constrained left (respectively, right) creation operators* as

$$B_i := P_{N_J} S_i|_{N_J} \quad \text{and} \quad W_i := P_{N_J} R_i|_{N_J},$$

respectively, for all $i = 1, \ldots, n$. Therefore $B = (B_1, \ldots, B_n)$ and $W = (W_1, \ldots, W_n)$ are n-tuples of constrained creation operators on N_J. A closed subspace $\mathcal{M} \subseteq N_J \otimes \mathcal{K}$, for some Hilbert space $\mathcal{K}$, is called a *submodule* if $(B_i \otimes I_{\mathcal{K}})\mathcal{M} \subseteq \mathcal{M}$ for all $i = 1, \ldots, n$.

The remarkable fact is that $B \in \mathcal{V}_J(N_J)$ and this constrained tuple B plays the role of model tuple for tuples of operators in noncommutative domains [20]. Moreover, we have Popescu's Beurling-Lax-Halmos type theorem for submodules of $N_J \otimes \mathcal{K}$ corresponding to the constrained tuple $(B_1 \otimes I_{\mathcal{K}}, \ldots, B_n \otimes I_{\mathcal{K}})$ on $N_J \otimes \mathcal{K}$ for Hilbert spaces $\mathcal{K}$:

Theorem 5.1. [20, Popescu, Theorem 1.2] *Let $J \subsetneq F_n^\infty$ be a weakly closed two-sided ideal, and let $\mathcal{K}$ be a Hilbert space. A closed subspace $\mathcal{M} \subseteq N_J \otimes \mathcal{K}$ is a submodule if and only if there exist a Hilbert space $\mathcal{G}$ and a partial isometry*

$$\Theta(W_1, \ldots, W_n) \in \mathscr{W}(W_1, \ldots, W_n) \overline{\otimes} B(\mathcal{G}, \mathcal{K})$$

such that $\mathcal{M} = \Theta(W_1, \ldots, W_n)(N_J \otimes \mathcal{G})$.

Recall here that $\mathscr{W}(B_1, \ldots, B_n)$ is the w^*-closed (or, weak operator topology closed, as they coincide in this particular situation) algebra generated by $\{I_{N_J}, B_1, \ldots, B_n\}$, and (see Popescu [20, page 396] and also see Arias and Popescu [3])

$$\mathscr{W}(B_1, \ldots, B_n) = P_{N_J} F_n^\infty|_{N_J} = \{\varphi(B_1, \ldots, B_n) : \varphi(S_1, \ldots, S_n) \in F_n^\infty\},$$

and

$$\mathscr{W}(B_1, \ldots, B_n)' = \mathscr{W}(W_1, \ldots, W_n) \text{ and } \mathscr{W}(W_1, \ldots, W_n)' = \mathscr{W}(B_1, \ldots, B_n).$$

Moreover, the noncommutative version of intertwiner lifting [16] implies that

$$\mathscr{W}(W_1, \ldots, W_n) \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*) = P_{N_J \otimes \mathcal{E}} [R_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*)]|_{N_J \otimes \mathcal{E}_*}, \tag{5.1}$$

for Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$. Note that a similar statement also holds for $\mathscr{W}(B_1, \ldots, B_n) \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$. The elements of $\mathscr{W}(W_1, \ldots, W_n)$ are called *constrained multi-analytic operators.* We will use the symbol Θ (or $\Theta(B_1, \ldots, B_n)$ and $\Theta(W_1, \ldots, W_n)$ if the presence of $(B_1, \ldots, B_n)$ and $(W_1, \ldots, W_n)$, respectively, is not clear from the context) to denote the constrained multi-analytic operators in $\mathscr{W}(B_1, \ldots, B_n)$ and $\mathscr{W}(W_1, \ldots, W_n)$.

The following result furnishes an analogue of Theorem 4.1 in the setting of noncommutative varieties.

Theorem 5.2. *Let $\mathcal{E}_*$ be a Hilbert space, $T \in \mathcal{B}(N_J \otimes \mathcal{E}_*)$, and let $\mathcal{M}$ be a closed subspace of $N_J \otimes \mathcal{E}_*$. Suppose $T(B_i \otimes I_{\mathcal{E}_*}) = (B_i \otimes I_{\mathcal{E}_*})T$ for all $i = 1, \ldots n$. The following statements are equivalent:*

(a) *$\mathcal{M}$ is a submodule of $N_J \otimes \mathcal{E}_*$ and $T\mathcal{M} \subseteq \mathcal{M}$.*

(b) *There exist a closed subspace $\mathcal{E}$ of $N_J \otimes \mathcal{E}_*$, a constrained multi-analytic partial isometry*

$$\Theta(W_1, \ldots, W_n) \in \mathscr{W}(W_1, \ldots, W_n)\overline{\otimes}\mathcal{B}(\mathcal{E}, \mathcal{E}_*),$$

and a constrained multi-analytic operator

$$\Phi(W_1, \ldots, W_n) \in \mathscr{W}(W_1, \ldots, W_n)\overline{\otimes}\mathcal{B}(\mathcal{E}),$$

such that

$$\mathcal{M} = \Theta(W_1, \ldots, W_n)(N_J \otimes \mathcal{E}),$$

and

$$T\Theta(W_1, \ldots, W_n) = \Theta(W_1, \ldots, W_n)\Phi(W_1, \ldots, W_n).$$

Proof. Again, $(b) \Rightarrow (a)$ follows from Douglas' range-inclusion theorem. We now prove that $(a) \Rightarrow (b)$. The idea is to apply Theorem 4.1 along with Popescu's non-commutative version of the commutant lifting theorem in an appropriate sense. Suppose $\mathcal{M}$ is a submodule of $N_J \otimes \mathcal{E}_*$ and $T\mathcal{M} \subseteq \mathcal{M}$. By the noncommutative commutant lifting theorem (see [16, Theorem 3.2]), there exists $\Psi \in R_n^\infty\overline{\otimes}\mathcal{B}(\mathcal{E}_*)$ such that

$$\Psi^*|_{N_J \otimes \mathcal{E}_*} = T^*. \tag{5.2}$$

Observe that $(F_n^2 \otimes \mathcal{E}_*) \ominus (N_J \otimes \mathcal{E}_*) = M_J \otimes \mathcal{E}_*$ is a submodule of $F_n^2 \otimes \mathcal{E}_*$. Define

$$\mathcal{M}_J = (M_J \otimes \mathcal{E}_*) \oplus \mathcal{M}.$$

Clearly, $\mathcal{M}_J \subseteq F_n^2 \otimes \mathcal{E}_*$. First, we claim that $\mathcal{M}_J$ is Ψ-invariant. Indeed, on the one hand

$$\Psi(M_J \otimes \mathcal{E}_*) \subseteq (M_J \otimes \mathcal{E}_*),$$

as $\Psi^*(N_J \otimes \mathcal{E}_*) \subseteq (N_J \otimes \mathcal{E}_*)$ and, on the other hand, since $\mathcal{M} \subseteq N_J \otimes \mathcal{E}_*$, we have

$$\begin{aligned}
\Psi P_{\mathcal{M}} &= (P_{M_J \otimes \mathcal{E}_*} + P_{N_J \otimes \mathcal{E}_*})\Psi P_{N_J \otimes \mathcal{E}_*} P_{\mathcal{M}} \\
&= \Big(P_{M_J \otimes \mathcal{E}_*}\Psi P_{N_J \otimes \mathcal{E}_*} + P_{N_J \otimes \mathcal{E}_*}\Psi P_{N_J \otimes \mathcal{E}_*}\Big) P_{\mathcal{M}} \\
&= \Big(P_{M_J \otimes \mathcal{E}_*}\Psi P_{N_J \otimes \mathcal{E}_*} + T\Big) P_{\mathcal{M}},
\end{aligned}$$

and hence $\Psi\mathcal{M} \subseteq \mathcal{M}_J$. Next we claim that $\mathcal{M}_J$ is a submodule of $F_n^2 \otimes \mathcal{E}_*$. The proof is similar to the proof of the above claim. Otherwise, one may argue, as in the first paragraph in [20, page 397], that

$$\mathcal{M}_J = \Big(F_n^2 \otimes \mathcal{E}_*\Big) \ominus \Big((N_J \otimes \mathcal{E}_*) \ominus \mathcal{M}\Big),$$

and since $(N_J \otimes \mathcal{E}_*) \ominus \mathcal{M}$ is invariant under $(B \otimes I_{\mathcal{E}_*})^*$, it follows that it is also invariant under $(S \otimes I_{\mathcal{E}_*})^*$, and hence $\mathcal{M}_J$ is a submodule of $F_n^2 \otimes \mathcal{E}_*$. On the other hand, for each $i = 1, \ldots, n$, we have

$$\begin{aligned}
P_{M_J \otimes \mathcal{E}_*}(S_i \otimes I_{\mathcal{E}_*})^*|_{\mathcal{M}_J} &= P_{M_J \otimes \mathcal{E}_*}(S_i \otimes I_{\mathcal{E}_*})^*|_{M_J \otimes \mathcal{E}_*} + P_{M_J \otimes \mathcal{E}_*}(S_i \otimes I_{\mathcal{E}_*})^*|_{\mathcal{M}} \\
&= P_{M_J \otimes \mathcal{E}_*}(S_i \otimes I_{\mathcal{E}_*})^*|_{M_J \otimes \mathcal{E}_*},
\end{aligned}$$

as

$$(S_i \otimes I_{\mathcal{E}_*})^*\mathcal{M} \subseteq (S_i \otimes I_{\mathcal{E}_*})^*(N_J \otimes \mathcal{E}_*) \subseteq N_J \otimes \mathcal{E}_*.$$

This implies

$$P_{\mathcal{M}_J}(S_i \otimes I_{\mathcal{E}_*})^*|_{\mathcal{M}_J} = P_{M_J\otimes\mathcal{E}_*}(S_i \otimes I_{\mathcal{E}_*})^*|_{M_J\otimes\mathcal{E}_*} + P_{\mathcal{M}}(S_i \otimes I_{\mathcal{E}_*})^*|_{\mathcal{M}_J},$$

and hence

$$\tilde{\mathcal{E}} = \mathcal{W} \oplus \bigcap_{i=1}^{n} \ker\Big(P_{\mathcal{M}}(S_i \otimes I_{\mathcal{E}_*})^*|_{\mathcal{M}_J}\Big), \tag{5.3}$$

where

$$\tilde{\mathcal{E}} = \bigcap_{i=1}^{n} \ker\Big(P_{\mathcal{M}_J}(S_i \otimes I_{\mathcal{E}_*})^*|_{\mathcal{M}_J}\Big), \tag{5.4}$$

and

$$\mathcal{W} = \bigcap_{i=1}^{n} \ker\Big(P_{M_J\otimes\mathcal{E}_*}(S_i \otimes I_{\mathcal{E}_*})^*|_{M_J\otimes\mathcal{E}_*}\Big), \tag{5.5}$$

are wandering subspaces of the tuples $(S \otimes I_{\mathcal{E}_*})|_{\mathcal{M}_J}$ and $(S \otimes I_{\mathcal{E}_*})|_{M_J\otimes\mathcal{E}_*}$ on $\mathcal{M}_J$ and $M_J \otimes \mathcal{E}_*$, respectively. We conclude therefore, by Theorem 4.1 along with the proof of Theorem 3.2 (and in particular, the construction of wandering subspace in (3.2)), that there exist an inner multi-analytic operator $\tilde{\Theta} \in R_n^\infty \bar{\otimes} \mathcal{B}(\tilde{\mathcal{E}}, \mathcal{E}_*)$ and a multi-analytic operator $\tilde{\Phi} \in R_n^\infty \bar{\otimes} \mathcal{B}(\tilde{\mathcal{E}})$ such that

$$\mathcal{M}_J = \tilde{\Theta}(F_n^2 \otimes \tilde{\mathcal{E}}),$$

and

$$M_J \otimes \mathcal{E}_* = \tilde{\Theta}|_{F_n^2\otimes\mathcal{W}}(F_n^2 \otimes \mathcal{W}), \tag{5.6}$$

and

$$\Psi\tilde{\Theta} = \tilde{\Theta}\tilde{\Phi},$$

and the Fourier coefficients of $\tilde{\Phi}$ are given by

$$\tilde{\varphi}_{\alpha^t} = P_{\tilde{\mathcal{E}}}(S^{\alpha *} \otimes I_{\mathcal{E}_*})\Psi|_{\tilde{\mathcal{E}}}, \tag{5.7}$$

for all $\alpha \in F_n^+$. Now we set

$$\mathcal{E} = \tilde{\mathcal{E}} \ominus \mathcal{W}. \tag{5.8}$$

Then

$$\mathcal{E} = \bigcap_{i=1}^{n} \ker\Big(P_{\mathcal{M}}(S_i \otimes I_{\mathcal{E}_*})^*|_{\mathcal{M}_J}\Big) \subseteq \mathcal{M}_J \subseteq F_n^2 \otimes \mathcal{E}_*.$$

Next we define

$$\Theta \in \mathscr{W}(W_1, \ldots, W_n)\bar{\otimes} B(\mathcal{E}, \mathcal{E}_*) \quad \text{and} \quad \Phi \in \mathscr{W}(W_1, \ldots, W_n) \otimes \mathcal{B}(\mathcal{E}),$$

by

$$\Theta = P_{N_J\otimes\mathcal{E}_*}\tilde{\Theta}|_{N_J\otimes\mathcal{E}},$$

and

$$\Phi = P_{N_J\otimes\mathcal{E}}\tilde{\Phi}|_{N_J\otimes\mathcal{E}}, \tag{5.9}$$

respectively. Now taking into account of the fact that $N_J \otimes \mathcal{E}_*$ is jointly co-invariant under $(R \otimes I_{\mathcal{E}_*})$ [20, Lemma 1.1] we have

$$\tilde{\Theta}^*(N_J \otimes \mathcal{E}_*) \subseteq N_J \otimes \mathcal{E},$$

or equivalently

$$\tilde{\Theta}^* P_{N_J\otimes\mathcal{E}_*} = P_{N_J\otimes\mathcal{E}}\tilde{\Theta}^* P_{N_J\otimes\mathcal{E}_*}. \tag{5.10}$$

Note also that

$$\Psi^*(N_J \otimes \mathcal{E}_*) \subseteq N_J \otimes \mathcal{E}_*,$$

as $\Psi^*|_{N_J\otimes\mathcal{E}_*} = T^* \in \mathcal{B}(N_J \otimes \mathcal{E}_*)$. Using these observations and the fact that $\tilde{\Theta}\tilde{\Phi} = \Psi\tilde{\Theta}$ we compute

$$\begin{aligned} P_{N_J\otimes\mathcal{E}}\tilde{\Phi}^*\tilde{\Theta}^*|_{N_J\otimes\mathcal{E}_*} &= P_{N_J\otimes\mathcal{E}}\tilde{\Theta}^*\Psi^*|_{N_J\otimes\mathcal{E}_*} \\ &= P_{N_J\otimes\mathcal{E}}\tilde{\Theta}^*|_{N_J\otimes\mathcal{E}_*}\Psi^*|_{N_J\otimes\mathcal{E}_*} \\ &= \Theta^*T^*, \end{aligned}$$

which implies

$$\begin{aligned} T\Theta &= P_{N_J\otimes\mathcal{E}_*}\tilde{\Theta}\tilde{\Phi}|_{N_J\otimes\mathcal{E}} \\ &= P_{N_J\otimes\mathcal{E}_*}\tilde{\Theta}I_{F^2_n\otimes\tilde{\mathcal{E}}}\tilde{\Phi}|_{N_J\otimes\mathcal{E}} \\ &= P_{N_J\otimes\mathcal{E}_*}\tilde{\Theta}(P_{F^2_n\otimes\mathcal{E}} + P_{F^2_n\otimes\mathcal{W}})\tilde{\Phi}|_{N_J\otimes\mathcal{E}}. \end{aligned}$$

By (5.6), we must have

$$\tilde{\Theta}(F^2_n \otimes \mathcal{W}) = M_J \otimes \mathcal{E}_* \perp N_J \otimes \mathcal{E}_*,$$

and hence $P_{N_J\otimes\mathcal{E}_*}\tilde{\Theta}P_{F^2_n\otimes\mathcal{W}} = 0$. We obtain

$$T\Theta = P_{N_J\otimes\mathcal{E}_*}\tilde{\Theta}P_{F^2_n\otimes\mathcal{E}}\tilde{\Phi}|_{N_J\otimes\mathcal{E}}$$

By (5.10), the later implies that

$$\begin{aligned} T\Theta &= P_{N_J\otimes\mathcal{E}_*}\tilde{\Theta}P_{N_J\otimes\mathcal{E}}P_{F^2_n\otimes\mathcal{E}}\tilde{\Phi}|_{N_J\otimes\mathcal{E}} \\ &= P_{N_J\otimes\mathcal{E}_*}\tilde{\Theta}P_{N_J\otimes\mathcal{E}}\tilde{\Phi}|_{N_J\otimes\mathcal{E}}, \end{aligned}$$

and hence $T\Theta = \Theta\Phi$. Finally, we use (5.10) again to obtain

$$\Theta\Theta^* = P_{N_J\otimes\mathcal{E}}\tilde{\Theta}\tilde{\Theta}^*|_{N_J\otimes\mathcal{E}} = P_{N_J\otimes\mathcal{E}}P_{\mathcal{M}_J}|_{N_J\otimes\mathcal{E}} = P_{\mathcal{M}}.$$

Therefore, Θ is a partial isometry and $\mathcal{M} = \operatorname{ran}\Theta$. We postpone the proof of the fact that $\mathcal{E} \subseteq N_J \otimes \mathcal{E}_*$ till Lemma 6.2. This completes the proof of the theorem. □

The submodule part of the above theorem reproves Popescu's version of constrained Beurling, Lax and Halmos theorem, namely Theorem 5.1 (or see [20, Popescu, Theorem 1.2]). Here, however, the present proof brings out more geometric flavour (like the fact that $\mathcal{E} \subseteq N_J \otimes \mathcal{E}_*$). This will be more evident in the following section.

Moreover, we will return to the constructions of $\tilde{\mathcal{E}}$ and $\mathcal{W}$ as in equations (5.3), (5.4) and (5.5) and the decomposition $\tilde{\mathcal{E}} = \mathcal{E} \oplus \mathcal{W}$ as in (5.8) in Section 6 when we discuss in more detail about the representation of the constrained multi-analytic map Φ.

Now we introduce constrained Fock $\boldsymbol{n}$-modules (quotient modules of Fock $\boldsymbol{n}$-modules).

Definition 5.3. Given $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ and weak operator topology closed two-sided proper ideal J_i in $F^\infty_{n_i}$, $i = 1, \dots, k$, the constrained Fock $\boldsymbol{n}$-module $N_{\boldsymbol{J_n}}$ is defined by

$$N_{\boldsymbol{J_n}} := N_{J_1} \otimes \cdots \otimes N_{J_k}.$$

Since $N_{\boldsymbol{J_n}} \subseteq F^2_{\boldsymbol{n}}$, the following operators on $N_{\boldsymbol{J_n}}$ are well defined:

$$\boldsymbol{B}_{ij} = P_{N_{\boldsymbol{J_n}}} \boldsymbol{S}_{ij}|_{N_{\boldsymbol{J_n}}} \quad \text{and} \quad \boldsymbol{W}_{ij} = P_{N_{\boldsymbol{J_n}}} \boldsymbol{R}_{ij}|_{N_{\boldsymbol{J_n}}},$$

for all $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$. Set $\boldsymbol{B}_{n_i} = (\boldsymbol{B}_{i1}, \ldots, \boldsymbol{B}_{in_i})$, the n_i-tuple on $N_{J_{\boldsymbol{n}}}$, and let $\boldsymbol{B} = (\boldsymbol{B}_{n_1}, \ldots, \boldsymbol{B}_{n_k})$. Similarly, define $\boldsymbol{W}_{n_i}$ and $\boldsymbol{W}$ on $N_{\boldsymbol{J_n}}$. Observe that

$$\mathbf{B}_{i,j} = I_{N_{J_1}} \otimes \cdots \otimes B_{ij} \otimes \cdots \otimes I_{N_{J_k}},$$

and

$$\mathbf{W}_{ij} = I_{N_{J_1}} \otimes \cdots \otimes W_{i,j} \otimes \cdots \otimes I_{N_{J_k}},$$

for all $i = 1, \ldots, k$ and $j = 1, \ldots, n_i$. Moreover, if $1 \leq p < q \leq k$ and $X \in \boldsymbol{B}_p$ and $Y \in \boldsymbol{B}_q$, then $XY^* = Y^*X$, that is, $\boldsymbol{B}_p$ *doubly commutes* with $\boldsymbol{B}_q$.

Clearly, $N_{\boldsymbol{J_n}}$ is a quotient module of the $\mathbb{C}\langle \boldsymbol{Z}\rangle_{\boldsymbol{n}}$-Hilbert module $F^2_{\boldsymbol{n}}$. From this point of view, a closed subspace $\mathcal{M} \subseteq N_{\boldsymbol{J_n}}$ is said to be a *submodule* if $X\mathcal{M} \subseteq \mathcal{M}$ for all $X \in \boldsymbol{B}_{n_i}$, $i = 1, \ldots, k$. The proof of the following corollary concerning submodules of $N_{\boldsymbol{J_n}}$ is now similar to that of Corollary 4.3.

Corollary 5.4. *Let J_i be a weak operator topology closed two-sided proper ideal in $F^\infty_{n_i}$, $i = 1, \ldots, k$, and let $\mathcal{M}$ be a closed subspace of the constrained Fock $\boldsymbol{n}$-module $N_{\boldsymbol{J_n}} = N_{J_1} \otimes \cdots \otimes N_{J_k}$. Suppose*

$$\mathcal{E}_{\boldsymbol{n}} = N_{J_2} \otimes \cdots \otimes N_{J_k}.$$

Then $\mathcal{M}$ is a submodule of the quotient module $N_{\boldsymbol{J_n}}$ if and only if there exist a Hilbert space $\mathcal{E}$, a constrained multi-analytic partial isometry

$$\Theta \in \mathscr{W}(\mathbf{W}_{11}, \ldots, \mathbf{W}_{1n_1}) \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_{\boldsymbol{n}}),$$

and a constrained multi-analytic operator

$$\Phi_{ij} \in \mathscr{W}(\mathbf{W}_{11}, \ldots, \mathbf{W}_{1n_1}) \overline{\otimes} \mathcal{B}(\mathcal{E}),$$

such that $\mathcal{M} = \Theta(N_{J_1} \otimes \mathcal{E})$ and $\mathbf{B}_{ij}\Theta = \Theta\Phi_{ij}$ for all $i = 2, \ldots, k$ and $j = 1, \ldots, n_i$.

Note, by the way, that the constrained multi-analytic operators

$$\Phi(W_1, \ldots, W_n) \in \mathscr{W}(W_1, \ldots, W_n) \overline{\otimes} \mathcal{B}(\mathcal{E}),$$

in Theorem 5.2 and

$$\Phi_{ij} \in \mathscr{W}(\mathbf{W}_{11}, \ldots, \mathbf{W}_{1n_1}) \overline{\otimes} B(\mathcal{E}),$$

in Corollary 5.4 are not canonical. This inconvenience is caused by the fact that Ψ in the proof of Theorem 5.2 is a lifting of the map T, and hence the choice of Φ is not uniquely determined, in general, by T. For now we will leave them alone and take up this issue again in the next section.

6. Drury Arveson $\boldsymbol{n}$-modules and the map Φ

In this section we continue our discussion of constrained Fock $\boldsymbol{n}$-modules by looking at special noncommutative varieties. More specifically, here we aim at representing submodules of Drury Arveson $\boldsymbol{n}$-modules (see the definition of Drury Arveson $\boldsymbol{n}$-modules below). Moreover, we will analyze the constrained multi-analytic map Φ of Corollary 5.4 and see that the representations of Φ for submodules of Drury Arveson $\boldsymbol{n}$-modules are more concrete and informative.

We first recall the construction of the Drury-Arveson module. Consider the Fock module F^2_n and let J denote the weakly closed two sided ideal generated by

$$\{S_pS_q - S_qS_p : p, q = 1, \dots, n\} \subseteq F^\infty_n. \tag{6.1}$$

Then the quotient N_J is the symmetric Fock space, $P_{N_J}S_i|_{N_J} = P_{N_J}W_i|_{N_J}$ for all $i = 1, \dots, n$, and $(P_{N_J}S_1|_{N_J}, \dots, P_{N_J}S_n|_{N_J})$ on N_J and the tuple of multiplication operators $(M_{z_1}, \dots, M_{z_n})$ on the Drury-Arveson space H^2_n are unitarily equivalent (we will often use this fact implicitly). Moreover

$$P_{N_J}F^\infty_n|_{P_{N_J}} = \mathscr{M}(H^2_n),$$

where $\mathscr{M}(H^2_n)$ denotes the set of multipliers of H^2_n (see [1] and [20]). Recall also that H^2_n is a reproducing kernel Hilbert space corresponding to the kernel function

$$K(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^n z_i \bar{w}_i)^{-1} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n),$$

where $\mathbb{B}^n$ denotes the open unit ball in $\mathbb{C}^n$. Moreover, $\mathscr{M}(H^2_n)$ is a commutative Banach algebra and is given by

$$\mathscr{M}(H^2_n) = \{\varphi \in \mathrm{Hol}(\mathbb{B}^n) : \varphi H^2_n \subseteq H^2_n\}.$$

We now consider the Fock $\boldsymbol{n}$-module $F^2_{\boldsymbol{n}} = F^2_{n_1} \otimes \cdots \otimes F^2_{n_k}$. Let $J_i \subseteq F^\infty_{n_i}$ denote the weakly closed two sided ideal generated by the commutants of the creation operators on $F^2_{n_i}$, $i = 1, \dots, k$, as in (6.1). Then the corresponding constrained Fock $\boldsymbol{n}$-module $N_{\boldsymbol{J_n}}$, also denoted by $H^2_{\boldsymbol{n}}$, is the tensor product of Drury-Arveson modules $\{H^2_{n_i}\}_{i=1}^k$, that is

$$H^2_{\boldsymbol{n}} = H^2_{n_1} \otimes \cdots \otimes H^2_{n_k}.$$

We call $H^2_{\boldsymbol{n}}$ the *Drury-Arveson $\boldsymbol{n}$-module.* Clearly

$$\boldsymbol{B}_{ij} = \boldsymbol{W}_{ij},$$

and, up to unitarily equivalence, they are equal to

$$\boldsymbol{M}_{z_{ij}} = I_{H^2_{n_1}} \otimes \cdots \otimes M_{z_{ij}} \otimes \cdots \otimes I_{H^2_{n_k}},$$

on $H^2_{\boldsymbol{n}}$, where $i = 1, \dots, k$ and $j = 1, \dots, n_i$. Also, for simplicity of notation let

$$\boldsymbol{M}_{n_i} := (\boldsymbol{M}_{z_{i1}}, \dots, \boldsymbol{M}_{z_{in_i}}) \qquad (i = 1, \dots, k).$$

Corollary 5.4, in the setting of Drury-Arveson $\boldsymbol{n}$-module, then yields the following:

Corollary 6.1. *Let $\boldsymbol{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, and let $\mathcal{M}$ be a closed subspace of the Drury-Arveson $\boldsymbol{n}$-module*

$$H^2_{\boldsymbol{n}} = H^2_{n_1} \otimes H^2_{n_2} \otimes \cdots \otimes H^2_{n_k}.$$

Suppose

$$\mathcal{E}_{\boldsymbol{n}} = H^2_{n_2} \otimes \cdots \otimes H^2_{n_k}.$$

Then $\mathcal{M}$ is a submodule of $H^2_{\boldsymbol{n}}$ if and only if there exist a Hilbert space $\mathcal{E}$, a partial isometry $\Theta \in \mathscr{M}(H^2_{n_1})\overline{\otimes}\mathcal{B}(\mathcal{E}, \mathcal{E}_{\boldsymbol{n}})$, and $\Phi_{ij} \in \mathscr{M}(H^2_{n_1})\overline{\otimes}\mathcal{B}(\mathcal{E})$ such that

$$\mathcal{M} = \Theta(H^2_{n_1} \otimes \mathcal{E})$$

and

$$\boldsymbol{M}_{z_{ij}}\Theta = \Theta\Phi_{ij},$$

for all $i = 2, \ldots, k$ and $j = 1, \ldots, n_i$.

As promised, we now return to address the representation of the constrained multi-analytic operator $\Phi(W_1, \ldots, W_n) \in \mathscr{W}(W_1, \ldots, W_n)\overline{\otimes}\mathcal{B}(\mathcal{E})$ in Theorem 5.2. And, at the end of this section, in Corollary 6.4, we will settle the issue of analytic representations of $\Phi_{ij} \in \mathscr{M}(H^2_{n_1})\overline{\otimes}\mathcal{B}(\mathcal{E})$. Here we will work with the same orthogonal decomposition

$$\tilde{\mathcal{E}} = \mathcal{E} \oplus \mathcal{W},$$

as in (5.8) in the proof of Theorem 5.2. Here

$$\tilde{\mathcal{E}} = \bigcap_{i=1}^{n} \Big(\ker(P_{\mathcal{M}_J}(S_i \otimes I_{\mathcal{E}_*})^*|_{\mathcal{M}_J})\Big), \quad \mathcal{E} = \bigcap_{i=1}^{n} \Big(\ker(P_{\mathcal{M}}(S_i \otimes I_{\mathcal{E}_*})^*|_{\mathcal{M}_J})\Big)$$

and

$$\mathcal{W} = \bigcap_{i=1}^{n} \Big(\ker(P_{M_J \otimes \mathcal{E}_*}(S_i \otimes I_{\mathcal{E}_*})^*|_{M_J \otimes \mathcal{E}_*})\Big),$$

are also as in (5.3), (5.4) and (5.5). In this setting, we have the following lemma which seems to be of independent interest:

Lemma 6.2. $\mathcal{E} \subseteq N_J \otimes \mathcal{E}_*$.

Proof. In what follows, $\bigvee$ denote the closed linear span in respective spaces. We observe that

$$\mathcal{E} = \tilde{\mathcal{E}} \ominus \mathcal{W} = \tilde{\mathcal{E}} \cap \mathcal{W}^\perp.$$

Now we compute

$$\begin{aligned}\mathcal{W} &= \bigcap_{i=1}^{n} \Big(\ker(P_{M_J \otimes \mathcal{E}_*}(S_i \otimes I_{\mathcal{E}_*})^*|_{M_J \otimes \mathcal{E}_*})\Big) \\ &= (M_J \otimes \mathcal{E}_*) \bigcap \Big(\bigvee_{i=1}^{n} \operatorname{ran}((S_i \otimes I_{\mathcal{E}_*})|_{M_J \otimes \mathcal{E}_*})\Big)^{\perp} \\ &= (N_J \otimes \mathcal{E}_*)^{\perp} \bigcap \Big(\bigvee_{i=1}^{n} \operatorname{ran}((S_i \otimes I_{\mathcal{E}_*})|_{M_J \otimes \mathcal{E}_*})\Big)^{\perp} \\ &= \Big((N_J \otimes \mathcal{E}_*) \bigvee \Big(\bigvee_{i=1}^{n} \operatorname{ran}((S_i \otimes I_{\mathcal{E}_*})|_{M_J \otimes \mathcal{E}_*})\Big)\Big)^{\perp},\end{aligned}$$

which implies

$$\mathcal{W}^{\perp} = (N_J \otimes \mathcal{E}_*) \bigvee \Big(\bigvee_{i=1}^{n} \operatorname{ran}((S_i \otimes I_{\mathcal{E}_*})|_{M_J \otimes \mathcal{E}_*})\Big).$$

On the other hand, since $M_J \otimes \mathcal{E}_*$ is a submodule of $F^2 \otimes \mathcal{E}_*$ we have

$$(N_J \otimes \mathcal{E}_*) \perp \Big(\bigvee_{i=1}^{n} (\operatorname{ran}(S_i \otimes I_{\mathcal{E}_*})|_{M_J \otimes \mathcal{E}_*})\Big),$$

and hence

$$\mathcal{W}^{\perp} = (N_J \otimes \mathcal{E}_*) \bigoplus \Big(\bigvee_{i=1}^{n} \operatorname{ran}((S_i \otimes I_{\mathcal{E}_*})|_{M_J \otimes \mathcal{E}_*})\Big).$$

Next we simplify $\tilde{\mathcal{E}}$. Observe that

$$\tilde{\mathcal{E}} = \bigcap_{i=1}^{n} \Big(\ker(P_{\mathcal{M}_J}(S_i \otimes I_{\mathcal{E}_*})^*|_{\mathcal{M}_J})\Big) = \mathcal{M}_J \bigcap \Big(\bigvee_{i=1}^{n} \operatorname{ran}((S_i \otimes I_{\mathcal{E}_*})|_{\mathcal{M}_J})\Big)^{\perp}.$$

In particular, $\tilde{\mathcal{E}} \subseteq \mathcal{M}_J$ and

$$\tilde{\mathcal{E}} \perp \Big(\bigvee_{i=1}^{n} \operatorname{ran}((S_i \otimes I_{\mathcal{E}_*})|_{\mathcal{M}_J})\Big).$$

Also since $M_J \otimes \mathcal{E}_* \subseteq \mathcal{M}_J$ we have

$$\Big(\bigvee_{i=1}^{n} \operatorname{ran}((S_i \otimes I_{\mathcal{E}_*})|_{M_J \otimes \mathcal{E}_*})\Big) \subseteq \Big(\bigvee_{i=1}^{n} \operatorname{ran}((S_i \otimes I_{\mathcal{E}_*})|_{\mathcal{M}_J})\Big),$$

and hence

$$\Big(\bigvee_{i=1}^{n} \operatorname{ran}((S_i \otimes I_{\mathcal{E}_*})|_{\mathcal{M}_J})\Big)^{\perp} \subseteq \Big(\bigvee_{i=1}^{n} \operatorname{ran}((S_i \otimes I_{\mathcal{E}_*})|_{M_J \otimes \mathcal{E}_*})\Big)^{\perp}.$$

This immediately leads to

$$\tilde{\mathcal{E}} \perp \Big(\bigvee_{i=1}^{n} \operatorname{ran}((S_i \otimes I_{\mathcal{E}_*})|_{M_J \otimes \mathcal{E}_*})\Big),$$

and hence, finally

$$\begin{aligned}\mathcal{E} &= \tilde{\mathcal{E}} \cap \mathcal{W}^{\perp}\\ &= \tilde{\mathcal{E}} \bigcap \Big((N_J \otimes \mathcal{E}_*) \bigoplus \Big(\bigvee_{i=1}^{n} \operatorname{ran}((S_i \otimes I_{\mathcal{E}_*})|_{M_J \otimes \mathcal{E}_*}) \Big) \Big)\\ &= \tilde{\mathcal{E}} \bigcap (N_J \otimes \mathcal{E}_*).\end{aligned}$$

In particular, $\mathcal{E} \subseteq N_J \otimes \mathcal{E}_*$, which completes the proof. □

Recall from (5.1) that if $\Phi(W_1, \ldots, W_n) \in \mathscr{W}(W_1, \ldots, W_n)\overline{\otimes}\mathcal{B}(\mathcal{E})$ is a constrained multi-analytic operator, then there exists (not necessarily unique) $\tilde{\Phi}(R_1, \ldots, R_n) \in R_n^{\infty}\overline{\otimes}\mathcal{B}(\mathcal{E})$ such that $\Phi = P_{N_J \otimes \mathcal{E}} \tilde{\Phi}|_{N_J \otimes \mathcal{E}}$. Evidently being a solution of the commutant lifting, the multi-analytic operator $\tilde{\Phi}$ is not unique and hence any possible definition of Fourier coefficients of Φ will be ambiguous (for instance, see (6.2) below). However, as we shall see soon, for constrained Fock $\boldsymbol{n}$-modules the situation is somewhat favourable.

First we turn to constrained multi-analytic operator Φ in Theorem 5.2. Here $\Phi = P_{N_J \otimes \mathcal{E}} \tilde{\Phi}|_{N_J \otimes \mathcal{E}}$, and $\tilde{\Phi} \in R_n^{\infty}\overline{\otimes}\mathcal{B}(\mathcal{E})$ (see (5.9)). We note that by Theorem 4.1, the Fourier coefficients of $\tilde{\Phi}$, as constructed in the proof of Theorem 5.2 (and also see (5.7)) are given by

$$\tilde{\varphi}_{\alpha^t} = P_{\tilde{\mathcal{E}}} (S \otimes I_{\mathcal{E}_*})^{\alpha *} \Psi|_{\tilde{\mathcal{E}}} \qquad (\alpha \in F_n^+).$$

In this case, we define the *Fourier coefficients of the constrained multi-analytic operator* $\Phi \in \mathscr{W}(W_1, \ldots, W_n)\overline{\otimes}\mathcal{B}(\mathcal{E})$ corresponding to $\tilde{\Phi}$ as

$$\varphi_{\alpha^t} := P_{\mathcal{E}} \tilde{\phi}_{\alpha^t}|_{\mathcal{E}} = P_{\mathcal{E}} (S \otimes I_{\mathcal{E}_*})^{\alpha *} \Psi|_{\mathcal{E}} \qquad (\alpha \in F_n^+).$$

We proceed now to describe the Fourier coefficients φ_{α^t} in detail as follows. Since, $\Psi^*|_{N_J \otimes \mathcal{E}_*} = T^*$ by (5.2), and $\mathcal{E} \subseteq N_J \otimes \mathcal{E}_*$ by Lemma 6.2, it follows that

$$\Psi|_{\mathcal{E}} = P_{N_J \otimes \mathcal{E}_*} \Psi|_{\mathcal{E}} + P_{M_J \otimes \mathcal{E}_*} \Psi|_{\mathcal{E}} = T|_{\mathcal{E}} + P_{M_J \otimes \mathcal{E}_*} \Psi|_{\mathcal{E}},$$

and hence

$$\varphi_{\alpha^t} = P_{\mathcal{E}} (S \otimes I_{\mathcal{E}_*})^{\alpha *} T|_{\mathcal{E}} + P_{\mathcal{E}} (S \otimes I_{\mathcal{E}_*})^{\alpha *} P_{M_J \otimes \mathcal{E}_*} \Psi|_{\mathcal{E}}, \tag{6.2}$$

for all $\alpha \in F_n^+$. Here note that the appearance of $P_{M_J \otimes \mathcal{E}_*} \Psi$ (here Ψ is a lifting of T as in (5.2)) is not so convenient.

We obviate the above inconvenience by restricting T to tensor product of operators. So we now move to the setting of Corollary 5.4. Here, we treat N_J as N_{J_1}, $\mathcal{E}_*$ as

$$\mathcal{E}_{\boldsymbol{n}} = N_{J_2} \otimes \cdots \otimes N_{J_k},$$

and T as $\mathbf{B}_{ij}$ on $N_{J_1} \otimes \mathcal{E}_{\boldsymbol{n}}$ for all $i = 2, \ldots, k$, and $j = 1, \ldots, n_i$. In this case, consequently one may choose a lifting Ψ_{ij} in $R_{n_1}^{\infty}\overline{\otimes}\mathcal{B}(\mathcal{E}_{\boldsymbol{n}})$ as the constant multi-analytic operator

$$Y_{ij} := I_{F^2_{n_1}} \otimes (I_{N_{J_2}} \otimes \cdots \otimes B_{ij} \otimes \cdots \otimes I_{N_{J_k}}),$$

for all $i = 2, \ldots, k$, and $j = 1, \ldots, n_i$. Let φ_{ij,α^t} be the α-th Fourier coefficient of the multi-analytic operator Φ_{ij} in $\mathscr{W}(W_1, \ldots, W_n)\overline{\otimes}\mathcal{B}(\mathcal{E})$, $i = 2, \ldots, k$, and $j = 1, \ldots, n_i$. Then (6.2) implies that

$$\varphi_{ij,\alpha^t} = P_{\mathcal{E}}(S_{n_1} \otimes I_{\mathcal{E}_{\boldsymbol{n}}})^{\alpha *}\boldsymbol{B}_{ij}|_{\mathcal{E}} + P_{\mathcal{E}}(S_{n_1} \otimes I_{\mathcal{E}_{\boldsymbol{n}}})^{\alpha *}P_{M_J \otimes \mathcal{E}_*}Y_{ij}|_{\mathcal{E}}.$$

Since, $\mathcal{E} \subseteq N_{J_1} \otimes \mathcal{E}_*$ by Lemma 6.2, and $N_{J_1} \otimes \mathcal{E}_*$ is invariant under Y_{ij} by construction, we have that $P_{M_J \otimes \mathcal{E}_*}Y_{ij}|_{\mathcal{E}} = 0$, and hence

$$\varphi_{ij,\alpha^t} = P_{\mathcal{E}}(S_{n_1} \otimes I_{\mathcal{E}_{\boldsymbol{n}}})^{\alpha *}\boldsymbol{B}_{ij}|_{\mathcal{E}} = P_{\mathcal{E}}\boldsymbol{B}_{n_1}^{\alpha *}\boldsymbol{B}_{ij}|_{\mathcal{E}},$$

as $\boldsymbol{B}_{ij}|_{\mathcal{E}} = P_{N_{J_1} \otimes \mathcal{E}_{\boldsymbol{n}}}\boldsymbol{B}_{ij}|_{\mathcal{E}}$ and $(S_{1q} \otimes I_{\mathcal{E}_{\boldsymbol{n}}})^*|_{N_{J_1} \otimes \mathcal{E}_{\boldsymbol{n}}} = \boldsymbol{B}_{1q}^*$ for all $q = 1, \ldots, n_1$. We have thus arrived at the following companion result to Corollary 5.4:

Corollary 6.3. *In the setting of Corollary* 5.4*, for each* $\alpha \in F_n^+$*,* $i = 2, \ldots, k$*, and* $j = 1, \ldots, n_i$*, the* α*-th Fourier coefficient of the constrained multi-analytic operator* Φ_{ij} *is given by*

$$\varphi_{ij,\alpha^t} = P_{\mathcal{E}}\boldsymbol{B}_{n_1}^{\alpha *}\boldsymbol{B}_{ij}|_{\mathcal{E}}.$$

Finally, in view of the above corollary, we now turn to analytic representations of $\Phi_{ij} \in \mathscr{M}(H^2_{n_1})\overline{\otimes}\mathcal{B}(\mathcal{E})$ in Corollary 6.1. So now on, we will be following the setting of Corollary 6.1.

Denote by σ the symmetrization map from $F_{n_1}^+$ to $\mathbb{Z}_+^{n_1}$. Then

$$\boldsymbol{B}_{n_1}^{\alpha} = P_{N_{J_1} \otimes \mathcal{E}_{\boldsymbol{n}}}(S_{n_1}^{\alpha} \otimes I_{\mathcal{E}_{\boldsymbol{n}}})|_{N_{J_1} \otimes \mathcal{E}_{\boldsymbol{n}}} = \boldsymbol{M}_{n_1}^{\sigma(\alpha)},$$

for all $\alpha \in F_{n_1}^+$. Then, by Corollary 5.4 we have

$$\varphi_{ij,\alpha^t} = P_{\mathcal{E}}(\boldsymbol{M}_{n_1})^{*\boldsymbol{m}}\boldsymbol{M}_{z_{ij}}|_{\mathcal{E}} \qquad (\alpha \in F_n^+)$$

where $\sigma(\alpha) = \boldsymbol{m}$. Moreover, a standard computation shows, for each $\boldsymbol{m} \in \mathbb{Z}_+^{n_1}$, that

$$\#\{\alpha \in F_{n_1}^+ : \sigma(\alpha) = \boldsymbol{m}\} = \frac{|\boldsymbol{m}|!}{\boldsymbol{m}!} := \frac{(\sum\limits_{i=1}^{n_1} m_i)!}{m_1! \cdots m_{n_1}!}.$$

Then, for each $\boldsymbol{z} \in \mathbb{B}^{n_1}$ we have

$$\begin{aligned}\Phi_{ij}(\boldsymbol{z}) &= \sum_{\boldsymbol{m} \in \mathbb{Z}_+^{n_1}} \frac{|\boldsymbol{m}|!}{\boldsymbol{m}!}(P_{\mathcal{E}}\boldsymbol{M}_{n_1}^{*\boldsymbol{m}}\boldsymbol{M}_{z_{ij}}|_{\mathcal{E}})\boldsymbol{z}^{\boldsymbol{m}} \\ &= P_{\mathcal{E}}\Big(\sum_{t \in \mathbb{Z}_+}\big(\sum_{\substack{\boldsymbol{m} \in \mathbb{Z}_+^{n_1} \\ |\boldsymbol{m}| = t}} \frac{|\boldsymbol{m}|!}{\boldsymbol{m}!}\boldsymbol{M}_{n_1}^{*\boldsymbol{m}}\boldsymbol{M}_{z_{ij}}\boldsymbol{z}^{\boldsymbol{m}}\big)\Big)|_{\mathcal{E}},\end{aligned}$$

and hence $\Phi_{ij}(\boldsymbol{z}) = P_{\mathcal{E}}\Big(I - \sum_{m=1}^{n_1} z_m \boldsymbol{M}_{z_{1m}}^*\Big)^{-1}\boldsymbol{M}_{z_{ij}}|_{\mathcal{E}}$. We have proved the following.

Corollary 6.4. *In the setting of Corollary* 6.1*, for each* $i = 2, \dots, k$ *and* $j = 1, \dots, n_i$*, the multiplier* $\Phi_{ij} \in \mathscr{M}(H^2_{n_1})\bar{\otimes}\mathcal{B}(\mathcal{E})$ *can be represented as*

$$\Phi_{ij}(\boldsymbol{z}) = P_{\mathcal{E}}\Big(I - \sum_{m=1}^{n_1} z_m \boldsymbol{M}^*_{z_{1m}}\Big)^{-1} \boldsymbol{M}_{z_{ij}}|_{\mathcal{E}} \qquad (\boldsymbol{z} \in \mathbb{B}^{n_1}).$$

7. An example and concluding remarks

Structure of isometries (that is, the von Neumann and Wold decomposition theorem), Beurling, Lax and Halmos theorem, Sarason's commutant lifting theorem, and the Sz.-Nagy and Foias analytic model theory have been inseparable companions in single variable operator theory and function theory. These concepts are increasingly accepted as stepping stones to the development of (both commutative and noncommutative) multivariable operator theory. However, there are a number of interesting and vital results that hold for single bounded linear operators but do not hold in general for commuting and noncommuting n-tuples, $n \geq 2$, of operators. Here we aim to present one such example.

First, we recall the noncommutative version of Beurling, Lax and Halmos theorem (see Theorem 3.2): Let $\mathcal{E}_*$ be a Hilbert space and let $\mathcal{M}$ be a closed subspace of $F^2_n \otimes \mathcal{E}_*$. Then $\mathcal{M}$ is a submodule of $F^2_n \otimes \mathcal{E}_*$ if and only if there exist a Hilbert space $\mathcal{E}$ and an inner multi-analytic operator $\Theta(R_1, \dots, R_n) : F^2_n \otimes \mathcal{E} \to F^2_n \otimes \mathcal{E}_*$ such that

$$\mathcal{M} = \Theta(R_1, \dots, R_n)\Big(F^2_n \otimes \mathcal{E}\Big).$$

If we assume in addition that $n = 1$, then

$$\dim \mathcal{E} \leq \dim \mathcal{E}_*.$$

This inequality plays a crucial role in single variable operator theory. Here, on the contrary, if $n > 1$, then we show that such dimension inequality do not hold in general, specifically we construct an inner multi-analytic operator $\Theta(R_1, \dots, R_n) \in R^\infty_n \bar{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$ such that $\dim \mathcal{E} > \dim \mathcal{E}_*$. Clearly and necessarily, here one must consider finite dimensional Hilbert spaces $\mathcal{E}_*$.

Example. Let $n > 1$ and let $\tilde{\mathcal{E}}_1, \dots, \tilde{\mathcal{E}}_n$ and $\mathcal{E}_*$ be Hilbert spaces. Suppose

$$\dim \tilde{\mathcal{E}}_i = \dim \mathcal{E}_* = m \ (< \infty),$$

for all $i = 1, \dots, n$, and let

$$\mathcal{E} = \bigoplus_{i=1}^{n} \tilde{\mathcal{E}}_i.$$

Let $\{e_{ij}\}_{j=1}^m$ be an orthonormal basis of $\tilde{\mathcal{E}}_i$, $i = 1, \dots, n$, and let $\{f_j\}_{j=1}^m$ be that of $\mathcal{E}_*$. Then

$$\dim \mathcal{E} = mn.$$

Now for each $i = 1, \dots, n$, we define linear operator $\theta_i : \mathcal{E} \to \mathcal{E}_*$ by

$$\theta_i(e_{pq}) = \begin{cases} f_q & \text{if } p = i \\ 0 & \text{if } p \neq i. \end{cases}$$

An easy calculation reveals that $\sum_{i=1}^n \theta_i^* \theta_i = I_{\mathcal{E}}$. Set

$$\Theta(R_1, \dots, R_n) = \sum_{i=1}^n R_i \otimes \theta_i.$$

Clearly

$$(S_i \otimes I_{\mathcal{E}_*})\Theta(R_1, \dots, R_n) = \Theta(R_1, \dots, R_n)(S_i \otimes I_{\mathcal{E}}),$$

for all $i = 1, \dots, n$, that is, $\Theta(R_1, \dots, R_n) \in F_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$. Moreover

$$\Theta(R_1, \dots, R_n)^* \Theta(R_1, \dots, R_n) = \sum_{i=1}^n R_i^* R_i \otimes \theta_i^* \theta = I_{F_n^2 \otimes \mathcal{E}},$$

that is, $\Theta(R_1, \dots, R_n)$ is an inner multi-analytic operator in $F_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$, where, on the other hand

$$mn = \dim \mathcal{E} > \dim \mathcal{E}_* = n.$$

It is now evident, from the above example point of view, to the least, that extensions of some of the concepts (for instance see the appendix in [12]) of submodules of the Hardy space over polydisc in its full generality is not possible in the context of multivariable (both commutative and noncommutative) operator theory and noncommutative varieties.

All the main results of this paper remain valid if we replace Fock $\boldsymbol{n}$-modules, constrained Fock $\boldsymbol{n}$-modules and Drury-Arveson $\boldsymbol{n}$-modules by the respective vector-valued counterparts and the proofs carry over verbatim.

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