

OPERATOR THEORY ON SYMMETRIZED BIDISC

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ABSTRACT. A commuting pair of operators (S, P) on a Hilbert space \mathcal{H} is said to be a Γ -contraction if the symmetrized bidisc

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1|, |z_2| \leq 1\}$$

is a spectral set of the tuple (S, P) . In this paper we develop some operator theory inspired by Agler and Young's results on a model theory for Γ -contractions.

We prove a Beurling-Lax-Halmos type theorem for Γ -isometries. Along the way we solve a problem in the classical one-variable operator theory, namely, a non-zero M_z -invariant subspace \mathcal{S} of $H_{\mathcal{E}^*}^2(\mathbb{D})$ is invariant under the analytic Toeplitz operator with the operator-valued polynomial symbol $p(z) = A + A^*z$ if and only if the Beurling-Lax-Halmos inner multiplier Θ of \mathcal{S} satisfies

$$(A + A^*z)\Theta = \Theta(B + B^*z),$$

for some unique operator B .

We use a "pull back" technique to prove that a completely non-unitary Γ -contraction (S, P) can be dilated to a pair

$$(((A + A^*M_z) \oplus U), (M_z \oplus M_{e^{it}})),$$

which is the direct sum of a Γ -isometry and a Γ -unitary on the Sz.-Nagy and Foias functional model of P , and that (S, P) can be realized as a compression of the above pair in the functional model \mathcal{Q}_P of P as

$$(\mathbf{P}_{\mathcal{Q}_P}((A + A^*M_z) \oplus U)|_{\mathcal{Q}_P}, \mathbf{P}_{\mathcal{Q}_P}(M_z \oplus M_{e^{it}})|_{\mathcal{Q}_P}).$$

Moreover, we show that this representation is unique. We prove that a commuting tuple (S, P) with $\|S\| \leq 2$ and $\|P\| \leq 1$ is a Γ -contraction if and only if there exists a compressed scalar operator X with the decompressed numerical radius not greater than one such that

$$S = X + PX^*.$$

In the commutant lifting set up, we obtain a unique and explicit solution to the lifting of S where (S, P) is a completely non-unitary Γ -contraction. Our results concerning the Beurling-Lax-Halmos theorem of Γ -isometries and the functional model of Γ -contractions answers a pair of questions of J. Agler and N. J. Young.

1. INTRODUCTION

The notion of spectral set was introduced by J. von Neumann in [20] where he proved that the closed unit disk $\overline{\mathbb{D}}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, is a spectral set of a bounded linear operator on a Hilbert space if and only if the operator is a contraction. Later in [17], Sz.-Nagy proved that a bounded linear operator is a contraction if and only if the operator has

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a unitary dilation. Therefore von Neumann's result can be derived from Sz.-Nagy's unitary dilation. Since then, one of the most celebrated problems in operator theory is to determine the class of commuting n -tuple of operators for which a normal ∂K -dilation exists, where $K \subseteq \mathbb{C}^n$ is compact and $n \geq 1$. We recall that a commuting tuple (T_1, \dots, T_n) on \mathcal{H} has a normal ∂K -dilation if there exists a tuple of commuting normal operators (N_1, \dots, N_n) on $\mathcal{K} \supseteq \mathcal{H}$ such that $\sigma_T(N_1, \dots, N_n) \subseteq \partial K$ and

$$\mathbf{P}_{\mathcal{H}} p(N_1, \dots, N_n)|_{\mathcal{H}} = p(T_1, \dots, T_n),$$

for all $p \in \mathbb{C}[z_1, \dots, z_n]$. Here $\mathbf{P}_{\mathcal{H}}$ is the orthogonal projection of \mathcal{K} onto \mathcal{H} . Many studies in this problem have been carried so far. In particular, it is known that the normal ∂K -dilation holds if K is the closure of an annulus [2] and fails when K is a triply connected domain in \mathbb{C} [14]. The theory becomes more subtle when the spectral set is assumed to be a subset of \mathbb{C}^n ($n > 1$).

On the other hand, it is well known that for $n \geq 2$, the von Neumann's inequality fails in general with the exception that a pair of commuting contractions can be dilated to a pair of commuting unitary operators [9]. One versions of von Neumann's inequality for domains like ball and general symmetric domains require to replace the sup norm of the polynomials by operator norm of certain natural multiplier algebras. Now, we define the notion of Γ -contractions.

A pair of commuting operators (S, P) on a Hilbert space \mathcal{H} is said to be a Γ -contraction if the symmetrized bidisc

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1|, |z_2| \leq 1\}.$$

is a spectral set of (S, P) . That is, for all polynomial $p \in \mathbb{C}[z_1, z_2]$,

$$\|p(S, P)\| \leq \sup_{z \in \Gamma} |p(z)|.$$

In particular, if (S, P) is a Γ -contraction then $\|S\| \leq 2$ and $\|P\| \leq 1$. Note also that the symmetrized bidisc Γ is the range of π restricted to the closed bidisc $\bar{\mathbb{D}}^2$ where $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is the proper holomorphic map defined by

$$\pi(z_1, z_2) = (z_1 + z_2, z_1 z_2),$$

for all $(z_1, z_2) \in \mathbb{C}^2$.

There is a significant difference between Γ and other bounded symmetric domains considered earlier by many researches in the development of analytic model theory (cf. [10]). For instance, Γ is polynomially convex [4] but non-convex and inhomogeneous. This in turns makes the theory of Γ -contraction more appealing and useful in the study of the classical and several variables operator theory.

In [7], Agler and Young developed a Γ -isometric dilation theory for Γ -contractions. In this paper, we develop an explicit Γ -isometric dilation and functional model of Γ -contractions. Furthermore, we provide a characterization of Γ -contractions which is compatible with the geometry of the domain Γ . Moreover, we obtain a characterization of invariant subspaces of Γ -isometries.

Our results improve, generalize and unify some recent known results ([11], [12]) and answer positively a pair of problems mentioned by Agler and Young (page 58, [7]).

Finally, it is worth mentioning that our classification results (cf. Theorems 6.2 and 6.3) imply that we must confine ourselves to the class of contractions and compressed scalar operators (see Section 6) in order to have a concrete realization of operator tuples with $\Omega \subseteq \mathbb{C}^n$ as a spectral set, where Ω is the symmetrized polydisc or a higher dimensional domain related to the symmetrized bidisc (cf. [1]).

The organization of the paper is as follows.

In Section 2, we recall some basic definitions and results in the theory of Γ -contractions.

In Section 3, we provide some basic classification results of pure Γ -isometries. We obtain a Beurling-Lax-Halmos type theorem characterizing joint invariant subspaces of a pure Γ -isometry.

In Section 4, we prove a factorization result concerning isometric dilation of a completely non-unitary contraction. Moreover, we use a "pull back" argument to the factorization and obtain a functional model for completely non-unitary Γ -contractions.

In Section 5, we show that the functional model of a completely non unitary Γ -contraction is unique.

In Section 6, we proceed to a new characterization of Γ -contractions.

In Section 7, we conclude with a number of results and remarks concerning Γ -isometric Hardy modules, isometrically isomorphic submodules of Γ -isometric Hardy modules and a solution to the commutant lifting problem.

In this paper, all Hilbert spaces are assumed to be separable and over the field of complex numbers.

2. PRELIMINARIES

In this section, we will gather together some of the necessary definitions and results on Γ -contractions which we will employ later in the paper. For more details about Γ -contractions, we refer readers to the seminal work of Agler and Young [3]-[8] (also see [11], [12]).

In what follows, we shall denote a pair of commuting operators by (S, P) , for "sum" and "product". However, it is far from true that a Γ -contraction is necessarily a sum and product of a pair of commuting contractions.

Let (S, P) be a pair of commuting operators on a Hilbert space \mathcal{H} . Then (S, P) is said to be

- (i) Γ -unitary if S and P are normal operators and the joint spectrum $\sigma(S, P)$ is contained in the distinguished boundary of Γ .
- (ii) Γ -isometry if (S, P) has a Γ -unitary extension.
- (iii) Γ -co-isometry if (S^*, P^*) is a Γ -isometry.

The following theorem is due to Agler and Young [7].

THEOREM 2.1. (Agler and Young) *Let (S, P) be a pair of commuting operators on a Hilbert space \mathcal{H} . Then the following statements are equivalent:*

- (i) (S, P) is a Γ -unitary.
- (ii) P is unitary and $S = S^*P$ and $\|S\| \leq 2$.

(iii) *There exists commuting unitary operators U_1, U_2 on \mathcal{H} such that*

$$S = U_1 + U_2, \quad P = U_1 U_2.$$

Note that for a Γ -isometry (S, P) on \mathcal{H} we have

$$S = \tilde{S}|_{\mathcal{H}} \quad \text{and} \quad P = \tilde{P}|_{\mathcal{H}},$$

where (\tilde{S}, \tilde{P}) is a Γ -unitary on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Consequently, a necessary condition for a pair of commuting operators (S, P) to be a Γ -isometry is that P is an isometry. A Γ -isometry (S, P) is said to be *pure Γ -isometry* if P is a pure isometry, that is, P does not have any unitary part. The following characterization result play an important role in the sequel.

THEOREM 2.2. (Agler and Young) *Let S, P be commuting operators on a Hilbert space \mathcal{H} . Then (S, P) is a pure Γ -isometry if and only if there exists a Hilbert space \mathcal{E} , a unitary operator $U : \mathcal{H} \rightarrow H_{\mathcal{E}}^2(\mathbb{D})$ and $A \in \mathcal{B}(\mathcal{E})$ such that $w(A) \leq 1$ and*

$$S = U^* M_{\varphi} U, \quad P = U^* M_z U,$$

where

$$\varphi(z) = A + A^* z, \quad z \in \mathbb{D}.$$

Here $w(A)$ is the numerical radius of the operator $A \in \mathcal{B}(\mathcal{E})$, that is,

$$w(A) = \sup\{|\langle Ah, h \rangle| : h \in \mathcal{E}, \|h\| \leq 1\}.$$

A contraction P on \mathcal{H} is said to be *completely non-unitary* (or *c.n.u.*) if there is no non-zero P -reducing subspace $\mathcal{H}_u \subseteq \mathcal{H}$ such that $T|_{\mathcal{H}_u}$ is unitary. It is known that a contraction P on \mathcal{H} can be uniquely decomposed as $P = P|_{\mathcal{H}_n} \oplus P|_{\mathcal{H}_u}$ where \mathcal{H}_n and \mathcal{H}_u are P -reducing subspaces of \mathcal{H} and $P|_{\mathcal{H}_n}$ is a c.n.u. contraction and $P|_{\mathcal{H}_u}$ is a unitary contraction. Moreover, let (S, P) be a Γ -contraction for some operator S on \mathcal{H} . Then \mathcal{H}_n and \mathcal{H}_u are S -reducing too and $(S|_{\mathcal{H}_u}, P|_{\mathcal{H}_u})$ is a Γ -unitary and $(S|_{\mathcal{H}_n}, P|_{\mathcal{H}_n})$ is a Γ -contraction (Theorem 2.8 in [7]). By virtue of this result, a Γ -contraction (S, P) is said to be *c.n.u.* if the contraction P is c.n.u.

Let $P \in \mathcal{B}(\mathcal{H})$ be a contraction and $V \in \mathcal{B}(\mathcal{K})$ be an isometry. If V is an isometric dilation of P , then there exists an isometry $\Pi : \mathcal{H} \rightarrow \mathcal{K}$ such that

$$\Pi P^* = V^* \Pi.$$

Conversely, if an isometry $\Pi : \mathcal{H} \rightarrow \mathcal{K}$ intertwine P^* and V^* , then that V is an isometric dilation of P . In the sequel, we shall identify an isometric dilation of a contraction P by either the dilation map V on the dilation space \mathcal{K} or by the isometry Π intertwining P^* and V^* . In either case, we call it an *isometric dilation* of the contraction P . Moreover, if the isometric dilation is minimal, that is, if

$$\mathcal{K} = \overline{\text{span}}\{V^m(\Pi\mathcal{H}) : m \in \mathbb{N}\},$$

then we say that Π is a *minimal isometric dilation* of P

We need to recall two dilation results. The Γ -isometric dilation of Γ -contractions is due to Agler and Young [7], while the isometric dilation of c.n.u. contractions is due to Sz.-Nagy [17] and Sz.-Nagy and Foias [19].

THEOREM 2.3. (Agler and Young) *Let (S, P) be a Γ -contraction on a Hilbert space \mathcal{H} . Then there exists a Hilbert space \mathcal{K} containing \mathcal{H} , a Γ -co-isometry (\tilde{S}, \tilde{P}) on \mathcal{K} and an orthogonal decomposition $\mathcal{K}_1 \oplus \mathcal{K}_2$ of \mathcal{K} such that:*

- (i) \mathcal{K}_1 and \mathcal{K}_2 are joint invariant subspaces of \tilde{S} and \tilde{P} , and $S = \tilde{S}|_{\mathcal{H}}, P = \tilde{P}|_{\mathcal{H}}$;
- (ii) \mathcal{K}_1 and \mathcal{K}_2 reduce both \tilde{S} and \tilde{P} ; and
- (iii) $(\tilde{S}|_{\mathcal{K}_1}, \tilde{P}|_{\mathcal{K}_1})$ is a pure Γ -isometry and $(\tilde{S}|_{\mathcal{K}_2}, \tilde{P}|_{\mathcal{K}_2})$ is a Γ -unitary.

THEOREM 2.4. (Sz.-Nagy) *Let P be a c.n.u. contraction on a Hilbert space \mathcal{H} . Then there exists a Hilbert space \mathcal{K} containing \mathcal{H} , an isometry V on \mathcal{K} and an orthogonal decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ of \mathcal{K} such that:*

- (i) \mathcal{K}_1 and \mathcal{K}_2 are invariant subspaces of V and $P = V|_{\mathcal{H}}$;
- (ii) \mathcal{K}_1 and \mathcal{K}_2 are reducing subspaces of V ;
- (iii) $V|_{\mathcal{K}_1}$ is a pure isometry and $V|_{\mathcal{K}_2}$ is a unitary; and
- (iv) the dilation V on \mathcal{K} is unique (up to unitary equivalence) when it is assumed to be minimal.

We like to point out the absence of the minimality property of the Γ -isometry in Theorem 2.3.

We still need to develop few more definitions and notations. Let P be a contraction on a Hilbert space \mathcal{H} . Then the defect operators of P are defined by

$$D_P = (I_{\mathcal{H}} - P^*P)^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad D_{P^*} = (I_{\mathcal{H}} - PP^*)^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H}),$$

and the defect spaces by

$$\mathcal{D}_P = \overline{\text{ran} D_P} \quad \text{and} \quad \mathcal{D}_{P^*} = \overline{\text{ran} D_{P^*}}.$$

The characteristic function $\Theta_P \in H_{\mathcal{B}(\mathcal{D}_P, \mathcal{D}_{P^*})}^{\infty}(\mathbb{D})$ is defined by

$$\Theta_P(z) = [-P + zD_{P^*}(I_{\mathcal{H}} - zP^*)^{-1}D_P]|_{\mathcal{D}_P}, \quad (z \in \mathbb{D})$$

which yields the multiplication operator $M_{\Theta_P} \in \mathcal{B}(H_{\mathcal{D}_P}^2(\mathbb{D}), H_{\mathcal{D}_{P^*}}^2(\mathbb{D}))$ defined by

$$(M_{\Theta_P}f)(z) = \Theta_P(z)f(z),$$

for all $f \in H_{\mathcal{D}_P}^2(\mathbb{D})$ and $z \in \mathbb{D}$. Note that

$$M_{\Theta_P}(M_z \otimes I_{\mathcal{D}_P}) = (M_z \otimes I_{\mathcal{D}_{P^*}})M_{\Theta_P}.$$

Define

$$\Delta_P(t) = [I_{\mathcal{D}_P} - \Theta_P(e^{it})^* \Theta_P(e^{it})]^{\frac{1}{2}}, \quad (t \in [0, 1])$$

on $L_{\mathcal{D}_P}^2(\mathbb{T})$ and

$$\mathcal{H}_P = H_{\mathcal{D}_{P^*}}^2(\mathbb{D}) \oplus \overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})},$$

and the subspace

$$\mathcal{S}_P = \{M_{\Theta_P}f \oplus \Delta_P f : f \in H_{\mathcal{D}_P}^2(\mathbb{D})\} \subseteq \mathcal{H}_P.$$

Notice that $M_z \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}}$ on \mathcal{H}_P is an isometry where M_z on $H_{\mathcal{D}_P}^2(\mathbb{D})$ is the pure part and $M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}}$ on $\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}$ is the unitary part in the sense of the Wold decomposition of isometries. Moreover, \mathcal{S}_P is invariant under $M_z \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}}$. Define the quotient space

$$\mathcal{Q}_P = \mathcal{H}_P \ominus \mathcal{S}_P.$$

Let \mathcal{S} be a closed subspace of a Hilbert space \mathcal{H} . We shall denote the orthogonal projection from \mathcal{H} onto \mathcal{S} by $\mathbf{P}_{\mathcal{S}}$.

THEOREM 2.5. (Sz.-Nagy and Foias) *Let P be a c.n.u. contraction on a Hilbert space \mathcal{H} . Then*

(i) *P is unitarily equivalent to $\mathbf{P}_{\mathcal{Q}_P}[M_z \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}}]|_{\mathcal{Q}_P}$.*

(ii) *The minimal isometric dilation of P can be identified with $M_z \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}}$ on \mathcal{H}_P .*

In the sequel, by virtue of the unitary $U : H_{\mathcal{E}}^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}) \otimes \mathcal{E}$ defined by

$$z^m \eta \mapsto z^m \otimes \eta, \quad (\eta \in \mathcal{E}, m \in \mathbb{N})$$

we shall often identify the vector valued Hardy space $H_{\mathcal{E}}^2(\mathbb{D})$ with $H^2(\mathbb{D}) \otimes \mathcal{E}$.

3. BEURLING-LAX-HALMOS REPRESENTATIONS OF Γ -ISOMETRIES

This section will focus on a characterization of joint invariant subspaces of pure Γ -isometries.

It is well known that the only invariant of pure unweighted unilateral shift operators is the multiplicity. That is, M_z on $H_{\mathcal{E}}^2(\mathbb{D})$ and M_z on $H_{\mathcal{F}}^2(\mathbb{D})$ are unitarily equivalent if and only if \mathcal{E} and \mathcal{F} are isomorphic Hilbert spaces. We begin with a characterization of pure Γ -isometries in terms of the symbols associated with them.

THEOREM 3.1. *Let $A \in \mathcal{B}(\mathcal{E})$ and $B \in \mathcal{B}(\mathcal{F})$. Then (M_{A+A^*z}, M_z) on $H_{\mathcal{E}}^2(\mathbb{D})$ and (M_{B+B^*z}, M_z) on $H_{\mathcal{F}}^2(\mathbb{D})$ are unitarily equivalent if and only if A and B are unitarily equivalent.*

Proof. Let $U : \mathcal{E} \rightarrow \mathcal{F}$ be a unitary operator such that $UA = BU$. Then the unitary operator

$$\tilde{U} = I_{H^2(\mathbb{D})} \otimes U : H^2(\mathbb{D}) \otimes \mathcal{E} \rightarrow H^2(\mathbb{D}) \otimes \mathcal{F},$$

intertwine the corresponding multiplication operators. Moreover,

$$\tilde{U}(I_{H^2(\mathbb{D})} \otimes A + M_z \otimes A^*) = (I_{H^2(\mathbb{D})} \otimes B + M_z \otimes B^*)\tilde{U}.$$

This proves the sufficiency part.

Conversely, let $\tilde{U} : H^2(\mathbb{D}) \otimes \mathcal{E} \rightarrow H^2(\mathbb{D}) \otimes \mathcal{F}$ be a unitary operator and

$$\tilde{U}(I_{H^2(\mathbb{D})} \otimes A + M_z \otimes A^*) = (I_{H^2(\mathbb{D})} \otimes B + M_z \otimes B^*)\tilde{U},$$

and

$$\tilde{U}(M_z \otimes I_{\mathcal{E}}) = (M_z \otimes I_{\mathcal{F}})\tilde{U}.$$

From the last equality it follows that $\tilde{U} = I_{H^2(\mathbb{D})} \otimes U$ for some unitary operator $U : \mathcal{E} \rightarrow \mathcal{F}$. Then

$$I_{H^2(\mathbb{D})} \otimes UAU^* + M_z \otimes UA^*U^* = I_{H^2(\mathbb{D})} \otimes B + M_z \otimes B^*$$

implies that $UAU^* = B$, and this completes the proof. \blacksquare

The following corollary is a simple but instructive characterization of pure Γ -isometries.

COROLLARY 3.2. *Let (S_i, P_i) be a pair of pure Γ -isometries on $H_{\mathcal{E}_i}^2(\mathbb{D})$ where $i = 1, 2$. Then (S_1, P_1) and (S_2, P_2) are unitarily equivalent if and only if*

$$(S_1^* - S_1 P_1^*) \cong (S_2^* - S_2 P_2^*).$$

Proof. Let

$$(S, P) = (I_{H^2(\mathbb{D})} \otimes A + M_z \otimes A^*, M_z \otimes I_{\mathcal{E}}),$$

be a pure Γ -isometry on $H_{\mathcal{E}}^2(\mathbb{D})$. Then

$$\begin{aligned} S^* - SP^* &= (I_{H^2(\mathbb{D})} \otimes A + M_z \otimes A^*)^* - (I_{H^2(\mathbb{D})} \otimes A + M_z \otimes A^*)(M_z \otimes I_{\mathcal{E}})^* \\ &= I_{H^2(\mathbb{D})} \otimes A^* + M_z^* \otimes A - M_z^* \otimes A - ((I_{H^2(\mathbb{D})} - \mathbf{P}_{\mathbb{C}}) \otimes A^*) \\ &= \mathbf{P}_{\mathbb{C}} \otimes A^*, \end{aligned}$$

where $\mathbf{P}_{\mathbb{C}}$ is the orthogonal projection from $H^2(\mathbb{D})$ onto the space of constant functions in $H^2(\mathbb{D})$. Consequently, by the previous theorem $S_1^* - S_1 P_1^*$ and $S_2^* - S_2 P_2^*$ are unitarily equivalent if and only if (S_1, P_1) and (S_2, P_2) are unitarily equivalent. This completes the proof. \blacksquare

A closed subspace $\mathcal{S} \neq \{0\}$ of $H_{\mathcal{E}_*}^2(\mathbb{D})$ is said to be $(A + A^* M_z, M_z)$ -invariant if \mathcal{S} is invariant under both $A + A^* M_z$ and M_z .

Let $\mathcal{S} \neq \{0\}$ be a closed subspace of $H_{\mathcal{E}_*}^2(\mathbb{D})$. By virtue of the Beurling-Lax-Halmos theorem \mathcal{S} is M_z -invariant if and only if there exists a Hilbert space \mathcal{E} and an inner function $\Theta \in H_{\mathcal{B}(\mathcal{E}, \mathcal{E}_*)}^\infty(\mathbb{D})$ such that

$$\mathcal{S} = M_{\Theta} H_{\mathcal{E}}^2(\mathbb{D}).$$

Moreover, the pair $\{\mathcal{E}, \Theta\}$ is unique in an appropriate sense (cf. see [19]).

Let \mathcal{S} be a non-zero $(A + A^* M_z, M_z)$ -invariant subspace of $H_{\mathcal{E}_*}^2(\mathbb{D})$. Then in particular, by the Beurling-Lax-Halmos theorem

$$\mathcal{S} = M_{\Theta} H_{\mathcal{E}}^2(\mathbb{D}),$$

for some Hilbert space \mathcal{E} and inner multiplier Θ . The next theorem will show that \mathcal{S} is $(A + A^* M_z, M_z)$ -invariant if and only if M_{Θ} intertwine $A + A^* M_z$ and $B + B^* M_z$ for some unique $B \in \mathcal{B}(\mathcal{E})$ with $w(B) \leq 1$.

Before we proceed with the formal statement and the proof let us remark that the classification result answers a question left open by Agler and Young in [7]. The proof is remarkably simple (a straightforward application of Theorem 2.2) and may be of independent interest. However, the intuitive idea behind this ‘‘guess’’ is that, $(A + A^* M_z, M_z)$ turns $H_{\mathcal{E}_*}^2(\mathbb{D})$ into a natural Hilbert module over $\mathbb{C}[z_1, z_2]$ (see Section 7).

THEOREM 3.3. *Let $\mathcal{S} \neq \{0\}$ be a closed subspace of $H_{\mathcal{E}_*}^2(\mathbb{D})$ and $A \in \mathcal{B}(\mathcal{E}_*)$ with $w(A) \leq 1$. Then \mathcal{S} is a (M_{A+A^*z}, M_z) -invariant subspace if and only if*

$$(A + A^* M_z) M_{\Theta} = M_{\Theta} (B + B^* M_z),$$

for some operator $B \in \mathcal{B}(\mathcal{E})$ with $w(B) \leq 1$, where $\mathcal{S} = M_\Theta H_\mathcal{E}^2(\mathbb{D})$ is the Beurling-Lax-Halmos representation of \mathcal{S} . Moreover, when such an operator B exists it is unique (up to unitary equivalence).

Proof. Let $\mathcal{S} \neq \{0\}$ be a (M_{A+A^*z}, M_z) -invariant subspace and

$$\mathcal{S} = M_\Theta H_\mathcal{E}^2(\mathbb{D}),$$

be the Beurling-Lax-Halmos representation of \mathcal{S} where $\Theta \in H_{\mathcal{B}(\mathcal{E}, \mathcal{E}_*)}^\infty(\mathbb{D})$ is an inner multiplier and \mathcal{E} is an auxiliary Hilbert space. Also

$$(A + A^*M_z)(M_\Theta H_\mathcal{E}^2(\mathbb{D})) \subseteq M_\Theta H_\mathcal{E}^2(\mathbb{D}),$$

implies that

$$(A + A^*M_z)M_\Theta = M_\Theta M_\Psi,$$

for some unique $\Psi \in H_{\mathcal{B}(\mathcal{E})}^\infty(\mathbb{D})$. Therefore,

$$M_\Theta^*(A + A^*M_z)M_\Theta = M_\Psi.$$

Multiplying both sides by M_z^* we have

$$M_z^*M_\Theta^*(A + A^*M_z)M_\Theta = M_z^*M_\Psi.$$

Then

$$M_\Theta^*(AM_z^* + A^*)M_\Theta = M_z^*M_\Psi.$$

Consequently, $M_z^*M_\Psi = M_\Theta^*$, or equivalently, $M_\Psi = M_\Theta^*M_z$. Since $\|M_\Psi\| \leq 2$, that (M_Ψ, M_z) is a Γ -isometry, and hence by Theorem 2.2, it follows that

$$M_\Psi = B + B^*M_z,$$

for some $B \in \mathcal{B}(\mathcal{E})$ and $w(B) \leq 1$, and uniqueness of B follows from that of Ψ .

The converse part is trivial, and the proof is complete. \blacksquare

To complete this section we will present the following variant of our Beurling-Lax-Halmos theorem for Γ -isometries.

THEOREM 3.4. *Let $\Theta \in H_{\mathcal{B}(\mathcal{E}, \mathcal{E}_*)}^\infty(\mathbb{D})$ be an inner function and $A \in \mathcal{B}(\mathcal{E}_*)$. Then $\mathcal{S} = M_\Theta H_\mathcal{E}^2(\mathbb{D}) \subseteq H_{\mathcal{E}_*}^2(\mathbb{D})$ is invariant under the Toeplitz operator with analytic polynomial symbol $A + A^*z$ if and only if there exists an operator $B \in \mathcal{B}(\mathcal{E})$ such that*

$$(A + A^*z)\Theta(z) = \Theta(z)(B + B^*z). \quad (z \in \mathbb{D})$$

Moreover, when such an operator B exists it is unique (up to unitary equivalence).

We like to point out that the above result is an application of the theory of Γ -contractions to the classical one-variable operator theory. Moreover, our result suggests a tentative connection between the theory of spectral sets and invariant subspaces of Toeplitz operator with analytic polynomial symbol. We will discuss some of these extensions at the end of this paper.

4. REPRESENTATION OF Γ -CONTRACTIONS

In this section we will show that a c.n.u. Γ -contraction can be realized as a compression of a Γ -isometry in the Sz.-Nagy and Foias model space \mathcal{Q}_P of the c.n.u. contraction P . Moreover, we show that the representation of S in \mathcal{Q}_P is given by a direct sum of a Γ -isometry and a Γ -unitary. Our method involves a "pull-back" technique of the Agler-Young's isometric dilation to the Sz.-Nagy and Foias minimal isometric dilation.

First note that if (S, P) is a Γ -contraction, then (S^*, P^*) is also a Γ -contraction, which is equivalent to saying that (S^*, P^*) has a Γ -isometric dilation. More precisely, let (S, P) be a Γ -contraction on a Hilbert space \mathcal{H} . Then there exists Hilbert spaces \mathcal{K}_1 and \mathcal{K}_2 and a pure Γ -isometry $(\tilde{S}_i, \tilde{P}_i)$ on \mathcal{K}_1 and a Γ -unitary $(\tilde{S}_u, \tilde{P}_u)$ on \mathcal{K}_2 and an isometry (see Theorem 2.3 or Theorem 3.2 in [7])

$$\Pi_{AY} : \mathcal{H} \rightarrow \mathcal{K}_1 \oplus \mathcal{K}_2,$$

such that

$$\Pi_{AY} S^* = (\tilde{S}_i \oplus \tilde{S}_u)^* \Pi_{AY} \quad \text{and} \quad \Pi_{AY} P^* = (\tilde{P}_i \oplus \tilde{P}_u)^* \Pi_{AY}.$$

On the other hand, by Theorem 2.5, we have the Sz.-Nagy and Foias isometric dilation

$$\Pi_{NF} : \mathcal{H} \rightarrow \mathcal{H}_P,$$

of the c.n.u. contraction P on \mathcal{H} with

$$\Pi_{NF} P^* = (M_z \oplus M_{eit} |_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})^* \Pi_{NF}.$$

Moreover, this dilation is minimal and hence unique.

The following factorization theorem provides a connection between the minimal isometric dilation to any other isometric dilation of a given contraction.

THEOREM 4.1. (Factorization of Dilations) *Let P be a c.n.u. contraction on a Hilbert space \mathcal{H} . Let V on \mathcal{K} be an isometric dilation of P corresponding to an isometry $\Pi : \mathcal{H} \rightarrow \mathcal{K}$. Then there exists a unique isometry $\Phi \in \mathcal{B}(\mathcal{H}_P, \mathcal{K})$ such that*

$$\Pi = \Phi \Pi_{NF},$$

and

$$\Phi (M_z \oplus M_{eit} |_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})^* = V^* \Phi.$$

Moreover, let $\mathcal{K} = H_{\mathcal{E}}^2(\mathbb{D}) \oplus \mathcal{K}_u$ and $V = M_z \oplus U$ be the Wold decomposition of V for some unitary $U \in \mathcal{B}(\mathcal{K}_u)$. Then

$$\Phi = (I_{H^2(\mathbb{D})} \otimes V_1) \oplus V_2,$$

for some isometries $V_1 \in \mathcal{B}(\mathcal{D}_{P^*}, \mathcal{E})$ and $V_2 \in \mathcal{B}(\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}, \mathcal{K}_u)$.

Proof. Since $\Pi_{NF} : \mathcal{H} \rightarrow \mathcal{H}_P$ is the minimal isometric dilation of P we have

$$\mathcal{H}_P = \bigvee_{m=0}^{\infty} (M_z \oplus M_{eit} |_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})^m (\Pi_{NF} \mathcal{H}).$$

Furthermore, the V -reducing subspace

$$\mathcal{K}_m := \bigvee_{m=0}^{\infty} V^m(\Pi\mathcal{H}) \subseteq \mathcal{K},$$

is the minimal isometric dilation space of P and hence there exists an isometry

$$\Phi : \mathcal{H}_p \rightarrow \mathcal{K}_m \oplus \mathcal{K}_r,$$

defined by

$$(4.1) \quad \Phi(M_z \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})^m(\Pi_{NF}h) = V^m(\Pi h),$$

for all $h \in \mathcal{H}$ and $m \in \mathbb{N}$, where $\mathcal{K}_r = \mathcal{K} \ominus \mathcal{K}_m$. Since

$$\begin{aligned} & \Phi(M_z \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})^*(M_z \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})^m \Pi_{NF} \\ &= \Phi(M_z \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})^{m-1} \Pi_{NF} = V^{m-1} \Pi = V^*(V^m \Pi) \\ &= V^* \Phi(M_z \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})^m \Pi_{NF}, \end{aligned}$$

for all $m \geq 1$ and

$$\Phi(M_z \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})^* \Pi_{NF} = \Phi \Pi_{NF} P^* = \Pi P^* = V^* \Pi,$$

it follows that

$$\Phi(M_z \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})^* = V^* \Phi.$$

To prove the last part, let $\mathcal{K} = H_{\mathcal{E}}^2(\mathbb{D}) \oplus \mathcal{K}_u$ and $V = M_z \oplus U$ for some unitary $U \in \mathcal{B}(\mathcal{K}_u)$. Let

$$\Phi = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \mathcal{H}_P = H_{\mathcal{D}_P^*}^2(\mathbb{D}) \oplus \overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})} \rightarrow \mathcal{K} = H_{\mathcal{E}}^2(\mathbb{D}) \oplus \mathcal{K}_u.$$

Then by the intertwining property of Φ with the conjugates of the multiplication operators we have that

$$X_1 M_z^* = M_z^* X_1, \quad X_4 M_{e^{it}}^*|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}} = U^* X_4,$$

and

$$X_2 M_{e^{it}}^*|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}} = M_z^* X_2, \quad X_3 M_z^* = U^* X_3.$$

Since both X_2^* and X_3^* intertwine a unitary and a pure isometry, it follows that (cf. Lemma 2.5 in [7])

$$X_2 = 0 \quad \text{and} \quad X_3 = 0.$$

Therefore

$$\Phi = \begin{bmatrix} X_1 & 0 \\ 0 & V_2 \end{bmatrix},$$

where $X_4 = V_2$. Finally, since

$$\text{ran} \Phi = \text{ran} X_1 \oplus \text{ran} V_2 \subseteq H_{\mathcal{E}}^2(\mathbb{D}) \oplus \mathcal{K}_u,$$

is a $(M_z \oplus U)$ -reducing subspace of \mathcal{K} and $\text{ran} X_1 \subseteq H_{\mathcal{E}}^2(\mathbb{D})$, it follows that $\text{ran} X_1$ is a M_z -reducing subspace of $H_{\mathcal{E}}^2(\mathbb{D})$. Consequently,

$$X_1 = I_{H^2(\mathbb{D})} \otimes V_1,$$

for some isometry $V_1 \in \mathcal{B}(\mathcal{D}_{P^*}, \mathcal{E})$.

Uniqueness of Φ follows from the equality (4.1). This completes the proof. \blacksquare

The above factorization result can be summarized in the following commutative diagram:

$$\begin{array}{ccc} & & \mathcal{H}_P \\ & \nearrow \Pi_{NF} & \downarrow \Phi \\ \mathcal{H} & \xrightarrow{\quad \Pi \quad} & \mathcal{K} \end{array}$$

where Φ is a unique isometry which intertwines the adjoints of the multiplication operators. Related results along this line can be found in the context of the commutant lifting theorem of contractions (cf. page 134 in [15] and page 133 in [16]). However, the minimal isometric dilation space in this consideration is the Schaffer's dilation space.

Before we proceed further let us recall the *Putnam's Corollary* [21]: If A and B are normal operators on Hilbert spaces \mathcal{H} and \mathcal{K} respectively and if C is a bounded linear operator from \mathcal{H} to \mathcal{K} such that $CA = BC$, then $CA^* = B^*C$.

The main result of this section is the following theorem concerning an analytic model of a c.n.u. Γ -contraction.

THEOREM 4.2. *Let (S, P) be a c.n.u. Γ -contraction on a Hilbert space \mathcal{H} . Then*

$$(NF\text{-AY}) \quad S \cong \mathbf{P}_{\mathcal{Q}_P}((A + A^*M_z) \oplus U)|_{\mathcal{Q}_P},$$

where $A \in \mathcal{B}(\mathcal{D}_{P^*})$ with $w(A) \leq 1$ and U is in $\mathcal{B}(\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})})$ such that

$$(U, M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}}),$$

is a Γ -unitary.

Proof. Let

$$\Pi_{NF} : \mathcal{H} \rightarrow \mathcal{H}_P = H_{\mathcal{D}_{P^*}}^2(\mathbb{D}) \oplus \overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})},$$

be the Sz.-Nagy and Foias minimal isometric dilation of the c.n.u. contraction P as in Theorem 2.5 and

$$\Pi_{AY} : \mathcal{H} \rightarrow \mathcal{K}_1 \oplus \mathcal{K}_2,$$

be the Agler-Young's Γ -isometric dilation of the c.n.u. Γ -contraction (S, P) as in Theorem 2.3, where $\mathcal{K}_1 = H_{\mathcal{E}}^2(\mathbb{D})$ is the pure part and \mathcal{K}_2 is the unitary part. By Theorem 4.1, we have the following commutative diagram

$$\begin{array}{ccc} & & \mathcal{H}_P \\ & \nearrow \Pi_{NF} & \downarrow \tilde{V} \\ \mathcal{H} & \xrightarrow{\quad} & \mathcal{K}_1 \oplus \mathcal{K}_2 \\ & \Pi_{AY} & \end{array}$$

where \tilde{V} is an isometry of the form

$$\tilde{V} = (I_{H^2(\mathbb{D})} \otimes \hat{V}_1) \oplus \hat{V}_2,$$

for some isometries $\hat{V}_1 \in \mathcal{B}(\mathcal{D}_{P^*}, \mathcal{E})$ and $\hat{V}_2 \in \mathcal{B}(\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}, \mathcal{K}_2)$. Moreover,

$$(4.2) \quad \hat{V}_2 M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}} = \tilde{P}_u^* \hat{V}_2.$$

Then

$$\Pi_{AY} = ((I_{H^2(\mathbb{D})} \otimes \hat{V}_1) \oplus \hat{V}_2) \Pi_{NF}.$$

Since

$$\Pi_{AY} P^* = (\tilde{P}_i \oplus \tilde{P}_u)^* \Pi_{AY} = ((M_z \otimes I_{\mathcal{E}}) \oplus \tilde{P}_u)^* \Pi_{AY},$$

and

$$\Pi_{AY} S^* = (\tilde{S}_i \oplus \tilde{S}_u)^* \Pi_{AY} = ((I_{H^2(\mathbb{D})} \otimes A + M_z \otimes A^*) \oplus \tilde{S}_u)^* \Pi_{AY},$$

we have

$$(4.3) \quad P^* = \Pi_{NF}^* ((M_z \otimes I_{\mathcal{E}}) \oplus \hat{V}_2^* \tilde{P}_u \hat{V}_2)^* \Pi_{NF},$$

and

$$(4.4) \quad S^* = \Pi_{NF}^* ((I_{\mathcal{E}} \otimes \hat{V}_1^* A \hat{V}_1 + M_z \otimes \hat{V}_1^* A^* \hat{V}_1) \oplus \hat{V}_2^* \tilde{S}_u \hat{V}_2)^* \Pi_{NF}.$$

By (4.2) and Putnam's Corollary, we have

$$\hat{V}_2 M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}} = \tilde{P}_u \hat{V}_2.$$

In particular, $\text{ran} \hat{V}_2$ is a \tilde{P}_u -reducing subspace, and

$$\hat{V}_2^* \tilde{P}_u \hat{V}_2 = M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}} \in \mathcal{B}(\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}).$$

Consequently,

$$\begin{aligned} \Pi_{NF} P^* \Pi_{NF}^* &= \mathbf{P}_{\mathcal{Q}_P} ((M_z \otimes I_{\mathcal{D}_{P^*}}) \oplus \hat{V}_2^* \tilde{P}_u \hat{V}_2)^*|_{\mathcal{Q}_P} \\ &= \mathbf{P}_{\mathcal{Q}_P} ((M_z \otimes I_{\mathcal{D}_{P^*}}) \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})^*|_{\mathcal{Q}_P}, \end{aligned}$$

and

$$\Pi_{NF} S^* \Pi_{NF}^* = \mathbf{P}_{\mathcal{Q}_P} ((I_{\mathcal{D}_{P^*}} \otimes \hat{V}_1^* A \hat{V}_1 + M_z \otimes \hat{V}_1^* A^* \hat{V}_1) \oplus \hat{V}_2^* \tilde{S}_u \hat{V}_2)^* |_{\mathcal{Q}_P}.$$

Therefore,

$$S^* \cong \mathbf{P}_{\mathcal{Q}_P} ((\tilde{A} + \tilde{A}^* M_z) \oplus \tilde{U}) |_{\mathcal{Q}_P},$$

where $\tilde{A} = \hat{V}_1^* A \hat{V}_1$ and $\tilde{U} = \hat{V}_2^* \tilde{S}_u \hat{V}_2$. Since $w(A) \leq 1$ we have that $w(\tilde{A}) \leq 1$. It remains to prove that

$$(\tilde{U}, \hat{V}_2^* \tilde{P}_u \hat{V}_2) = (\tilde{U}, M_{e^{it}} |_{\frac{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})$$

is a Γ -unitary. Since $(\tilde{S}_u, \tilde{P}_u)$ is a Γ -unitary, we conclude that

$$\tilde{S}_u = \tilde{S}_u^* \tilde{P}_u.$$

Using the fact that the range of \hat{V}_2 is \tilde{P}_u -reducing, it follows that

$$\tilde{S}_u \hat{V}_2 = \tilde{S}_u^* \tilde{P}_u \hat{V}_2 = (\tilde{S}_u^* \hat{V}_2) (\hat{V}_2^* \tilde{P}_u \hat{V}_2).$$

Hence

$$\hat{V}_2^* \tilde{S}_u \hat{V}_2 = (\hat{V}_2^* \tilde{S}_u \hat{V}_2)^* (\hat{V}_2^* \tilde{P}_u \hat{V}_2),$$

which implies that

$$(\hat{V}_2^* \tilde{S}_u \hat{V}_2, \hat{V}_2^* \tilde{P}_u \hat{V}_2) = (\tilde{U}, \hat{V}_2^* \tilde{P}_u \hat{V}_2) = (\tilde{U}, M_{e^{it}} |_{\frac{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})$$

is a Γ -unitary. This finishes the proof of the theorem. \blacksquare

The following result shows that the representation of $S - PS^*$ in the Sz.-Nagy and Foias quotient space \mathcal{Q}_P is the compression of the scalar operator A on \mathcal{Q}_P .

COROLLARY 4.3. *With notations as in Theorem 4.2, representation of the operator $S^* - SP^*$ in \mathcal{Q}_P is given by*

$$S^* - SP^* \cong \mathbf{P}_{\mathcal{Q}_P} (\mathbf{P}_{\mathbb{C}} \otimes A^*) |_{\mathcal{Q}_P}.$$

Proof. Since

$$\begin{aligned} & [\mathbf{P}_{\mathcal{Q}_P} ((A + A^* M_z) \oplus U) |_{\mathcal{Q}_P}]^* - [\mathbf{P}_{\mathcal{Q}_P} ((A + A^* M_z) \oplus U) |_{\mathcal{Q}_P}] \mathbf{P}_{\mathcal{Q}_P} ([M_z \oplus M_{e^{it}}] |_{\mathcal{Q}_P})^* \\ &= \mathbf{P}_{\mathcal{Q}_P} [((A^* + A M_z^*) \oplus U^*) - ((A + A^* M_z) \oplus U)(M_z^* \oplus M_{e^{it}}^*)] |_{\mathcal{Q}_P} \\ &= \mathbf{P}_{\mathcal{Q}_P} (\mathbf{P}_{\mathbb{C}} \otimes A^*) |_{\mathcal{Q}_P}, \end{aligned}$$

we have

$$S^* - SP^* \cong \mathbf{P}_{\mathcal{Q}_P} (\mathbf{P}_{\mathbb{C}} \otimes A^*) |_{\mathcal{Q}_P}.$$

This completes the proof. \blacksquare

5. UNIQUE REPRESENTATION OF Γ -CONTRACTIONS

In this section we will discuss the uniqueness of the Γ -isometry and the Γ -unitary part in the representation (NF-AY) of a c.n.u. Γ -contraction.

We begin by recalling one way to construct the minimal isometric dilation of a c.n.u. contraction. More details can be found, for instance, in the monograph by Foias and Frazho (Page 137 in [15]). Let $P \in \mathcal{B}(\mathcal{H})$ be a c.n.u. contraction. Then

$$\tilde{X}_P := SOT - \lim_{m \rightarrow \infty} P^m P^{*m},$$

is a positive operator on \mathcal{H} . Let X_P be the positive square root of \tilde{X}_P . Then

$$\|X_P h\|^2 = \lim_{m \rightarrow \infty} \|P^{*m} h\|^2,$$

and

$$\|X_P h\| = \|X_P P^* h\|,$$

for all $h \in \mathcal{H}$. Consequently, there exists an isometry $V_1 \in \mathcal{B}(\overline{X_P \mathcal{H}})$ such that

$$V_1 X_P = X_P P^*.$$

Let V_2 on \mathcal{K}_u be the minimal unitary extension of V_1 so that

$$V_2 X_P = X_P P^*.$$

Define $\Pi : \mathcal{H} \rightarrow H_{\mathcal{D}_{P^*}}^2(\mathbb{D}) \oplus \mathcal{K}_u$ by

$$\Pi h = D_{P^*}(I - zP^*)^{-1}h \oplus X_P h,$$

for all $h \in \mathcal{H}$. Then Π is an isometry and

$$\Pi P^* = (M_z^* \oplus U^*)\Pi,$$

where $U = V_2^*$. Moreover, Π is minimal and

$$\Pi^*((\mathbb{S}_w \otimes \eta) \oplus 0) = (I - \bar{w}P)^{-1}D_{P^*}\eta,$$

for all $\eta \in \mathcal{D}_{P^*}$, where \mathbb{S} is the Szegő kernel on the open unit disk defined by

$$\mathbb{S}_w(z) = (1 - z\bar{w})^{-1},$$

for all $z, w \in \mathbb{D}$.

In the proof of the following theorem, we shall identify (by virtue of Theorem 4.1 where Φ is a unitary) the minimal isometric dilation Π_{NF} with the one described above.

THEOREM 5.1. *Let P be a c.n.u. contraction on a Hilbert space \mathcal{H} and $A \in \mathcal{B}(\mathcal{D}_{P^*})$. Then*

$$D_{P^*} A D_{P^*} \cong \mathbf{P}_{\mathcal{Q}_P}(\mathbf{P}_{\mathbb{C}} \otimes A)|_{\mathcal{Q}_P}.$$

Proof. Let $\mathbf{ev}_0 : (H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}) \oplus \overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})} \rightarrow \mathcal{D}_{P^*}$ be the evaluation operator defined by

$$\mathbf{ev}_0(f \oplus g) = f(0),$$

for all $f \oplus g \in (H^2(\mathbb{D}) \otimes \mathcal{D}_{P^*}) \oplus \overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}$. Then

$$\mathbf{ev}_0 \Pi_{NF} h = (D_{P^*}(I - zP^*)^{-1}h)(0) = D_{P^*} h,$$

for all $h \in \mathcal{H}$. From this we readily obtain

$$\mathbf{e}v_0\Pi_{NF} = D_{P^*}.$$

Moreover,

$$\Pi_{NF}^*(\mathbf{P}_{\mathbb{C}} \otimes A)\eta = \Pi_{NF}^*(1 \otimes A\eta) = \Pi_{NF}^*((\mathbb{S}_0 \otimes A\eta) \oplus 0) = D_{P^*}A\eta,$$

for all $\eta \in \mathcal{D}_{P^*}$. Thus,

$$\Pi_{NF}^*(\mathbf{P}_{\mathbb{C}} \otimes A) = D_{P^*}A\mathbf{e}v_0.$$

Consequently,

$$\Pi_{NF}^*(\mathbf{P}_{\mathbb{C}} \otimes A)\Pi_{NF} = D_{P^*}A\mathbf{e}v_0\Pi_{NF} = D_{P^*}AD_{P^*}.$$

Then the result follows from the fact that

$$\Pi_{NF}\Pi_{NF}^*(\mathbf{P}_{\mathbb{C}} \otimes A)\Pi_{NF}\Pi_{NF}^* = \Pi_{NF}(D_{P^*}AD_{P^*})\Pi_{NF}^*.$$

■

The following corollary is immediate.

COROLLARY 5.2. *Let P be a c.n.u. contraction and $A \in \mathcal{B}(\mathcal{D}_{P^*})$. Then $A = 0$ if and only if*

$$\mathbf{P}_{\mathcal{Q}_P}(\mathbf{P}_{\mathbb{C}} \otimes A)|_{\mathcal{Q}_P} = 0.$$

Now we are ready to prove the uniqueness of the Γ -isometric part in (NF-AY).

THEOREM 5.3. *Let (S, P) be a c.n.u. Γ -contraction on a Hilbert space \mathcal{H} . Then the operator $A \in \mathcal{B}(\mathcal{D}_{P^*})$ in the representation (NF-AY) is unique. That is, if*

$$S \cong \mathbf{P}_{\mathcal{Q}_P}((\tilde{A} + \tilde{A}^*M_z) \oplus \tilde{U})|_{\mathcal{Q}_P},$$

where $(\tilde{U}, M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})$ is a Γ -unitary and $\tilde{A} \in \mathcal{B}(\mathcal{D}_{P^*})$, $w(\tilde{A}) \leq 1$ and $\tilde{U} \in \mathcal{B}(\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})})$, then $A = \tilde{A}$.

Proof. Let

$$\mathbf{P}_{\mathcal{Q}_P}((A + A^*M_z) \oplus U)|_{\mathcal{Q}_P} = \mathbf{P}_{\mathcal{Q}_P}((\tilde{A} + \tilde{A}^*M_z) \oplus \tilde{U})|_{\mathcal{Q}_P}$$

where $(\tilde{U}, M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})$ is a Γ -unitary and $\tilde{A} \in \mathcal{B}(\mathcal{D}_{P^*})$, $w(\tilde{A}) \leq 1$ and $\tilde{U} \in \mathcal{B}(\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})})$.

By Corollary 4.3, we have

$$\mathbf{P}_{\mathcal{Q}_P}(\mathbf{P}_{\mathbb{C}} \otimes A)|_{\mathcal{Q}_P} = \mathbf{P}_{\mathcal{Q}_P}(\mathbf{P}_{\mathbb{C}} \otimes \tilde{A})|_{\mathcal{Q}_P}.$$

This and Corollary 5.2 implies that

$$A = \tilde{A}.$$

This completes the proof. ■

The following result plays an important role in the proof of the uniqueness of \tilde{U} in (NF-AY).

PROPOSITION 5.4. *Let X be a bounded linear operator on $\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}$ where $XM_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}} = M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}}X$ and $X|_{\mathcal{Q}_P} = 0$. Then $X = 0$.*

Proof. Let $X|_{\mathcal{Q}_P} = 0$. Then

$$\text{ran}X^* = \text{ran}(0 \oplus X^*) \subseteq \mathcal{Q}_P^\perp = \overline{\text{span}}\{\Theta_P f \oplus \Delta_P f : f \in H_{\mathcal{D}_P}^2(\mathbb{D})\}.$$

If $\Delta_P f \in \text{ran}X^*$ for some $f \in H_{\mathcal{D}_P}^2(\mathbb{D})$ then $\Theta_P f = 0$, or equivalently, $\Theta_P^* \Theta_P f = 0$. Therefore,

$$\Delta_P^2 f = f,$$

and hence

$$\Delta_P(\text{ran}X^*) \subseteq H_{\mathcal{D}_P}^2(\mathbb{D}).$$

Also by

$$X M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}} = M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}} X,$$

we conclude that $\overline{\Delta_P(\text{ran}X^*)} \subseteq H_{\mathcal{D}_P}^2(\mathbb{D})$ is a $M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}}$ -reducing subspace. Consequently,

$$\Delta_P(\text{ran}X^*) = \{0\}.$$

Since $X \in \mathcal{B}(\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})})$, we have $X = 0$, which completes the proof. \blacksquare

From the previous proposition we readily obtain the desired uniqueness of \tilde{U} in (NF-AY).

COROLLARY 5.5. *Let (S, P) be a c.n.u. Γ -contraction on a Hilbert space \mathcal{H} . Then the operator $U \in \mathcal{B}(\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})})$ in the representation of S in (NF-AY) is unique.*

Proof. Let

$$\mathbf{P}_{\mathcal{Q}_P}((A + A^* M_z) \oplus U)|_{\mathcal{Q}_P} = \mathbf{P}_{\mathcal{Q}_P}((A + A^* M_z) \oplus \tilde{U})|_{\mathcal{Q}_P},$$

where $(\tilde{U}, M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})$ is a Γ -unitary and $\tilde{A} \in \mathcal{B}(\mathcal{D}_{P^*})$, $w(\tilde{A}) \leq 1$ and $\tilde{U} \in \mathcal{B}(\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})})$.

Then

$$\mathbf{P}_{\mathcal{Q}_P}[0 \oplus (U - \tilde{U})]|_{\mathcal{Q}_P} = 0.$$

By Proposition 5.4 with $X = (U - \tilde{U})^*$, we have

$$U = \tilde{U}.$$

This completes the proof. \blacksquare

Combining the above corollary with Theorems 4.2 and 5.3, we obtain the unique representation of a c.n.u. Γ -contraction (S, P) in the model space \mathcal{Q}_P .

THEOREM 5.6. *Let (S, P) be a c.n.u. Γ -contraction on a Hilbert space \mathcal{H} . Then the representing operators A and U in (NF-AY) are unique. That is, if*

$$S \cong \mathbf{P}_{\mathcal{Q}_P}((\tilde{A} + \tilde{A}^* M_z) \oplus \tilde{U})|_{\mathcal{Q}_P},$$

where $(\tilde{U}, M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})$ is a Γ -unitary and $\tilde{A} \in \mathcal{B}(\mathcal{D}_{P^*})$, $w(\tilde{A}) \leq 1$ and $\tilde{U} \in \mathcal{B}(\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})})$.

Then $A \cong \tilde{A}$ and $U \cong \tilde{U}$. Moreover,

$$((A + A^* M_z) \oplus U, M_z \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}}),$$

is the minimal isometric dilation of the Γ -contraction (S, P) .

Therefore, a c.n.u. Γ -contraction (S, P) on \mathcal{H} is uniquely determined by Θ_P (and hence by \mathcal{Q}_P) and the *representing operators* A and U .

6. A CHARACTERIZATION OF Γ -CONTRACTIONS

Let $(s, p) \in \mathbb{C}^2$. Then (s, p) is in the symmetrized bidisc Γ if and only if $|p| \leq 1$ and that

$$s = \beta + p\bar{\beta},$$

for some $\beta \in \mathbb{C}$ such that $|\beta| \leq 1$ (see [3]). In this section we generalize the scalar characterization of elements in Γ to the class of Γ -contractions on Hilbert spaces.

We begin by recalling the Schaffer isometric dilation of a contraction P on \mathcal{H} . In this case, the dilation space is defined by $\mathcal{K}_P := \mathcal{H} \oplus H_{\mathcal{D}_P}^2(\mathbb{D})$. Let

$$V_P = \begin{bmatrix} P & 0 \\ \mathbf{D}_P & M_z \end{bmatrix},$$

where $\mathbf{D}_P : \mathcal{H} \rightarrow H_{\mathcal{D}_P}^2(\mathbb{D})$ is the constant function defined by

$$(\mathbf{D}_P h)(z) = D_P h,$$

for all $h \in \mathcal{H}$ and $z \in \mathbb{D}$. That is,

$$V_P(h \oplus f) = Ph \oplus (D_P h + M_z f),$$

for all $h \oplus f \in \mathcal{K}_P$. Then V_P is an isometry and the map

$$\Pi_{S_c} : \mathcal{H} \rightarrow \mathcal{K}_P = \mathcal{H} \oplus H_{\mathcal{D}_P}^2(\mathbb{D}),$$

defined by

$$\Pi_{S_c} h = h \oplus 0,$$

for all $h \in \mathcal{H}$, satisfies

$$\Pi_{S_c} P^* = V_P^* \Pi_{S_c}.$$

The isometric dilation Π_{S_c} is known as the *Schaffer dilation* of the contraction P .

The following result summarizes Theorems 4.2, 4.3 and 4.4 in [11]. For completeness and the reader's convenience, we supply a proof. Moreover, our view is slightly different and the proof is considerably short and simple.

THEOREM 6.1. *Let (S, P) be a Γ -contraction. Then the Schaffer dilation of P satisfies*

$$\Pi_{S_c} S^* = W_A^* \Pi_{S_c},$$

for some Γ -isometry (W_A, V_P) which is uniquely determined by the operator $A \in \mathcal{B}(\mathcal{D}_P)$ such that $S - S^*P = D_P A D_P$ and $w(A) \leq 1$. Conversely, let (S, P) be a commuting tuple where P is a contraction and $\|S\| \leq 2$ and $S - S^*P = D_P A D_P$ for some $A \in \mathcal{B}(\mathcal{D}_P)$ with $w(A) \leq 1$. Then (S, P) is a Γ -contraction.

Proof. Let (S, P) be a Γ -contraction. First, we assume that (S, P) is c.n.u. Therefore P is a c.n.u. contraction and by the factorization of dilations, Theorem 4.1, we have an isometry $\Phi : \mathcal{H}_P \rightarrow \mathcal{K}_P$ such that

$$\Pi_{S_C} = \Phi \Pi_{NF}.$$

The Schaffer dilation Π_{S_C} is minimal means that Φ is unitary. Then

$$(W, V) := (\Phi((A + A^*M_z) \oplus U)\Phi^*, \Phi(M_z \oplus M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}})\Phi^*),$$

is a Γ -isometry on \mathcal{K}_P and

$$\Pi_{S_C} P^* = V^* \Pi_{S_C}, \quad \text{and} \quad \Pi_{S_C} S^* = W^* \Pi_{S_C}.$$

Finally, by taking the orthogonal direct sum of the unitary part with the c.n.u. part, it follows that Π_{S_C} is the minimal isometric dilation of the Γ -contraction (S, P) . By the uniqueness of the Schaffer dilation of contractions we therefore identify that V with V_P .

Next we will show that

$$(6.1) \quad W = \begin{bmatrix} S & 0 \\ \mathbf{A}^* \mathbf{D}_P & A + A^* M_z \end{bmatrix},$$

for some $A \in \mathcal{B}(\mathcal{D}_{P^*})$ with $w(A) \leq 1$. To see this, assume

$$W = \begin{bmatrix} S & 0 \\ W_3 & W_4 \end{bmatrix},$$

and compute

$$W^* V_P = \begin{bmatrix} S^* P + W_3^* D_P & W_3^* M_z \\ W_4^* \mathbf{D}_P & W_4^* M_z \end{bmatrix}.$$

Since (W, V_P) is a Γ -isometry, we have $W^* V_P = W$ and so

$$(6.2) \quad \begin{bmatrix} S^* P + W_3^* D_P & W_3^* M_z \\ W_4^* \mathbf{D}_P & W_4^* M_z \end{bmatrix} = \begin{bmatrix} S & 0 \\ W_3 & W_4 \end{bmatrix}.$$

By $W_4^* M_z = W_4$ and that $\|W_4\| \leq 2$ we have

$$W_4 = A + A^* M_z,$$

for some $A \in \mathcal{B}(\mathcal{D}_P)$ and $w(A) \leq 1$. Also

$$W_3 = W_4^* \mathbf{D}_P = (A + A^* M_z)^* \mathbf{D}_P = \mathbf{A}^* \mathbf{D}_P,$$

which yields the desired representation of W . In the above equality we used the fact that

$$(A M_z^* \mathbf{D}_P h)(z) = M_z^* D_P h = 0,$$

and

$$(A^* \mathbf{D}_P h)(z) = A^* D_P h,$$

for all $h \in \mathcal{H}$ and $z \in \mathbb{D}$.

Now we will show that A is uniquely determined by (S, P) . For that, equating the $(1, 1)$ -th entries in (6.2) we have

$$S^* P + W_3^* D_P = S.$$

Hence

$$S - S^*P = W_3^*D_{P^*} = D_P A D_P,$$

and that A is uniquely determined by (S, P) .

Therefore, if (S, P) is a Γ -contraction on \mathcal{H} then there exists a unique $A \in \mathcal{B}(\mathcal{D}_{P^*})$ with $w(A) \leq 1$ and $S - S^*P = D_P A D_P$ such that $\Pi_{S_c} : \mathcal{H} \rightarrow \mathcal{K}$ satisfies

$$\Pi_{S_c} S^* = W_A^* \Pi_{S_c},$$

where W_A is the operator matrix in (6.1).

On the other hand, given a commuting tuple (S, P) on \mathcal{H} , where P is a contraction and $\|S\| \leq 2$ and $S - S^*P = D_P A D_P$ for some $B \in \mathcal{B}(\mathcal{D}_{P^*})$ (and hence, unique) the Schaffer dilation of P satisfies

$$\Pi_{S_c} S^* = W_A^* \Pi_{S_c},$$

where W_A is the operator matrix in (6.1). Moreover, by the given conditions, it is easy to check that $W_A^* V_P = W_A$. Since $\|W_A\| = r(W_A) \leq 2$, (see page 598 in [11]) we obtain that (W_A, V_P) is a Γ -isometry, that is, Π_{S_c} is a Γ -isometric dilation of (S, P) . Here we are using the fact that if W on \mathcal{H} commutes with an isometry V and $W^*V = W$ then W is a hyponormal operator and that $r(W) = \|W\|$ (see Theorem 1 in [23]). Consequently, (S, P) is a Γ -contraction. This completes the proof. \blacksquare

Let us remark that the equality $S - S^*P = D_P A D_P$ of a Γ -contraction (S, P) also follows by applying Corollary 4.3 and Theorem 5.1 and the Wold decomposition theorem of Γ -contractions to the Γ -contraction (S^*, P^*) .

Let (S, P) be a Γ -unitary. Since the only way to obtain a Γ -unitary is to symmetrize a pair of commuting unitary operators, say U and U_1 , we let $S = U + U_1$ and $P = U U_1$. Then $U_1 = U^*P$ and hence $S = U + U^*P$. Therefore, a Γ -unitary can be represented by $(U_1 + U_1^*U, U)$ for some commuting unitary operators U_1 and U (see Theorem 2.5 in [11]).

Let \mathcal{E} be a Hilbert space and U be a unitary operator on some Hilbert space \mathcal{K} and \mathcal{Q} be a $(M_z \oplus U)^*$ -invariant subspace of $H_{\mathcal{E}}^2(\mathbb{D}) \oplus \mathcal{K}$. A bounded linear operator X on \mathcal{Q} is said to be *compressed scalar* if

$$X = P_{\mathcal{Q}}(A \oplus U_1)|_{\mathcal{Q}},$$

for some $A \in \mathcal{B}(\mathcal{E})$ and a unitary $U_1 \in \mathcal{B}(\mathcal{K})$ such that $U U_1 = U_1 U$. Given a compressed scalar operator $X = P_{\mathcal{Q}}(A \oplus U_1)|_{\mathcal{Q}}$ on \mathcal{Q} , the *decompressed numerical radius* of X , denoted by $w_{\mathcal{Q}}(X)$, is defined by $w(A)$, the numerical radius of the symbol operator A .

The realization of Γ -unitary along with the Γ -isometric dilation theorem, Theorem 5.6, and Theorem 6.1 enables us to prove the following characterization of Γ -contractions. Here, and in what follows, if P on \mathcal{H} is a c.n.u. contraction then we will assume that $\mathcal{H} = \mathcal{Q}_P$, the Nagy-Foias quotient space.

THEOREM 6.2. *Let P be a c.n.u. contraction and $S \in \mathcal{B}(\mathcal{Q}_P)$ commutes with P and $\|S\| \leq 2$. Then (S, P) is a Γ -contraction on \mathcal{Q}_P if and only if*

$$S = X + P X^*,$$

for some compressed scalar operator $X \in \mathcal{B}(\mathcal{Q}_P)$ with $w_{\mathcal{Q}_P}(X) \leq 1$.

Proof. Let (S, P) be a c.n.u. Γ -contraction and let

$$((A + A^*M_z) \oplus (U_1 + U_1^*U), M_z \oplus U)$$

on \mathcal{H}_P be the Γ -isometric dilation of (S, P) (see Theorem 5.6), where $U = M_{e^{it}}|_{\overline{\Delta_P L_{\mathcal{D}_P}^2(\mathbb{T})}}$ and $w(A) \leq 1$. Now

$$\begin{aligned} (A + A^*M_z) \oplus (U_1 + U_1^*U) &= (A \oplus U_1) + (A^*M_z \oplus U_1^*U) \\ &= (A \oplus U_1) + (M_z \oplus U)(A \oplus U_1)^*. \end{aligned}$$

Identifying (S, P) with

$$(\mathbf{P}_{\mathcal{Q}_P}((A + A^*M_z) \oplus (U_1 + U_1^*U))|_{\mathcal{Q}_P}, \mathbf{P}_{\mathcal{Q}_P}(M_z \oplus U)|_{\mathcal{Q}_P}),$$

on the quotient space \mathcal{Q}_P , we have

$$\begin{aligned} S^* &= ((A + A^*M_z) \oplus (U_1 + U_1^*U))^*|_{\mathcal{Q}_P} \\ &= ((A \oplus U_1)^* + (A \oplus U_1)(M_z \oplus U)^*)|_{\mathcal{Q}_P} \\ &= \mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1)^*|_{\mathcal{Q}_P} + \mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1)\mathbf{P}_{\mathcal{Q}_P}(M_z \oplus U)^*|_{\mathcal{Q}_P} \\ &= \mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1)^*|_{\mathcal{Q}_P} + \mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1)|_{\mathcal{Q}_P}\mathbf{P}_{\mathcal{Q}_P}(M_z \oplus U)^*|_{\mathcal{Q}_P} \\ &= X^* + XP^*, \end{aligned}$$

that is,

$$S = X + PX^*,$$

where

$$X := \mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1)|_{\mathcal{Q}_P} \in \mathcal{B}(\mathcal{Q}_P),$$

is the compressed scalar operator. Finally, since $w(A) \leq 1$, the decompressed numerical radius of X is not greater than one.

For the converse part, let

$$S = X + PX^*,$$

for some compressed scalar operator $X = \mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1)|_{\mathcal{Q}_P} \in \mathcal{B}(\mathcal{Q}_P)$ with $w(A) \leq 1$. Then we calculate

$$\begin{aligned} S^* - SP^* &= (X + PX^*)^* - (X + PX^*)P^* \\ &= X^* + XP^* - (XP^* + PX^*P^*) \\ &= X^* - PX^*P^*. \end{aligned}$$

On the other hand, since

$$\begin{aligned} A \oplus U_1 &= (I_{H^2(\mathbb{D})} \otimes A) \oplus U_1 \\ &= ((\mathbf{P}_{\mathbb{C}} + M_z M_z^*) \otimes A) \oplus U_1 \\ &= ((\mathbf{P}_{\mathbb{C}} \otimes A) \oplus 0) + ((M_z \otimes I_{\mathcal{D}_{P^*}}) \oplus U)((I_{H^2(\mathbb{D})} \otimes A) \oplus U_1)((M_z \otimes I_{\mathcal{D}_{P^*}}) \oplus U)^*, \end{aligned}$$

and

$$\begin{aligned}
 & \mathbf{P}_{\mathcal{Q}_P} [((M_z \otimes I_{\mathcal{D}_{P^*}}) \oplus U)((I_{H^2(\mathbb{D})} \otimes A) \oplus U_1)((M_z \otimes I_{\mathcal{D}_{P^*}}) \oplus U)^*] |_{\mathcal{Q}_P} \\
 &= \mathbf{P}_{\mathcal{Q}_P} ((M_z \otimes I_{\mathcal{D}_{P^*}}) \oplus U)(\mathbf{P}_{\mathcal{Q}_P^\perp} + \mathbf{P}_{\mathcal{Q}_P})((I_{H^2(\mathbb{D})} \otimes A) \oplus U_1)\mathbf{P}_{\mathcal{Q}_P}((M_z \otimes I_{\mathcal{D}_{P^*}}) \oplus U)^* |_{\mathcal{Q}_P} \\
 &= \mathbf{P}_{\mathcal{Q}_P} ((M_z \otimes I_{\mathcal{D}_{P^*}}) \oplus U)\mathbf{P}_{\mathcal{Q}_P}((I_{H^2(\mathbb{D})} \otimes A) \oplus U_1)\mathbf{P}_{\mathcal{Q}_P}((M_z \otimes I_{\mathcal{D}_{P^*}}) \oplus U)^* |_{\mathcal{Q}_P} \\
 &= [\mathbf{P}_{\mathcal{Q}_P}((M_z \otimes I_{\mathcal{D}_{P^*}}) \oplus U)] |_{\mathcal{Q}_P} [\mathbf{P}_{\mathcal{Q}_P}((I_{H^2(\mathbb{D})} \otimes A) \oplus U_1) |_{\mathcal{Q}_P}] [\mathbf{P}_{\mathcal{Q}_P}((M_z \otimes I_{\mathcal{D}_{P^*}}) \oplus U)^*] |_{\mathcal{Q}_P} \\
 &= P[\mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1) |_{\mathcal{Q}_P}] P^*,
 \end{aligned}$$

we have

$$\begin{aligned}
 \mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1) |_{\mathcal{Q}_P} &= \mathbf{P}_{\mathcal{Q}_P}((I_{H^2(\mathbb{D})} \otimes A) \oplus U_1) |_{\mathcal{Q}_P} \\
 &= \mathbf{P}_{\mathcal{Q}_P}(\mathbf{P}_{\mathbb{C}} \otimes A) |_{\mathcal{Q}_P} + P[\mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1)\mathbf{P}_{\mathcal{Q}_P}] P^*.
 \end{aligned}$$

The last equality shows that

$$X = \mathbf{P}_{\mathcal{Q}_P}(\mathbf{P}_{\mathbb{C}} \otimes A) |_{\mathcal{Q}_P} + PXP^*,$$

and hence

$$X^* - PXP^* = \mathbf{P}_{\mathcal{Q}_P}(\mathbf{P}_{\mathbb{C}} \otimes A^*) |_{\mathcal{Q}_P}.$$

It now follows that

$$S^* - SP^* = \mathbf{P}_{\mathcal{Q}_P}(\mathbf{P}_{\mathbb{C}} \otimes A^*) |_{\mathcal{Q}_P}.$$

In view of Theorems 5.1 and 6.1, we conclude that (S, P) is a Γ -contraction. This completes the proof of the theorem. \blacksquare

Let (S, P) be a Γ -contraction. Then by the Wold decomposition theorem of Γ -contractions (see the paragraph after Theorem 2.2 or Theorem 2.8 in [7]) the tuple can be decomposed as the direct sum of a c.n.u. Γ -contraction and a Γ -unitary on the Wold decomposition space of the contraction P . Consequently, the above characterization is valid for any Γ -contraction on the Wold decomposition space of the contraction where the operator X is the direct sum of a compressed scalar operator with the decompressed numerical radius not greater than one with a unitary operator. For C_0 case, the necessary part of the above result was obtained in [11] in a different point of view.

Our next theorem concerns the uniqueness of the compressed scalar operators of the representations of Γ -contractions in Theorem 6.2.

THEOREM 6.3. *Let (S, P) be a c.n.u. Γ -contraction on \mathcal{Q}_P . Then there exists a unique compressed scalar operator X with $w_{\mathcal{Q}_P}(X) \leq 1$ such that*

$$S = X + PXP^*.$$

Proof. The representation $S = X + PX^*$ for some compressed scalar $X = \mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1)|_{\mathcal{Q}_P}$ follows from Theorem 6.2. For the remaining part, we calculate

$$\begin{aligned}
S &= X + PX^* \\
&= \mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1)|_{\mathcal{Q}_P} + \mathbf{P}_{\mathcal{Q}_P}(M_z \oplus U)\mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1)^*|_{\mathcal{Q}_P} \\
&= \mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1)|_{\mathcal{Q}_P} + \mathbf{P}_{\mathcal{Q}_P}(M_z \oplus U)(I - P_{\mathcal{Q}_P^\perp})(A \oplus U_1)^*|_{\mathcal{Q}_P} \\
&= \mathbf{P}_{\mathcal{Q}_P}(A \oplus U_1)|_{\mathcal{Q}_P} + \mathbf{P}_{\mathcal{Q}_P}(M_z \oplus U)(A \oplus U_1)^*|_{\mathcal{Q}_P} \\
&= \mathbf{P}_{\mathcal{Q}_P}[(A \oplus U_1) + (A \oplus U_1)^*(M_z \oplus U)]|_{\mathcal{Q}_P} \\
&= \mathbf{P}_{\mathcal{Q}_P}[(A + A^*M_z) \oplus (U_1 + U_1^*U)]|_{\mathcal{Q}_P},
\end{aligned}$$

where $U = M_{e^{it}}|_{\overline{\Delta_P L^2_{\mathcal{D}_P}(\mathbb{T})}}$. Consequently,

$$((A + A^*M_z) \oplus (U_1 + U_1^*U), M_z \oplus U)$$

is a Γ -isometric dilation of (S, P) and hence the result follows from the uniqueness part of the Γ -isometric dilation of Theorem 5.6. \blacksquare

7. CONCLUDING REMARKS

(I) Γ -contractive Hilbert modules:

Let $\{T_1, T_2\}$ be a pair of commuting operators in $\mathcal{B}(\mathcal{H})$. Then \mathcal{H} is a *Hilbert module* (see [13]) over $\mathbb{C}[z_1, z_2]$ where

$$p \cdot h = p(T_1, T_2)h,$$

for all $p \in \mathbb{C}[z_1, z_2]$ and $h \in \mathcal{H}$. A Hilbert module \mathcal{H} over $\mathbb{C}[z_1, z_2]$ is said to be Γ -contractive *Hilbert module* if the ordered pair (T_1, T_2) is a Γ -contraction.

Examples of Γ -contractive Hilbert modules are :

- (i) Γ -isometries,
- (ii) Γ -unitaries,
- (iii) direct sum of (i) and (ii).

Finally, by Theorem 2.3,

(iv) a Γ -contractive Hilbert module can be realized as a compression of any one of (i), (ii) or (iii) to a joint co-invariant subspace.

Now we turn to the class of pure Γ -isometries. Given a pure Γ -isometry $(A + A^*M_z, M_z)$ on $H_{\mathcal{E}_*}^2(\mathbb{D})$ for some $A \in \mathcal{B}(\mathcal{E}_*)$ with $w(A) \leq 1$, we say that $H_{\mathcal{E}_*}^2(\mathbb{D})$ is a Γ -isometric *Hardy module* with symbol A , where

$$p \cdot h = p(A + A^*M_z, M_z)h,$$

for all $p \in \mathbb{C}[z_1, z_2]$ and $h \in \mathcal{H}$. Let $H_{\mathcal{E}_*}^2(\mathbb{D})$ be a Γ -isometric Hardy module with symbol A and \mathcal{S} be a closed subspace of $H_{\mathcal{E}_*}^2(\mathbb{D})$. Then \mathcal{S} is said to be a submodule of the Γ -isometric Hardy module $H_{\mathcal{E}_*}^2(\mathbb{D})$ if \mathcal{S} is invariant under $A + A^*M_z$ and M_z .

Here we will present our Beurling-Lax-Halmos type theorem (Theorem 3.3) in the Hilbert modules language.

THEOREM 7.1. *Let $\mathcal{S} \neq \{0\}$ be a closed subspace of $H_{\mathcal{E}_*}^2(\mathbb{D})$. Then \mathcal{S} is a submodule of the Γ -isometric Hardy module $H_{\mathcal{E}_*}^2(\mathbb{D})$ with symbol A if and only if there exists a Γ -isometric Hardy module $H_{\mathcal{E}}^2(\mathbb{D})$ with a symbol B on \mathcal{E} and an isometric module map*

$$U : H_{\mathcal{E}}^2(\mathbb{D}) \longrightarrow H_{\mathcal{E}_*}^2(\mathbb{D}),$$

such that $\mathcal{S} = UH_{\mathcal{E}}^2(\mathbb{D})$. Moreover, when such a Γ -isometric Hardy module exists it is unique (up to unitary equivalence).

One consequence of the Beurling-Lax-Halmos theorem is that a non-zero submodule of the Hardy module $H_{\mathcal{E}_*}^2(\mathbb{D})$ is unitarily equivalent to $H_{\mathcal{E}}^2(\mathbb{D})$ for some Hilbert space \mathcal{E} . This phenomenon is no longer true in general when one consider the Hardy modules over the unit ball or the unit polydisc in \mathbb{C}^n , $n \geq 2$. However, a non-zero submodule of a Γ -isometric Hardy module is unitarily equivalent to a Γ -isometric Hardy module.

COROLLARY 7.2. *A non-zero submodule of a Γ -isometric Hardy module is isometrically isomorphic with a Γ -isometric Hardy module.*

Let $\mathcal{S} = M_{\Theta}H_{\mathcal{E}}^2(\mathbb{D})$ be a M_z -invariant subspace of $H_{\mathcal{E}_*}^2(\mathbb{D})$ for some inner multiplier $\Theta \in H_{\mathcal{B}(\mathcal{E}, \mathcal{E}_*)}^{\infty}(\mathbb{D})$ and that \mathcal{S} be invariant under the multiplication operator M_p where p is a $\mathcal{B}(\mathcal{E}_*, \mathcal{E})$ -valued analytic polynomial. Then

$$p\Theta = \Theta\Psi,$$

for some unique $\Psi \in H_{\mathcal{B}(\mathcal{E}_*, \mathcal{E})}^{\infty}(\mathbb{D})$.

Problem: What is the representation of the unique multiplier Ψ ? Under what conditions that Ψ will be a polynomial, or a polynomial of the same degree of p ?

Theorem 3.4 implies that the question has a complete answer when

$$p(z) = A + A^*z.$$

One possible approach to solve this problem is to consider first the finite dimension case, that is, $\mathcal{E}_* = \mathbb{C}^m$ for $m > 1$.

Also one can formulate the above problem in the Hilbert modules point of view. In this case, an isometric module map may yield a natural candidate for Ψ . At present, we do not have any positive result along that line.

(II) Complete unitary invariants:

We now turn to the task of determining a complete set of unitary invariants of the class of c.n.u. Γ -contraction.

Let (S_1, P_1) on \mathcal{H}_1 and (S_2, P_2) on \mathcal{H}_2 be a pair of Γ -contractions. We consider the representation of (S_i, P_i) in \mathcal{Q}_{P_i} . Then by Theorem 6.3, there exists unique compressed scalar operators $X_i \in \mathcal{B}(\mathcal{Q}_{P_i})$ with $w_{\mathcal{Q}_{P_i}}(X)$, $i = 1, 2$, such that

$$S_i = X_i + P_i X_i^*. \quad (i = 1, 2)$$

Furthermore, recall that (see the proof of Theorem 6.2)

$$S_i^* - S_i P_i^* = X_i^* = P_{\mathcal{Q}_{P_i}}(A_i^* \oplus U_i^*)|_{\mathcal{Q}_{P_i}} \in \mathcal{B}(\mathcal{Q}_P).$$

We are now ready to prove that (X, P) is a complete set of unitary invariants for the class of c.n.u. contractions $(S, P) = (X + PX^*, P)$, realized in the model space \mathcal{Q}_P .

THEOREM 7.3. *(X, P) is a complete set of unitary invariants for the class of c.n.u. Γ contractions. That is, for (S_i, P_i) on \mathcal{Q}_{P_i} , a pair of c.n.u. Γ -contractions on the model space \mathcal{Q}_{P_i} , $(S_1, P_1) \cong (S_2, P_2)$ if and only if*

$$(X_1, P_1) \cong (X_2, P_2).$$

Proof. Let $(X_1, P_1) \cong (X_2, P_2)$, that is,

$$X_2 = \tau X_1 \tau^*, \quad \text{and} \quad P_2 = \tau P_1 \tau^*,$$

for some unitary operator $\tau \in \mathcal{B}(\mathcal{Q}_{P_1}, \mathcal{Q}_{P_2})$. Then

$$S_2 = X_2 + P_2 X_2^* = \tau(X_1 + P_1 X_1^*)\tau^*,$$

and hence

$$(S_1, P_1) \cong (S_2, P_2).$$

Conversely, let

$$S_2 = \eta S_1 \eta^* \quad \text{and} \quad P_2 = \eta P_1 \eta^*,$$

for some unitary operator $\eta \in \mathcal{B}(\mathcal{Q}_{P_1}, \mathcal{Q}_{P_2})$. Then

$$X_2^* = S_2^* - S_2 P_2^* = \eta(S_1^* - S_1 P_1^*)\eta^* = \eta X_1^* \eta^*,$$

that is, $(X_1, P_1) \cong (X_2, P_2)$. ■

(III) Solving the commutant lifting theorem:

The commutant lifting theorem was first proved by D. Sarason [22] and then in complete generality by Sz.-Nagy and Foias (see [18] and [19]). Since then, it has been identified as one of the most useful results in operator theory. Here we recall a special case of the commutant lifting theorem. Let P be a c.n.u. contraction on \mathcal{H} . Let X commute with P . First, we identify P with the compression of the multiplication operator on the Sz.-Nagy and Foias model space \mathcal{Q}_P . Next, we consider the representation of X on \mathcal{Q}_P , which we again denote by X . Then the commutant lifting theorem implies the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}_P & \xrightarrow{\tilde{X}} & \mathcal{H}_P \\ \mathbf{P}_{\mathcal{Q}_P} \downarrow & & \downarrow \mathbf{P}_{\mathcal{Q}_P} \\ \mathcal{Q}_P & \xrightarrow{X} & \mathcal{Q}_P \end{array}$$

where \tilde{X} commutes with the multiplication operator on \mathcal{H}_P and $\|X\| = \|\tilde{X}\|$. It is usually a difficult problem to find a solution \tilde{X} to a given X in the commutator of P . One way to explain one of our main results, namely, Theorem 5.6 is that if S commutes with a given

c.n.u. contraction and if Γ is a spectral set of (S, P) (that is, (S, P) is a Γ -contraction) then the solution to the commutant lifting theorem is unique and explicit.

Therefore the results of this paper along with the seminal work of Agler and Young ([3] - [8]) is an evidence of solving the commutant lifting theorem uniquely and explicitly for a class of commutators of a contraction.

Another possible approach to obtain some of the results of this paper is to develop an independent proof of the characterization result, Theorem 6.2. Here, however, we do not pursue this direction. Also we believe that our methods will be applicable not only to other studies, but also demonstrate one way to set up and solve the commutant lifting theorem in a more general framework.

Finally, following the work of Sz.-Nagy and Foias and by virtue of our results, one can develop a H^∞ -functional calculus on $\{(z_1 + z_2, z_1 z_2) : |z_1|, |z_2| < 1\}$ of a c.n.u. Γ -contraction. Moreover, a study of invariant subspaces of Γ -contractions can be carried out. This will be considered in future work.

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