

ORTHOGONAL DECOMPOSITIONS AND TWISTED ISOMETRIES II

NARAYAN RAKSHIT, JAYDEB SARKAR, AND MANSI SURYAWANSHI

ABSTRACT. We classify tuples of (not necessarily commuting) isometries that admit von Neumann-Wold decomposition. We introduce the notion of twisted isometries for tuples of isometries and prove the existence of orthogonal decomposition for such tuples. The former classification is partially inspired by a result that was observed more than three decades ago by Gaspar and Suci. And the latter result generalizes Popovici's orthogonal decompositions for pairs of commuting isometries to general tuples of twisted isometries which also includes the case of tuples of commuting isometries. Our results unify all the known orthogonal decomposition related results in the literature.

1. INTRODUCTION

This article is concerned with orthogonal decompositions and representations of n -tuples of (not necessarily commuting) isometries acting on Hilbert spaces. We assume throughout that $n(> 1)$ is a natural number and Hilbert spaces are separable and over \mathbb{C} . The present article can be considered as a sequel to [13], although fairly different, more general in spirit, and almost an independent read.

The starting point for our analysis is the classical von Neumann–Wold decompositions of isometries on Hilbert spaces. Let \mathcal{H} be an arbitrary but fixed Hilbert space. A bounded linear operator V on \mathcal{H} (in short $V \in \mathcal{B}(\mathcal{H})$) is said to be an isometry if $\|Vf\| = \|f\|$ for all $f \in \mathcal{H}$, or equivalently, $V^*V = I_{\mathcal{H}}$. Of course, a trivial example of an isometry is unitary, whereas a less trivial example is a shift operator. Recall that an isometry V is a *shift* if

$$\text{SOT} - \lim_{m \rightarrow \infty} V^{*m} = 0.$$

These examples of isometries are rather typical:

Theorem 1.1 (von Neumann–Wold decomposition). *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. Then $\mathcal{H}_{\emptyset} := \bigcap_{m=0}^{\infty} V^m \mathcal{H}$ reduces V . Moreover, $V|_{\mathcal{H}_{\emptyset}}$ is a unitary and $V|_{\mathcal{H}_{\{1\}}}$ is a shift, where*

$$\mathcal{H}_{\{1\}} := \mathcal{H}_{\emptyset}^{\perp} = \bigoplus_{m=0}^{\infty} V^m(\ker V^*).$$

Therefore, V gives rise to an orthogonal direct sum of two closed subspaces (one of them is possibly zero) with respect to which V admits a 2×2 diagonal block matrix whose diagonal blocks are shift and unitary operators. This simple description of isometries is one of the

2010 *Mathematics Subject Classification.* 46L65, 47A20, 46L05, 81S05, 30H10, 46J15, 15B99.

Key words and phrases. Isometries, von Neumann and Wold decompositions, shift operators, wandering subspaces, invariant subspaces, Hardy space, polydisc.

most fundamental results in linear analysis. For instance, Theorem 1.1 plays an essential role in prediction theory [4], time series analysis [10], stochastic process [11], operator models [17], C^* -algebras [2, 18, 20], etc.

Theorem 1.1 raises natural and important questions, such as how to determine whether an n -tuple of isometries admits (meaningful) orthogonal direct sum decomposition, and even if such decomposition exists, what are the representations of tuples of isometries, whether such representations are unique or canonical, and of course, what are the invariants. The general problem seems completely inaccessible as even the structure of commuting pairs of isometries is notoriously complicated and largely unknown. However, this problem has been settled for certain classes of isometries. Notably, in [15], Śłociński demonstrated a completely satisfactory theory for pairs of doubly commuting isometries. In order to be more precise, let us clarify the meaning of orthogonal decompositions of tuples of isometries. The motivation comes from von Neumann [19], Wold [21], Śłociński [15], and Popovici [12].

Definition 1.2. An n -tuple of isometries $V = (V_1, \dots, V_n)$ acting on \mathcal{H} admits an orthogonal decomposition if there exist 2^n closed subspaces $\{\mathcal{H}_A\}_{A \subseteq I_n}$ of \mathcal{H} (some of these subspaces may be trivial) such that

- (1) \mathcal{H}_A reduces V for all $A \subseteq I_n$,
- (2) $\mathcal{H} = \bigoplus_{A \subseteq \{1, \dots, n\}} \mathcal{H}_A$, and
- (3) $V_i|_{\mathcal{H}_A}$, $i \in A$, is a shift, and $V_j|_{\mathcal{H}_A}$, $j \in A^c$, is a unitary for all $A \subsetneq \{1, \dots, n\}$.

If, in addition, (3) holds for $A = \{1, \dots, n\}$, then we say that V admits a von Neumann–Wold decomposition.

The meaning of condition (1) is that \mathcal{H}_A reduces V_i for all $i = 1, \dots, n$. At this point, we pause to warn the reader that, in the definition of orthogonal decomposition; we do not impose any particular condition on the reducing subspace $\mathcal{H}_{\{1, \dots, n\}}$.

Śłociński proved that pairs of doubly commuting isometries admit von Neumann–Wold decomposition. Recall that an n -tuple of isometries (V_1, \dots, V_n) on \mathcal{H} is said to be *doubly commuting* if

$$(1.1) \quad V_i^* V_j = V_j V_i^* \quad (i \neq j).$$

In particular, if (V_1, V_2) is a pair of doubly commuting isometries, then there exist four closed joint (V_1, V_2) -reducing subspaces \mathcal{H}_A , $A \subseteq \{1, 2\}$, such that

$$(1.2) \quad \mathcal{H} = \bigoplus_{A \subseteq \{1, 2\}} \mathcal{H}_A,$$

and $V_i|_{\mathcal{H}_A}$ is a shift or unitary according to $i \in A$ or $i \notin A$, respectively. Śłociński's observation (also see Suciu [16]) provided much of the inspiration for the subsequent development of orthogonal decompositions of tuples of isometries. For example, see [3, 8, 14] for doubly commuting analogue in higher dimensions and other general settings.

Now, passing to general tuples of isometries, there are two instances that are relevant to us: Popovici [12] proved that a pair of commuting isometries admits an orthogonal decomposition, just as in (1.2) above, but with the additional property that the fourth part $(V_1, V_2)|_{\mathcal{H}_{\{1, 2\}}}$ is a weak bi-shift. Here (and similarly for general tuples of operators) $(V_1, V_2)|_{\mathcal{H}_{\{1, 2\}}}$ refers

to the pair $(V_1|_{\mathcal{H}_{\{1,2\}}}, V_2|_{\mathcal{H}_{\{1,2\}}})$ on $\mathcal{H}_{\{1,2\}}$. Weak bi-shifts are rather complicated and encode the information of the complexity of pairs of commuting isometries. Secondly, motivated by Heisenberg group C^* -algebras (and also [5]), in [13] we introduced the notion of doubly twisted isometries (see Definition 1.3 below) and prove that a tuple of doubly twisted isometries always admits a von Neumann–Wold decomposition. Of course, doubly twisted isometries are fairly noncommutative objects, and therefore, it is curious to observe that the existence of the von Neumann–Wold decomposition is not completely a commutative feature. Note that doubly twisted isometries were referred to as tuples of \mathcal{U}_n -twisted isometries in [13].

In this paper, we study orthogonal decompositions at a higher level. First, we propose a general framework for noncommuting tuples of isometries and classify tuples admitting von Neumann–Wold decomposition. Our classification unifies all the existing results on von Neumann–Wold decompositions of tuple of isometries. Our results restricted to the commuting tuples of operators recover the classification of Gaspar and Suciu [3]. In other words, we point out that the idea of Gaspar and Suciu works for tuples of noncommuting isometries, which is fairly relevant as the existence and subsequent (direct or indirect) applications of orthogonal decomposition to C^* -algebras appear to be fruitful (although we do not pursue this direction here). See [6, 7, 9, 20], and also see the central paper [1].

Secondly, we introduce the notion of twisted isometries and prove that an n -tuple of twisted isometries $V = (V_1, \dots, V_n)$ admits an orthogonal decomposition with the additional property that $V|_{\mathcal{H}_{\{1, \dots, n\}}}$ is an n -tuple of twisted weak shift. This decomposition is more general than that of [12] as well as [13]. On one hand, our orthogonal decompositions for commuting tuples work for n -tuples, $n \geq 2$, and on the other hand, the generalizations of wandering subspaces and weak bi-shifts for n -tuples of isometries seem to be of interest as these ideas are not straight extensions of existing two variable theory.

Let us now turn to the technical part: Throughout the paper, when we refer to a *twist* on \mathcal{H} , we mean $\binom{n}{2}$ commuting unitaries $\{U_{ij}\}_{1 \leq i < j \leq n}$ on \mathcal{H} such that

$$U_{ji} := U_{ij}^* \quad (1 \leq i < j \leq n).$$

We now recall the notion of doubly twisted isometries which was introduced in [13] and referred to as \mathcal{U}_n -twisted isometry.

Definition 1.3 (Doubly twisted isometries). An n -tuple of isometries (V_1, \dots, V_n) on \mathcal{H} is said to be doubly twisted with respect to a twist $\{U_{st}\}_{s < t} \subseteq \mathcal{B}(\mathcal{H})$ if $V_k \in \{U_{st}'\}_{s < t}$ and

$$V_i^* V_j = U_{ij}^* V_j V_i^*,$$

for all $i, j, k = 1, \dots, n$, and $i \neq j$.

We often suppress the twist $\{U_{ij}\}_{i < j}$ and simply say that V is a doubly twisted isometry. In [13], we proved the existence of von Neumann–Wold decompositions for doubly twisted isometries. We further note that for a doubly twisted isometry (V_1, \dots, V_n) , we necessarily have that $V_i V_j = U_{ij} V_j V_i$ for all $i \neq j$ (cf. [13, Lemma 3.1]). This motivates:

Definition 1.4 (Twisted isometries). An n -tuple of isometries $V = (V_1, \dots, V_n)$ on \mathcal{H} is said to be a twisted isometry with respect to a twist $\{U_{st}\}_{s < t} \subseteq \mathcal{B}(\mathcal{H})$ if $V_k \in \{U_{st}'\}_{s < t}$ and

$$V_i V_j = U_{ij} V_j V_i,$$

for all $i, j, k = 1, \dots, n$, and $i \neq j$. The tuple V is said to be a twisted isometry if it is a twisted isometry corresponding to some twist.

The particular case $U_{ij} = I_{\mathcal{H}}$ for all $i < j$ yields tuples of commuting isometries. As pointed out previously, in the case of pairs of commuting isometries, Popovici [12] introduced orthogonal decompositions replacing Słociński's doubly commuting shift part with weak bi-shift. A pair of commuting isometries is called *weak bi-shift* if

$$V_1|_{\bigcap_{i \geq 0} \ker V_2^* V_1^i}, V_2|_{\bigcap_{j \geq 0} \ker V_1^* V_2^j}, \text{ and } V_1 V_2,$$

are shifts. In this context, we remark that a pair of shifts (V_1, V_2) is doubly commuting if and only if $V_1|_{\ker V_2^*}$, $V_2|_{\ker V_1^*}$ and $V_1 V_2$ are shifts [12]. However, as we will see, for a general n -tuple, one needs a little more care in extending the ideas of Popovici. After some preparation on the geometric structure of twisted shifts, in Definition 5.5, we introduce the notion of twisted weak shifts, a twisted counterpart of Popovici's weak bi-shifts. In Theorem 6.2, we prove that a twisted isometry admits an orthogonal decomposition with $V|_{\mathcal{H}_{I_n}}$ as a twisted weak shift. In summary, an n -tuple of twisted isometry $V = (V_1, \dots, V_n)$ gives rise to a direct sum decomposition just as in the case of doubly commuting isometry but the restriction of V on \mathcal{H}_{I_n} is (the twisted weak shift) is rather complicated and encode the information of the complexity of tuples of twisted (or even commuting) isometries. In other words, the first $2^n - 1$ subspaces of the direct summand are simple and enjoy similar properties to that of Słociński decomposition, and the Popovici analogue of the 2^n -th summand is rather the challenging part.

The rest of the paper is organized as follows. In Section 2, we classify tuples of isometries that admit von Neumann–Wold decomposition. The existence of von Neumann–Wold decompositions of tuples of doubly commuting isometries then follows as a simple corollary.

Section 3 revisits von Neumann–Wold decompositions of doubly twisted isometries, which was first observed in [13]. The present proof follows as an application of the classification of von Neumann–Wold decompositions for general tuples of isometries. On one hand, this makes the paper an independent read and easy reference for the remaining results. On the other hand, this connects the recent developments with the techniques which were observed more than three decades ago by Gaspar and Suciu [3].

Section 4 sets the stage for wandering subspaces for twisted isometries. This section is central to the theory of orthogonal decomposition of twisted isometries.

In Section 5, we introduce the notion of twisted weak shifts, the twisted counterpart of Popovici's weak bi-shifts. The idea of twisted weak shifts comes from a classification of twisted shifts (see Proposition 5.4). Section 6 deals with orthogonal decompositions of twisted isometries.

2. CHARACTERIZATIONS OF VON NEUMANN-WOLD DECOMPOSITIONS

In this section, we work in the category of noncommuting tuples of isometries. More specifically, we classify tuples of isometries that admit von Neumann–Wold decomposition.

Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. To simplify the notation, we set

$$(2.1) \quad \mathcal{H}_{V,u} = \bigcap_{m \in \mathbb{Z}_+} V^m \mathcal{H}, \text{ and } \mathcal{H}_{V,s} = \bigoplus_{m \in \mathbb{Z}_+} V^m(\ker V^*),$$

the unitary part and the shift part, respectively, of V (see Theorem 1.1). Therefore, $\mathcal{H} = \mathcal{H}_{V,s} \oplus \mathcal{H}_{V,u}$, where $V|_{\mathcal{H}_{V,s}}$ is a shift and $V|_{\mathcal{H}_{V,u}}$ is a unitary.

The following notation will be convenient throughout this paper:

$$I_n = \{1, \dots, n\}.$$

Let $V = (V_1, \dots, V_n)$ be an n -tuple of isometries acting on \mathcal{H} . Recall from Definition 1.2 that V admits a von Neumann–Wold decomposition if there exist 2^n closed subspaces $\{\mathcal{H}_A\}_{A \subseteq I_n}$ of \mathcal{H} such that $\mathcal{H} = \bigoplus_{A \subseteq I_n} \mathcal{H}_A$, and

- (1) \mathcal{H}_A reduces V for all $A \subseteq I_n$, and
- (2) $V_i|_{\mathcal{H}_A}$, $i \in A$, is a shift, and $V_j|_{\mathcal{H}_A}$, $j \in A^c$, is a unitary for all $A \subseteq I_n$.

We are now ready for the characterization of tuples of isometries admitting von Neumann–Wold decomposition.

Theorem 2.1. *Let $V = (V_1, \dots, V_n)$ be an n -tuple of isometries on \mathcal{H} . The following are equivalent:*

- (1) V admits a von Neumann–Wold decomposition.
- (2) $\mathcal{H}_{V_i,u}$ reduces V_j for all $i, j \in I_n$.
- (3) $\mathcal{H}_{V_i,s}$ reduces V_j for all $i, j \in I_n$.

Proof. Evidently, it is enough to prove that (1) and (2) are equivalent. Let V admits a von Neumann–Wold decomposition $\mathcal{H} = \bigoplus_{A \subseteq I_n} \mathcal{H}_A$. Fix $i \in I_n$. By the definition of von Neumann–Wold decomposition and Theorem 1.1, we have

$$\mathcal{H}_{V_i,s} = \bigoplus_{\substack{B \subseteq I_n \\ i \in B}} \mathcal{H}_B, \text{ and } \mathcal{H}_{V_i,u} = \bigoplus_{\substack{B \subseteq I_n \\ i \notin B}} \mathcal{H}_B.$$

Since \mathcal{H}_A reduces V_j for all $j \in I_n$ and $A \subseteq I_n$, it follows that $\mathcal{H}_{V_i,u}$ reduces V_j for all $i, j \in I_n$. For the converse, suppose $\mathcal{H}_{V_i,u}$ (and hence $\mathcal{H}_{V_i,s}$ too) reduces V_j for all $i, j \in I_n$. For each $A \subseteq I_n$, define

$$(2.2) \quad \mathcal{H}_A = \left[\bigcap_{i \in A} \mathcal{H}_{V_i,s} \right] \cap \left[\bigcap_{j \in A^c} \mathcal{H}_{V_j,u} \right].$$

Clearly, \mathcal{H}_A reduces V_i for all $i \in I_n$ and $A \subseteq I_n$. It then remains to show that $\mathcal{H} = \bigoplus_{A \subseteq I_n} \mathcal{H}_A$. Since $\bigoplus_{A \subseteq I_n} \mathcal{H}_A \subseteq \mathcal{H}$, it is enough to prove that

$$\mathcal{H} = \mathcal{H}_{V_1,s} \oplus \mathcal{H}_{V_1,u} \subseteq \bigoplus_{A \subseteq I_n} \mathcal{H}_A.$$

For this, we need a general observation: Given $A \subseteq I_m \subsetneq I_n$ and $j \notin I_m$, we denote by \tilde{A} the set A itself but as a subset of $I_m \cup \{j\}$. We claim that

$$\mathcal{H}_A \subseteq \mathcal{H}_{\tilde{A} \cup \{j\}} \oplus \mathcal{H}_{\tilde{A}}.$$

Note that \mathcal{H}_A reduces V_j , and also $\ker V_j^*|_{\mathcal{H}_A} = \ker V_j^* \cap \mathcal{H}_A$. By applying the von Neumann-Wold Decomposition to the isometry $V_j|_{\mathcal{H}_A}$, we have

$$\mathcal{H}_A = \left[\bigoplus_{k_j \in \mathbb{Z}_+} V_j^{k_j} (\ker V_j^* \cap \mathcal{H}_A) \right] \bigoplus \left[\bigcap_{k_j \in \mathbb{Z}_+} V_j^{k_j} \mathcal{H}_A \right].$$

For each $k_j \in \mathbb{Z}_+$, it is obvious that

$$V_j^{k_j} (\ker V_j^* \cap \mathcal{H}_A) \subseteq V_j^{k_j} \mathcal{H}_A \subseteq \mathcal{H}_A,$$

and

$$V_j^{k_j} (\ker V_j^* \cap \mathcal{H}_A) \subseteq V_j^{k_j} (\ker V_j^*).$$

The latter inclusion yields

$$\bigoplus_{k_j \in \mathbb{Z}_+} V_j^{k_j} (\ker V_j^* \cap \mathcal{H}_A) \subseteq \bigoplus_{k_j \in \mathbb{Z}_+} V_j^{k_j} (\ker V_j^*) = \mathcal{H}_{V_j, s},$$

which along with the former inclusion gives

$$\bigoplus_{k_j \in \mathbb{Z}_+} V_j^{k_j} (\ker V_j^* \cap \mathcal{H}_A) \subseteq \mathcal{H}_A \cap \mathcal{H}_{V_j, s} = \mathcal{H}_{\tilde{A} \cup \{j\}}.$$

We also have

$$\bigcap_{k_j \in \mathbb{Z}_+} V_j^{k_j} \mathcal{H}_A \subseteq \mathcal{H}_A \cap \left(\bigcap_{k_j \in \mathbb{Z}_+} V_j^{k_j} \mathcal{H} \right) = \mathcal{H}_{\tilde{A}}.$$

This implies $\mathcal{H}_A \subseteq \mathcal{H}_{\tilde{A} \cup \{j\}} \oplus \mathcal{H}_{\tilde{A}}$ and proves the claim. Applying this to $A \subseteq I_m \subsetneq I_n$ and $j, k \notin I_m$, we obtain

$$\mathcal{H}_A \subseteq \mathcal{H}_{\tilde{A} \cup \{j\} \cup \{k\}} \oplus \mathcal{H}_{\tilde{A} \cup \{j\}} \oplus \mathcal{H}_{\tilde{A} \cup \{k\}} \oplus \mathcal{H}_{\tilde{A}},$$

where $\tilde{A} = A$ but a subset of $I_m \cup \{j, k\}$. Consider the von-Neumann Wold decomposition for V_1 on \mathcal{H} :

$$\mathcal{H} = \mathcal{H}_{V_1, s} \oplus \mathcal{H}_{V_1, u} = \mathcal{H}_{\{1\}} \oplus \mathcal{H}_{\emptyset},$$

where index sets $\{1\}$ and \emptyset on the right side are subsets of $I = \{1\}$. Applying the above recipe repeatedly to $\mathcal{H}_{\{1\}}$ one sees that

$$\mathcal{H}_{\{1\}} \subseteq \bigoplus_{A \subseteq J} \mathcal{H}_{\{\tilde{1}\} \cup A},$$

where $J = \{2, \dots, n\}$, and $\{\tilde{1}\} = \{1\}$ but a subset of I_n . Similarly, $\mathcal{H}_{\emptyset} \subseteq \bigoplus_{A \subseteq J} \mathcal{H}_A$. Then

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{\{1\}} \oplus \mathcal{H}_{\emptyset} \\ &\subseteq \left(\bigoplus_{A \subseteq J} \mathcal{H}_{\{\tilde{1}\} \cup A} \right) \oplus \left(\bigoplus_{A \subseteq J} \mathcal{H}_A \right) \\ &= \bigoplus_{A \subseteq I_n} \mathcal{H}_A, \end{aligned}$$

completes the proof of the theorem. □

In the case of commuting tuples of isometries, the above result was stated by Gaspar and Suci (see [3, Theorem 2]). The proof in [3] was mentioned only in the case of $n = 3$. It is very curious to observe that the commutative theme of Gaspar and Suci also works for noncommuting tuples of isometries.

Two particular cases of the above classification are worthy of special attention: doubly commuting tuples of isometries (also see [3]) and doubly twisted isometries. Of course, the latter notion is more general than the former. We conclude this section with the case of doubly commuting isometries.

Before proceeding further, it is useful to make some standard observations about isometries. Given a closed subspace $\mathcal{S} \subseteq \mathcal{H}$, denote by $P_{\mathcal{S}}$ the orthogonal projection of \mathcal{H} onto \mathcal{S} .

Lemma 2.2. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. Then*

- (1) $P_{\mathcal{H}_{V,s}} = \text{SOT} - \sum_{m \in \mathbb{Z}_+} V^m P_{\ker V^*} V^{*m}$.
- (2) $P_{\mathcal{H}_{V,u}} = \text{SOT} - \lim_{m \rightarrow \infty} V^m V^{*m}$.

Proof. The proof follows from representations of unitary and shift parts of isometries as in (2.1). □

Let $V = (V_1, \dots, V_n)$ be a tuple of doubly commuting isometries on \mathcal{H} (see (1.1)). By Lemma 2.2, it follows that

$$P_{\mathcal{H}_{V_i,u}} = \text{SOT} - \lim_{m \rightarrow \infty} V_i^m V_i^{*m} \quad (i \in I_n).$$

In particular, if $h \in \mathcal{H}$ and $i \in I_n$, then $h \in \mathcal{H}_{V_i,u}$ if and only if $P_{\mathcal{H}_{V_i,u}} h = h$, whereas $h \in \mathcal{H}_{V_i,s}$ if and only if $P_{\mathcal{H}_{V_i,u}} h = 0$. Clearly, $\mathcal{H}_{V_i,u}$ reduces V_i for all $i \in I_n$. Moreover, for all $i \neq j$ in I_n , we have

$$(V_i^m V_i^{*m}) V_j = V_j (V_i^m V_i^{*m})$$

that is

$$P_{\mathcal{H}_{V_i,u}} V_j = V_j P_{\mathcal{H}_{V_i,u}}.$$

This implies immediately that $\mathcal{H}_{V_i,u}$ reduces V_j for all $i, j \in I_n$. Theorem 2.1 then implies that V admits von Neumann-Wold decomposition. This assertion was observed earlier in [14]. The assertion also follows from Corollary 3.2.

It is now an interesting problem to represent the direct summands \mathcal{H}_A of the von Neumann-Wold decomposition as described in (2.2). The answer is not clear in this generality and even it is unclear what conditions we would need to impose to get concrete representations of \mathcal{H}_A . In the following section, we will discuss this in the setting of doubly twisted isometries.

3. DOUBLY TWISTED ISOMETRIES

In this section, we deal with doubly twisted isometries (see Definition 1.3). The goal here is to recover, as an application of Theorem 2.1, the existence of the von Neumann-Wold decomposition for doubly twisted isometries along with representations of \mathcal{H}_A 's as described in (2.2). Needless to say, the main results of this section are not new (cf. [13, Theorem 3.6]), but the techniques involved are more algebraic. We believe that the present approach has more potential in other general frameworks.

Let (V_1, \dots, V_n) be a doubly twisted isometry on \mathcal{H} . We know that

$$(3.1) \quad V_i^* V_j = U_{ij}^* V_j V_i^*, \text{ and } V_i V_j = U_{ij} V_j V_i,$$

for all $i \neq j$ and $i, j \in I_n$. For each $i \neq j$, we then have

$$(3.2) \quad U_{ij} = V_i^* V_j^* V_i V_j,$$

and

$$V_i(V_j V_j^*) = U_{ij} V_j V_i V_j^* = U_{ij} V_j (U_{ij}^* V_j^* V_i) = (V_j V_j^*) V_i.$$

In the above, we have used the fact that $V_p \in \{U_{st}\}'_{s < t}$ for all $p \in I_n$, and $U_{st}^* = U_{ts}$, $t > s$. Therefore

$$(3.3) \quad V_i(V_j V_j^*) = (V_j V_j^*) V_i \quad (i \neq j).$$

Let $\mathcal{N}_\emptyset = \mathcal{H}$, and let

$$\mathcal{N}_A = \bigcap_{i \in A} \ker V_i^* \quad (A \neq \emptyset).$$

Also set $\mathcal{N}_i = \mathcal{N}_{\{i\}} = \ker V_i^*$ for all $i \in I_n$. In view of the above observation, we have

$$(3.4) \quad (V_i V_i^*)(V_j V_j^*) = (V_j V_j^*)(V_i V_i^*) \quad (i \neq j).$$

Since $I - V_i V_i^* = P_{\mathcal{N}_i}$ for all $i \in I_n$, it follows that $\{P_{\mathcal{N}_i}\}_{i \in I_n}$ is a family of commuting orthogonal projections. This implies

$$P_{\mathcal{N}_A} = \prod_{i \in A} P_{\mathcal{N}_i} \quad (A \subseteq I_n, A \neq \emptyset).$$

Moreover, (3.3) implies that \mathcal{N}_A reduces V_j for all $j \notin A$. Since $V_i \in \{U_{st}\}'_{s < t}$, it follows that $(I - V_i V_i^*)U_{st} = U_{st}(I - V_i V_i^*)$ for all $i \in I_n$ and hence, by the factorization of $P_{\mathcal{N}_A}$ above, we have that \mathcal{N}_A reduces U_{st} and $U_{st}\mathcal{N}_A = \mathcal{N}_A$ for all $s \neq t$ and $A \subseteq I_n$. We summarize all these observations in the following lemma (also see Lemma 3.3 and Lemma 3.5 in [13]).

Lemma 3.1. *Let (V_1, \dots, V_n) be a doubly twisted isometry, and let $A \subseteq I_n$. Then:*

- (1) $\{P_{\mathcal{N}_i}\}_{i \in I_n}$ is a family of commuting orthogonal projections.
- (2) $P_{\mathcal{N}_A} = \prod_{i \in A} P_{\mathcal{N}_i}$, $A \neq \emptyset$.
- (3) \mathcal{N}_A reduces V_j for all $j \notin A$, and $A \neq I_n$.
- (4) \mathcal{N}_A reduces U_{st} and $U_{st}\mathcal{N}_A = \mathcal{N}_A$ for all $s \neq t$.

We are now ready for the second application of Theorem 2.1:

Corollary 3.2. *Doubly twisted isometries admit von Neumann-Wold decomposition.*

Proof. Let (V_1, \dots, V_n) be a doubly twisted isometry on \mathcal{H} . Fix $i, j \in I_n$, and suppose $i \neq j$. Since $V_i V_j^m = V_j^m V_i U_{ij}^m$ for all $m \in \mathbb{Z}_+$, and $U_{ij}\mathcal{H} = \mathcal{H}$ and $V_i\mathcal{H} \subseteq \mathcal{H}$, it follows that

$$V_i \mathcal{H}_{V_j, u} = V_i \left(\bigcap_{m \in \mathbb{Z}_+} V_j^m \mathcal{H} \right) = \bigcap_{m \in \mathbb{Z}_+} V_j^m V_i U_{ij}^m \mathcal{H} \subseteq \bigcap_{m \in \mathbb{Z}_+} V_j^m \mathcal{H}$$

that is, $V_i \mathcal{H}_{V_j, u} \subseteq \mathcal{H}_{V_j, u}$. Similarly, $V_i^* \mathcal{H}_{V_j, u} \subseteq \mathcal{H}_{V_j, u}$. This proves that $\mathcal{H}_{V_j, u}$ reduces V_i . Finally, the fact that $\mathcal{H}_{V_i, u}$ reduces V_i follows from the von Neumann-Wold decomposition of V_i . Thus the assertion follows from Theorem 2.1. \square

Now we turn to the problem of representations of the direct summands of the von Neumann-Wold decomposition. First, observe that Lemma 3.1 and part (2) of Lemma 2.2 implies:

Lemma 3.3. *Let (V_1, \dots, V_n) be a doubly twisted isometry, and let $A \subsetneq I_n$ be a nonempty subset. Then $P_{\mathcal{H}_{V_j,s}}, P_{\mathcal{H}_{V_j,u}} \in \{P_{\mathcal{N}_A}\}'$ for all $j \in A^c$.*

We then have the commutativity of orthogonal projections:

Lemma 3.4. *Let (V_1, \dots, V_n) be a doubly twisted isometry. Then $\{P_{\mathcal{H}_{V_i,s}}, P_{\mathcal{H}_{V_j,u}}\}_{i,j \in I_n}$ is a family of commuting orthogonal projections.*

Proof. Clearly, $P_{\mathcal{H}_{V_i,s}}P_{\mathcal{H}_{V_i,u}} = 0 = P_{\mathcal{H}_{V_i,u}}P_{\mathcal{H}_{V_i,s}}$ for all $i \in I_n$. Therefore, assume that $i \neq j$. Lemma 2.2 and Lemma 3.3 then imply

$$P_{\mathcal{H}_{V_i,s}}P_{\mathcal{H}_{V_j,s}} = \left(\sum_{m \in \mathbb{Z}_+} V_i^m P_{\mathcal{N}_i} V_i^{*m} \right) P_{\mathcal{H}_{V_j,s}} = P_{\mathcal{H}_{V_j,s}} \left(\sum_{m \in \mathbb{Z}_+} V_i^m P_{\mathcal{N}_i} V_i^{*m} \right) = P_{\mathcal{H}_{V_j,s}}P_{\mathcal{H}_{V_i,s}}.$$

In the above, we have also used the fact that $\mathcal{H}_{V_j,s}$ reduces V_i . A similar computation then yields $P_{\mathcal{H}_{V_i,s}}P_{\mathcal{H}_{V_j,u}} = P_{\mathcal{H}_{V_j,u}}P_{\mathcal{H}_{V_i,s}}$ and $P_{\mathcal{H}_{V_i,u}}P_{\mathcal{H}_{V_j,u}} = P_{\mathcal{H}_{V_j,u}}P_{\mathcal{H}_{V_i,u}}$, and completes the proof of the lemma. \square

Similar technique applies to the splitting of product of isometries, co-isometries, and orthogonal projections. For each $m \geq 1$, denote by $k = (k_1, \dots, k_m)$ the multi-index in \mathbb{Z}_+^m .

Lemma 3.5. *Let (V_1, \dots, V_n) be a doubly twisted isometry, $A \subseteq I_n$, $A \neq \emptyset$, and let $k \in \mathbb{Z}_+^{|A|}$. Then*

$$\prod_{i \in A} \left(V_i^{k_i} P_{\mathcal{N}_i} V_i^{*k_i} \right) = \left(\prod_{i \in A} V_i^{k_i} \right) P_{\mathcal{N}_A} \left(\prod_{i \in A} V_i^{k_i} \right)^*.$$

Proof. Let $i, j \in I_n$, and suppose $i \neq j$. Note that (see (3.1))

$$V_i^{*p} V_j^q = U_{ij}^{*pq} V_j^q V_i^{*p} \quad (p, q \in \mathbb{Z}_+).$$

Since $U_{ij} \in \{V_i, V_j\}'$ and $\mathcal{N}_s = \ker V_s^*$ reduces V_t for all $t \neq s$ (see part (3) of Lemma 3.1), for each $k_i, k_j \in \mathbb{Z}_+$, it follows that

$$\begin{aligned} (V_i^{k_i} P_{\mathcal{N}_i} V_i^{*k_i})(V_j^{k_j} P_{\mathcal{N}_j} V_j^{*k_j}) &= (U_{ij}^{*k_i k_j}) V_i^{k_i} P_{\mathcal{N}_i} V_j^{k_j} V_i^{*k_i} P_{\mathcal{N}_j} V_j^{*k_j} \\ &= (U_{ij}^{*k_i k_j}) V_i^{k_i} V_j^{k_j} P_{\mathcal{N}_i} P_{\mathcal{N}_j} V_i^{*k_i} V_j^{*k_j} \\ &= (U_{ij}^{*k_i k_j})(V_i^{k_i} V_j^{k_j}) P_{\mathcal{N}_{\{i,j\}}} (V_i^{k_i} V_j^{k_j})^* (U_{ij}^{k_i k_j}) \\ &= (V_i^{k_i} V_j^{k_j}) P_{\mathcal{N}_{\{i,j\}}} (V_i^{k_i} V_j^{k_j})^*. \end{aligned}$$

This verifies the conclusion for $A = \{i, j\}$. The lemma now follows by induction. \square

We set the following convention. Given an n -tuple of bounded linear operators $T = (T_1, \dots, T_n)$ on \mathcal{H} and $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, we define T^k by

$$T^k := T_1^{k_1} \dots T_n^{k_n}.$$

Now we discuss the unitary part of doubly twisted isometries:

Lemma 3.6. *Let $V = (V_1, \dots, V_n)$ be a doubly twisted isometry and let \mathcal{S} reduces V . Then*

$$\bigcap_{k \in \mathbb{Z}_+^n} V^k \mathcal{S} = \bigcap_{i \in I_n} \bigcap_{k_i \in \mathbb{Z}_+} V_i^{k_i} \mathcal{S}.$$

Proof. It suffices to prove the assertion for $n = 2$; the general case then easily follows by induction. Let $V = (V_1, V_2)$ be a doubly twisted isometry and let U be the corresponding twist. By (3.1) and (3.2), we know that $V_1 V_2 = U V_2 V_1$ and $U = V_1^* V_2^* V_1 V_2$. In particular, \mathcal{S} reduces U . It is now easy to see that

$$\bigcap_{k \in \mathbb{Z}_+^2} V^k \mathcal{S} \subseteq \bigcap_{i \in I_2} \bigcap_{k_i \in \mathbb{Z}_+} V_i^{k_i} \mathcal{S}.$$

For the reverse inclusion, let $y \in (\bigcap_{k_1 \in \mathbb{Z}_+} V_1^{k_1} \mathcal{S}) \cap (\bigcap_{k_2 \in \mathbb{Z}_+} V_2^{k_2} \mathcal{S})$. For each $k_1, k_2 \in \mathbb{Z}_+$, there exist x_{k_1} and x_{k_2} in \mathcal{S} such that $y = V_1^{k_1} x_{k_1} = V_2^{k_2} x_{k_2}$. Then

$$x_{k_1} = V_1^{*k_1} V_2^{k_2} x_{k_2} = V_2^{k_2} (V_1^{*k_1} U^{*k_1 k_2} x_{k_2}) = V_2^{k_2} x,$$

where $x = V_1^{*k_1} U^{*k_1 k_2} x_{k_2} \in \mathcal{S}$. Therefore, $y = V_1^{k_1} V_2^{k_2} x \in \bigcap_{k \in \mathbb{Z}_+^2} V^k \mathcal{S}$, which completes the proof of the lemma. \square

The following conventions will be in effect throughout: For an n -tuple of bounded linear operators $T = (T_1, \dots, T_n)$ on \mathcal{H} and $A = \{m_1 < \dots < m_p\} \subseteq I_n$, we define

$$T_A = (T_{m_1}, \dots, T_{m_p}),$$

and, for each $k = (k_1, \dots, k_p) \in \mathbb{Z}_+^p$, define

$$(3.5) \quad T_A^k := T_{m_1}^{k_1} \dots T_{m_p}^{k_p}.$$

Now we are ready to compute the closed subspaces \mathcal{H}_A in (2.2).

Theorem 3.7. *Let $V = (V_1, \dots, V_n)$ be a doubly twisted isometry. Then V admits a von Neumann-Wold decomposition $\mathcal{H} = \bigoplus_{A \subseteq I_n} \mathcal{H}_A$, where*

$$\mathcal{H}_A = \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \left(\bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l \mathcal{N}_A \right) \quad (A \subseteq I_n).$$

Proof. By Corollary 3.2, we already know the existence of the von Neumann-Wold decomposition. We also know that (see (2.2) in the proof of Theorem 2.1)

$$\mathcal{H}_A = \left(\bigcap_{i \in A} \mathcal{H}_{V_i, s} \right) \cap \left(\bigcap_{j \in A^c} \mathcal{H}_{V_j, u} \right) \quad (A \subseteq I_n).$$

Fix $A \subseteq I_n$, and suppose $A \neq \emptyset$. Lemma 3.4 implies $\mathcal{H}_A = \left(\prod_{i \in A} P_{\mathcal{H}_{V_i, s}} \right) \left(\prod_{j \in A^c} P_{\mathcal{H}_{V_j, u}} \right) \mathcal{H}$. We compute

$$\begin{aligned} \prod_{i \in A} P_{\mathcal{H}_{V_i, s}} &= \prod_{i \in A} \left(\sum_{k_i \in \mathbb{Z}_+} V_i^{k_i} P_{\mathcal{N}_i} V_i^{*k_i} \right) \\ &= \sum_{k \in \mathbb{Z}_+^{|A|}} \left(\prod_{i \in A} V_i^{k_i} P_{\mathcal{N}_i} V_i^{*k_i} \right) \\ &= \sum_{k \in \mathbb{Z}_+^{|A|}} \left(\prod_{i \in A} V_i^{k_i} \right) P_{\mathcal{N}_A} \left(\prod_{i \in A} V_i^{k_i} \right)^*, \end{aligned}$$

where the last equality follows from Lemma 3.5. If $A = I_n$, then the above equality gives desired representation of \mathcal{H}_{I_n} . Suppose $A \subsetneq I_n$. By Lemma 3.3, we know that $P_{\mathcal{H}_{V_j, u}} P_{\mathcal{N}_A} = P_{\mathcal{N}_A} P_{\mathcal{H}_{V_j, u}}$ for all $j \in A^c$. Also, since $P_{\mathcal{H}_{V_j, u}} V_i = V_i P_{\mathcal{H}_{V_j, u}}$ for all $i \in A$ and $j \in A^c$, we have

$$\begin{aligned} \left(\prod_{i \in A} P_{\mathcal{H}_{V_i, s}} \right) \left(\prod_{j \in A^c} P_{\mathcal{H}_{V_j, u}} \right) &= \left(\sum_{k \in \mathbb{Z}_+^{|A|}} \left(\prod_{i \in A} V_i^{k_i} \right) P_{\mathcal{N}_A} \left(\prod_{i \in A} V_i^{k_i} \right)^* \right) \left(\prod_{j \in A^c} P_{\mathcal{H}_{V_j, u}} \right) \\ &= \sum_{k \in \mathbb{Z}_+^{|A|}} \left(\prod_{i \in A} V_i^{k_i} \right) \left(\prod_{j \in A^c} P_{\mathcal{H}_{V_j, u}} \right) P_{\mathcal{N}_A} \left(\prod_{i \in A} V_i^{k_i} \right)^*. \end{aligned}$$

Consequently

$$\mathcal{H}_A = \text{ran} \left(\sum_{k \in \mathbb{Z}_+^{|A|}} \left(\prod_{i \in A} V_i^{k_i} \right) \left(\prod_{j \in A^c} P_{\mathcal{H}_{V_j, u}} \right) P_{\mathcal{N}_A} \right).$$

Finally, note that

$$\text{ran} \left(\left(\prod_{j \in A^c} P_{\mathcal{H}_{V_j, u}} \right) P_{\mathcal{N}_A} \right) = \bigcap_{j \in A^c} P_{\mathcal{H}_{V_j, u}} \mathcal{N}_A = \bigcap_{j \in A^c} \left(\bigcap_{k_j \in \mathbb{Z}_+} V_j^{k_j} (\mathcal{N}_A) \right) = \bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l (\mathcal{N}_A),$$

where the last equality follows from Lemma 3.6. Therefore

$$\mathcal{H}_A = \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \left(\bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l (\mathcal{N}_A) \right).$$

If $A = \emptyset$, then $\mathcal{N}_A = \mathcal{H}$, and hence using above equality we get

$$\mathcal{H}_\emptyset = \bigcap_{l \in \mathbb{Z}_+^n} V_{I_n}^l \mathcal{H},$$

which completes the proof of the theorem. \square

As already pointed out, this was first observed in [13]. The present proof is an application of Theorem 2.1 and is conceptually different. Moreover, the techniques and ideas used in this section will be useful in describing the structure of twisted isometries.

4. WANDERING SUBSPACES

Let $V = (V_1, \dots, V_n)$ be a tuple of isometries on \mathcal{H} , and let \mathcal{S} be a closed subspace of \mathcal{H} . We say that \mathcal{S} satisfies *wandering property* (or \mathcal{S} is a wandering subspace) for V if

$$V^k \mathcal{S} \perp V^l \mathcal{S},$$

for all $k \neq l$ in \mathbb{Z}_+^n . If, in addition

$$\mathcal{H} = \bigoplus_{k \in \mathbb{Z}_+^n} V^k \mathcal{S},$$

then we say that \mathcal{S} is a *generating wandering subspace* for V . Generating wandering subspaces play a vital role in representing single isometries. Indeed, if $n = 1$, then $\ker V^*$ is the generating wandering subspace which represents the pure part of an isometry V (see Theorem 1.1). Wandering subspaces for doubly twisted isometries are also explicit and play important role in the structure of such tuples (compare Theorem 3.7 with the identity (5.2)). However, as we will see in this section, the analog of wandering subspaces for twisted isometries requires some more care. We begin with the following definition (recall (3.5)).

Definition 4.1. Let $V = (V_1, \dots, V_n)$ be a tuple of isometries on \mathcal{H} . The weak A -wandering subspace for V is defined by

$$\mathcal{E}_A := \bigcap_{i \in A} \bigcap_{B \subseteq \{i\}^c} \bigcap_{k \in \mathbb{Z}_+^{|B|}} \left(\ker V_i^* V_B^k \right),$$

whenever $A \subseteq I_n$ and $A \neq \emptyset$, whereas $\mathcal{E}_\emptyset := \mathcal{H}$.

In the above, $\{i\}^c := I_n \setminus \{i\}$. First, we clarify that weak A -wandering subspaces of twisted isometries indeed satisfy wandering property.

Lemma 4.2. *Let $V = (V_1, \dots, V_n)$ be a twisted isometry. Then \mathcal{E}_A satisfies the wandering property for V for all $A \subseteq I_n$, $A \neq \emptyset$.*

Proof. Let $|A| = p$ and suppose $A := \{m_1 < \dots < m_p\}$. Let $k = (k_1, \dots, k_p)$ and $l = (l_1, \dots, l_p)$ be in \mathbb{Z}_+^p . Suppose $k \neq l$, and let j be the minimum of all $i \in \{1, \dots, p\}$ such that $k_i \neq l_i$. Assume, without loss of generality, that $k_j < l_j$. For $x, y \in \mathcal{E}_A$, we have

$$\begin{aligned} \langle V_A^k x, V_A^l y \rangle &= \langle V_{m_j}^{k_j} \dots V_{m_p}^{k_p} x, V_{m_j}^{l_j} \dots V_{m_p}^{l_p} y \rangle \\ &= \langle V_{m_j}^* (V_{m_{j+1}}^{k_{j+1}} \dots V_{m_p}^{k_p}) x, V_{m_j}^{l_j - k_j - 1} \dots V_{m_p}^{l_p} y \rangle. \end{aligned}$$

Note that (see Definition 4.1)

$$x \in \mathcal{E}_A = \bigcap_{i \in A} \bigcap_{B \subseteq \{i\}^c} \bigcap_{k \in \mathbb{Z}_+^{|B|}} \left(\ker V_i^* V_B^k \right).$$

Therefore, if we set $B := \{m_{j+1}, \dots, m_p\}$, then $B \subseteq \{m_j\}^c$, and consequently

$$x \in \ker \left(V_{m_j}^* V_B^{\bar{k}} \right),$$

where $\tilde{k} = (k_{j+1}, \dots, k_p) \in \mathbb{Z}_+^{|B|}$. Consequently, $V_{m_j}^* V_B^{\tilde{k}} x = 0$, which implies

$$\begin{aligned} \langle V_A^k x, V_A^l y \rangle &= \langle V_{m_j}^* V_B^{\tilde{k}} x, V_{m_j}^{l_j - k_j - 1} \dots V_{m_p}^{l_p} y \rangle \\ &= 0, \end{aligned}$$

and completes the proof of the lemma. \square

Before we get into the working class subspaces satisfying wandering property, we prove some lemmas. In the remaining part of this section we assume that $V = (V_1, \dots, V_n)$ is a twisted isometry corresponding to a twist $\{U_{ij}\}_{i < j}$. The following simple observation will turn out to be an indispensable tool in what follows.

Lemma 4.3. *For each $k \in \mathbb{Z}_+^n$ and $i \in I_n$, there exists a monomial $\eta_{i,k} \in \mathbb{C}[z_1, \dots, z_n]$ such that*

$$V_i V^k = V^k V_i \eta_{i,k}(U).$$

Proof. By the definition of twisted isometries, we have $V_i V_j = U_{ij} V_j V_i$ for all $i \neq j$. Therefore

$$\begin{aligned} V_i V^k &= (U_{i1}^{k_1} \dots U_{i(i-1)}^{k_{i-1}}) (U_{i(i+1)}^{k_{i+1}} \dots U_{in}^{k_n}) V^k V_i \\ &= \eta_{i,k}(U) V^k V_i \\ &= V^k V_i \eta_{i,k}(U), \end{aligned}$$

where

$$\eta_{i,k} = z_1^{k_1} \dots z_{i-1}^{k_{i-1}} z_{i+1}^{k_{i+1}} \dots z_n^{k_n} \in \mathbb{C}[z_1, \dots, z_n],$$

and $\eta_{i,k}(U)$ refers to the polynomial functional calculus. \square

In the above, the polynomial functional calculus $\eta_{i,k}(U)$ is given by

$$\eta_{i,k}(U) = U_{i1}^{k_1} \dots U_{i(i-1)}^{k_{i-1}} U_{i(i+1)}^{k_{i+1}} \dots U_{in}^{k_n}.$$

We will adopt this convention throughout the remainder of this paper. Moreover, if i and k are clear from the context, then we simply denote a monomial in U by $\eta(U)$ instead of $\eta_{i,k}(U)$.

Before proceeding, we observe the useful identity

$$(4.1) \quad U_{ij} = V_i^* V_j^* V_i V_j \quad (i \neq j).$$

This follows from the definition of twisted isometries that $V_i V_j = U_{ij} V_j V_i$ and the fact that $V_i \in \{U_{st}\}_{s < t}$ for all $i \neq j$.

Lemma 4.4. *$U_{ij} \mathcal{E}_A = \mathcal{E}_A$ and $U_{ij} \mathcal{N}_A = \mathcal{N}_A$ for all $i \neq j$ and $A \subseteq I_n$.*

Proof. If $A = \emptyset$, then $\mathcal{E}_\emptyset = \mathcal{H} = \mathcal{N}_\emptyset$, and the desired equality is clear. Suppose $A \neq \emptyset$. By assumption, we have

$$(4.2) \quad U_{ij} V_k = V_k U_{ij}, \text{ and } U_{ij} V_k^* = V_k^* U_{ij},$$

for all $i, j, k \in I_n$ and $i \neq j$. Fix $A \subseteq I_n$, $i \neq j$, and suppose $x \in \mathcal{E}_A$. We know that $x \in \ker V_t^* V_B^k$ for all $t \in A$, $B \subseteq \{t\}^c$ and $k \in \mathbb{Z}_+^{|B|}$. Therefore,

$$V_t^* V_B^k U_{ij} x = U_{ij} V_t^* V_B^k x = 0,$$

and similarly, $V_t^* V_B^k U_{ij}^* x = 0$. This implies that U_{ij} reduces \mathcal{E}_A . Since U_{ij} is unitary, it follows that $U_{ij} \mathcal{E}_A = \mathcal{E}_A$. The second equality $U_{ij} \mathcal{N}_A = \mathcal{N}_A$ follows from (4.2). \square

Since $V_i V_j = U_{ij} V_j V_i = V_j V_i U_{ij}$ for all $i \neq j$, the above lemma implies that:

Lemma 4.5. $V_i V_j \mathcal{E}_A = V_j V_i \mathcal{E}_A$ for all $A \subseteq I_n$.

We also have the following invariance property:

Lemma 4.6. $V_j \mathcal{E}_A \subseteq \mathcal{E}_A$ for all $A \subsetneq I_n$ and $j \in A^c$.

Proof. We know that $\mathcal{E}_\emptyset = \mathcal{H}$. Then obviously $V_j \mathcal{E}_\emptyset \subseteq \mathcal{E}_\emptyset$. Suppose $A \neq \emptyset$. Let $x \in \mathcal{E}_A$. Since $j \in A^c$, for any $B \subseteq \{i\}^c$, $k \in \mathbb{Z}_+^{|B|}$ and $i \in A$, we have, $V_i^* V_B^k V_j(x) = 0$. This implies $V_j \mathcal{E}_A \subseteq \mathcal{E}_A$ and completes the proof of the lemma. \square

Although the weakly wandering subspace \mathcal{E}_A satisfies the wandering property, for the sake of appropriate orthogonal decompositions of twisted isometries, we require to identify a suitable subspace of \mathcal{E}_A :

Definition 4.7. Let $V = (V_1, \dots, V_n)$ be a twisted isometry. For each $A \subseteq I_n$, the A -wandering subspace for V is defined by

$$\mathcal{W}_A = \bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l \mathcal{E}_A.$$

Clearly, $\mathcal{W}_A \subseteq \mathcal{E}_A$ for all $A \subseteq I_n$. In particular, we have $V_A^k \mathcal{W}_A \perp V_A^l \mathcal{W}_A$ for all $k \neq l$ in $\mathbb{Z}_+^{|A|}$, $A \neq \emptyset$, and $A \subseteq I_n$. Therefore, the orthogonal sum

$$(4.3) \quad \mathcal{H}_{V,A} := \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \mathcal{W}_A,$$

is well-defined for all $A \neq \emptyset$. We also set $\mathcal{H}_{V,\emptyset} := \mathcal{W}_\emptyset$, that is

$$\mathcal{H}_{V,\emptyset} := \mathcal{W}_\emptyset = \bigcap_{k \in \mathbb{Z}_+^n} V_{I_n}^k \mathcal{H}.$$

By the commutativity property of V_k 's and U_{ij} 's as in (4.2) and Lemma 4.4, it follows that

$$(4.4) \quad U_{ij} \mathcal{H}_{V,A} = \mathcal{H}_{V,A},$$

for all $i \neq j$ and $A \subseteq I_n$. Moreover, we have:

Lemma 4.8. Let $V = (V_1, \dots, V_n)$ be a twisted isometry. Then:

- (1) $V_j \mathcal{W}_A = \mathcal{W}_A$ for all $A \subsetneq I_n$ and $j \in I_n \setminus A$.
- (2) $U_{st} \mathcal{W}_A = \mathcal{W}_A$ for all $s < t$ and $A \subseteq I_n$.
- (3) $V_j \mathcal{H}_{V,A} = \mathcal{H}_{V,A}$ for all $A \subsetneq I_n$ and $j \in I_n \setminus A$.

Proof. Fix $A \subsetneq I_n$ and $j \in I_n \setminus A$. Clearly, $\mathcal{W}_A \subseteq V_j \mathcal{W}_A$. For the nontrivial inclusion, observe that by Lemma 4.3, there exists a monomial $\eta_1 \in \mathbb{C}[z_1, \dots, z_{n-|A|}]$ such that

$$\begin{aligned} V_j \left(\bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l \mathcal{E}_A \right) &= \bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l V_j \eta_1(U) \mathcal{E}_A \\ &= \bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l V_j \mathcal{E}_A \\ &\subseteq \bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l \mathcal{E}_A. \end{aligned}$$

where the last equality and the inclusion follow from Lemma 4.4 and Lemma 4.6, respectively. Hence $V_j \mathcal{W}_A = \mathcal{W}_A$, which completes the proof of part (1). Part (2) follows from the Lemma 4.4 along with the fact that U_{ij} commutes with V_k for all $i \neq j$ and k .

Now we prove that $V_j \mathcal{H}_{V,A} = \mathcal{H}_{V,A}$, for all $A \subsetneq I_n$ and $j \in A^c$. Fix $A \subsetneq I_n$ and $j \in A^c$. Note that for each $k \in \mathbb{Z}_+^{|A|}$, there exists a monomial η_k such that $V_j V_A^k = V_A^k V_j \eta_k(U)$. In view of $\eta_k(U) \mathcal{W}_A = \mathcal{W}_A$, we compute

$$\begin{aligned} V_j \mathcal{H}_{V,A} &= V_j \left(\bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \mathcal{W}_A \right) \\ &= \left(\bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k V_j \eta_k(U) \mathcal{W}_A \right) \\ &= \left(\bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k V_j \mathcal{W}_A \right). \end{aligned}$$

Since $V_j \mathcal{W}_A = \mathcal{W}_A$, by part (1), it follows that $V_j \mathcal{H}_{V,A} = \mathcal{H}_{V,A}$. \square

Wandering subspaces will play a key role in the remaining part of the paper.

5. TWISTED WEAK SHIFTS

In this section, we will introduce the notion of twisted weak shifts. This will appear to be the right generalization of Popovici's weak bi-shifts for pairs of commuting isometries. We begin with the definition of twisted shifts which was introduced in [13].

Definition 5.1. A twisted shift is an n -tuple of doubly twisted isometry (V_1, \dots, V_n) such that V_i is a shift for all $i = 1, \dots, n$.

Suppose $V = (V_1, \dots, V_n)$ is a doubly twisted isometry on \mathcal{H} . In view of Theorem 3.7, V admits a von Neumann-Wold decomposition $\mathcal{H} = \bigoplus_{A \subseteq I_n} \mathcal{H}_A$, where

$$\mathcal{H}_A = \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \left(\bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l \mathcal{N}_A \right) \quad (A \subseteq I_n).$$

Therefore, V is a twisted shift if and only if $\mathcal{H}_A = \{0\}$ for all $A \neq I_n$. Equivalently, \mathcal{H} admits the following decomposition

$$\mathcal{H} = \bigoplus_{k \in \mathbb{Z}_+^n} V^k(\mathcal{N}),$$

where $\mathcal{N} =: \mathcal{N}_{I_n} = \bigcap_{i \in I_n} \ker V_i^*$.

The typical example of twisted shifts is built up from commuting unitary operators and shifts on the Hardy space $H^2(\mathbb{D}^n)$. Recall that $H^2(\mathbb{D}^n)$ is the Hilbert space of all square summable analytic functions on the unit polydisc \mathbb{D}^n . Given a Hilbert space \mathcal{E} , we denote by $H_{\mathcal{E}}^2(\mathbb{D}^n)$ the \mathcal{E} -valued Hardy space over \mathbb{D}^n . Then $(M_{z_1}, \dots, M_{z_n})$ defines a doubly commuting shifts on $H_{\mathcal{E}}^2(\mathbb{D}^n)$, where $M_{z_i}f = z_i f$ for all $f \in H_{\mathcal{E}}^2(\mathbb{D}^n)$. It is often convenient to identify $H_{\mathcal{E}}^2(\mathbb{D}^n)$ with $H^2(\mathbb{D}^n) \otimes \mathcal{E}$. We need a definition:

Definition 5.2. Let \mathcal{E} be a Hilbert space, $U \in \mathcal{B}(\mathcal{E})$ be a unitary, and let $j \in \{1, \dots, n\}$. The j -th diagonal operator with symbol U is the unitary operator $D_j[U]$ on $H_{\mathcal{E}}^2(\mathbb{D}^n)$ defined by

$$D_j[U](z^k \eta) = z^k (U^{k_j} \eta) \quad (k \in \mathbb{Z}_+^n, \eta \in \mathcal{E}).$$

Let \mathcal{E} be a Hilbert space, and let $\{U_{ij}\}_{i < j}$ be a twist on \mathcal{E} . Then $\{I_{H^2(\mathbb{D}^n)} \otimes U_{ij}\}_{i < j}$ defines a twist on $H_{\mathcal{E}}^2(\mathbb{D}^n)$ (or on $H^2(\mathbb{D}^n) \otimes \mathcal{E}$, to be more specific). Define

$$V_i = \begin{cases} M_{z_1} & \text{if } i = 1 \\ M_{z_i} \left(D_1[U_{i1}] D_2[U_{i2}] \cdots D_{i-1}[U_{ii-1}] \right) & \text{otherwise.} \end{cases}$$

A routine computation (cf. [13]) then reveals that (V_1, \dots, V_n) is a doubly twisted shift on $H_{\mathcal{E}}^2(\mathbb{D}^n)$. This is essentially a model example of twisted shifts. Observe that, since a twist is made by a family of commuting unitaries and a twisted shift is made of a twist, it is immediate that the class of doubly twisted isometries is larger than the doubly non-commuting isometries [5].

In order to formulate the notion of twisted weak shifts, we need a characterization of twisted shifts. Suppose $V = (V_1, \dots, V_n)$ is a twisted isometry. Clearly, $\prod_{i \in I_n} V_i$ is an isometry. The following lemma, in particular, explains the unitary part of $\prod_{i \in I_n} V_i$ in terms of \mathcal{H}_\emptyset as in Definition 4.7.

Lemma 5.3. *Let $V = (V_1, \dots, V_n)$ be a twisted isometry on \mathcal{H} , and let $\mathcal{S} \subseteq \mathcal{H}$ reduces V . Then*

$$\bigcap_{m \in \mathbb{Z}_+} (V_1 \cdots V_n)^m \mathcal{S} = \bigcap_{k \in \mathbb{Z}_+^n} V^k \mathcal{S}.$$

Proof. We prove it only for $n = 2$ as the remaining part can easily be proven by induction for any positive integer $n \geq 2$. Suppose $V = (V_1, V_2)$. Evidently $\bigcap_{m \in \mathbb{Z}_+} (V_1 V_2)^m \mathcal{S} \supseteq \bigcap_{k_1, k_2 \in \mathbb{Z}_+} V_1^{k_1} V_2^{k_2} \mathcal{S}$. For the reverse inclusion, suppose $x \in \bigcap_{m \in \mathbb{Z}_+} (V_1 V_2)^m \mathcal{S}$. Let U be the corresponding twist for (V_1, V_2) . Since \mathcal{S} reduce V_1 and V_2 , it follows that \mathcal{S} also reduces U (see (4.1)). Let $k_1, k_2 \in \mathbb{Z}_+$ and suppose $k_1 < k_2$. There exists $h \in \mathcal{S}$ (depending on k_2) such

that $x = (V_1 V_2)^{k_2} h$. By Lemma 4.3 again, there exist monomials η_1 and η_2 such that

$$\begin{aligned} x &= (V_1 V_2)^{k_1} (V_1 V_2)^{k_2 - k_1} h \\ &= V_1^{k_1} V_2^{k_1} (V_1 V_2)^{k_2 - k_1} (\eta_1(U^*) h) \\ &= V_1^{k_1} V_2^{k_2} (V_1^{k_2 - k_1} \eta_2(U) \eta_1(U^*) h). \end{aligned}$$

Since $\eta_2(U) \eta_1(U^*) \mathcal{S} \subseteq \mathcal{S}$, we have $x \in V_1^{k_1} V_2^{k_2} \mathcal{S}$, which proves the reverse inclusion. \square

We are now ready for the characterization of twisted shifts.

Proposition 5.4. *Let $V = (V_1, \dots, V_n)$ be a doubly twisted isometry on \mathcal{H} . Then V is a twisted shift if and only if $V_i|_{\mathcal{N}_{\{i\}^c}}$ and $V_j V_k$ are shifts for all $i, j, k \in I_n$ and $j \neq k$.*

Proof. If V is a twisted shift, then, by the fact that $\mathcal{N}_{\{i\}^c}$ reduces V_i (see Lemma 3.1), it follows that $V_i|_{\mathcal{N}_{\{i\}^c}}$ is a shift. Moreover, by Lemma 5.3, we have

$$\bigcap_{m \in \mathbb{Z}_+} (V_j V_k)^m \mathcal{H} = \bigcap_{m_j, m_k \in \mathbb{Z}_+} (V_j^{m_j} V_k^{m_k}) \mathcal{H} \subseteq \bigcap_{m_j \in \mathbb{Z}_+} V_j^{m_j} \mathcal{H} = \{0\},$$

as V_j is a shift. Therefore, $V_j V_k$ is a shift for all $j \neq k$. For the converse, suppose $V_i|_{\mathcal{N}_{\{i\}^c}}$ and $V_j V_k$ are shifts for all $j \neq k$ and $i \in I_n$. Our goal is to prove that V_i is a shift, that is, $\bigcap_{k_i \geq 0} V_i^{k_i} \mathcal{H} = 0$ for all $i \in I_n$. To this end, fix $i \in I_n$. Note that

$$\tilde{V} = (V_1, \dots, V_{i-1}, V_{i+1}, \dots, V_n),$$

is an $(n-1)$ -tuple of doubly twisted isometry with respect to the twist

$$\tilde{U} = \{U_{pq} : p < q \text{ and } p, q \neq i\}.$$

Therefore, by Theorem 3.7, \tilde{V} admits von Neumann-Wold decomposition $\mathcal{H} = \bigoplus_{A \subseteq \{i\}^c} \mathcal{H}_A$, where

$$\mathcal{H}_A = \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \left(\bigcap_{l \in \mathbb{Z}_+^{n-1-|A|}} V_{J \setminus A}^l(\mathcal{N}_A) \right) \quad (A \subseteq \{i\}^c),$$

and $J = I_n \setminus \{i\}$. Fix $A \subseteq \{i\}^c$ and $k_i \in \mathbb{Z}_+$. Suppose $m = n-1-|A|$. By Lemma 4.3, for each $k \in \mathbb{Z}_+^{|A|}$, there exists a monomial η_k such that $V_i^{k_i} V_A^k = V_A^k V_i^{k_i} \eta_k(U)$. We compute

$$\begin{aligned} V_i^{k_i} \mathcal{H}_A &= V_i^{k_i} \left(\bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \left(\bigcap_{l \in \mathbb{Z}_+^m} V_{J \setminus A}^l(\mathcal{N}_A) \right) \right) \\ &= \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k V_i^{k_i} \eta_k(U) \left(\bigcap_{l \in \mathbb{Z}_+^m} V_{J \setminus A}^l(\mathcal{N}_A) \right) \\ &= \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k V_i^{k_i} \left(\bigcap_{l \in \mathbb{Z}_+^m} V_{J \setminus A}^l(\mathcal{N}_A) \right) \\ &= \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \left(\bigcap_{l \in \mathbb{Z}_+^m} V_i^{k_i} V_{J \setminus A}^l(\mathcal{N}_A) \right), \end{aligned}$$

where the last but one equality follows from the fact that

$$\eta_k(U) \left(\bigcap_{l \in \mathbb{Z}_+^m} V_{J \setminus A}^l(\mathcal{N}_A) \right) = \bigcap_{l \in \mathbb{Z}_+^m} V_{J \setminus A}^l(\mathcal{N}_A).$$

Therefore

$$\bigcap_{k_i \in \mathbb{Z}_+} V_i^{k_i} \mathcal{H}_A = \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \left(\bigcap_{k_i \in \mathbb{Z}_+, l \in \mathbb{Z}_+^m} V_i^{k_i} V_{J \setminus A}^l(\mathcal{N}_A) \right).$$

If $A \subsetneq J = \{i\}^c$, there exists $j \in \{i\}^c \setminus A$, such that

$$(5.1) \quad \bigcap_{k_i \in \mathbb{Z}_+, l \in \mathbb{Z}_+^m} V_i^{k_i} V_{J \setminus A}^l(\mathcal{N}_A) = \bigcap_{k_i, l_j \in \mathbb{Z}_+, l' \in \mathbb{Z}_+^{m-1}} V_i^{k_i} V_j^{l_j} \left(V_{J \setminus A \cup \{j\}}^{l'}(\mathcal{N}_A) \right),$$

where $l = (l_1, \dots, l_m) \in \mathbb{Z}_+^m$. Applying Lemma 5.3, we have

$$\bigcap_{k_i, l_j \in \mathbb{Z}_+} V_i^{k_i} V_j^{l_j} \left(V_{J \setminus A \cup \{j\}}^{l'}(\mathcal{N}_A) \right) \subseteq \bigcap_{k_i, l_j \in \mathbb{Z}_+} V_i^{k_i} V_j^{l_j} \mathcal{H} = \bigcap_{m \in \mathbb{Z}_+} (V_i V_j)^m \mathcal{H} = \{0\},$$

for all $l' \in \mathbb{Z}_+^{m-1}$. Then (5.1) implies

$$\bigcap_{k_i \in \mathbb{Z}_+, l \in \mathbb{Z}_+^m} V_i^{k_i} V_{J \setminus A}^l(\mathcal{N}_A) = \{0\}.$$

Then the equality preceding (5.1) yields $\bigcap_{i \in \mathbb{Z}_+} V_i^{k_i} \mathcal{H}_A = \{0\}$ for all $A \subsetneq \{i\}^c$. Therefore

$$\begin{aligned} \bigcap_{k_i \in \mathbb{Z}_+} V_i^{k_i} \mathcal{H} &= \bigcap_{k_i \in \mathbb{Z}_+} V_i^{k_i} \mathcal{H}_{\{i\}^c} \\ &= \bigoplus_{k \in \mathbb{Z}_+^{n-1}} V_{\{i\}^c}^k \left(\bigcap_{k_i \in \mathbb{Z}_+} V_i^{k_i}(\mathcal{N}_{\{i\}^c}) \right). \end{aligned}$$

Since $V_i|_{\mathcal{N}_{\{i\}^c}}$ is a shift by assumption, we obtain $\bigcap_{k_i \in \mathbb{Z}_+} V_i^{k_i} \mathcal{H} = 0$, which completes the proof of the proposition. \square

Now we turn to twisted isometries. Let $V = (V_1, \dots, V_n)$ be a twisted isometry. Recall from Definition 4.1 the A -weak wandering subspace for V is given by

$$\mathcal{E}_A = \bigcap_{i \in A} \bigcap_{B \subseteq \{i\}^c} \bigcap_{k \in \mathbb{Z}_+^{|B|}} \ker V_i^* V_B^k,$$

for all nonempty $A \subseteq I_n$, and $\mathcal{E}_\emptyset = \mathcal{H}$. Also recall from Definition 4.7 that the A -wandering subspace for V is given by

$$\mathcal{W}_A = \bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l \mathcal{E}_A \quad (A \subseteq I_n).$$

Assume for a moment that V is doubly twisted. Fix $i \in A$. Let $B \subseteq \{i\}^c$ and $k \in \mathbb{Z}_+^{|B|}$. Since $i \notin B$ and V is doubly twisted, by Lemma 4.3, there exists a monomial η such that

$$V_i^* V_B^k = (\eta(U) V_B^k) V_i^*.$$

Since $\eta(U)V_B^k$ is an isometry, it follows that $\mathcal{E}_A = \bigcap_{i \in A} \ker V_i^* = \mathcal{N}_A$, and hence

$$(5.2) \quad \mathcal{W}_A = \bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l \mathcal{N}_A \quad (A \subseteq I_n),$$

the wandering subspace of doubly twisted isometries (see Theorem 3.7). In view of this observation and Proposition 5.4, we are now in a position to define a weaker version of shift that fits appropriately in orthogonal decompositions of twisted isometries.

Definition 5.5 (Twisted weak shift). A twisted isometry $V = (V_1, \dots, V_n)$ is said to be a twisted weak shift if

- (1) $V_i|_{\mathcal{E}_{\{i\}}^c}$ is a shift for all $i \in I_n$, and
- (2) $V_j V_k|_{\mathcal{E}_{\{j,k\}}^c}$ is a shift for all $j, k \in I_n$ and $j \neq k$.

Lemma 4.6 ensures that the above definition is consistent. Moreover, in the case of pairs of commuting isometries, the above definition coincides with Popovici's weak bi-shift.

6. ORTHOGONAL DECOMPOSITIONS OF TWISTED ISOMETRIES

In this section, we prove that a twisted isometry $V = (V_1, \dots, V_n)$ admits orthogonal decompositions in the sense of Definition 1.2. Moreover, we prove that $V|_{\mathcal{H}_{I_n}}$ is a twisted weak shift. For the case of commuting pairs of isometries, our result recovers the Popovici decomposition. First, we prove that V indeed admits an orthogonal decomposition. Recall that the A -wandering subspace for V is defined by (see Definition 4.7)

$$\mathcal{W}_A = \bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l \mathcal{E}_A \quad (A \subseteq I_n),$$

where (see Definition 4.1) $\mathcal{E}_\emptyset = \mathcal{H}$ and

$$\mathcal{E}_A := \bigcap_{i \in A} \bigcap_{B \subseteq \{i\}^c} \bigcap_{k \in \mathbb{Z}_+^{|B|}} \left(\ker V_i^* V_B^k \right),$$

for all $A \subseteq I_n$ such that $A \neq \emptyset$.

Proposition 6.1. *Let $V = (V_1, \dots, V_n)$ be a twisted isometry. For each $A \subsetneq I_n$, define*

$$\mathcal{H}_{V,A} := \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \mathcal{W}_A.$$

Then the following holds:

- (1) $\mathcal{H}_{V,A}$ reduces V_i for all $i \in I_n$.
- (2) $V_i|_{\mathcal{H}_{V,A}}$ is a shift for all $i \in A$.
- (3) $V_i|_{\mathcal{H}_{V,A}}$ is a unitary for all $i \notin A$.
- (4) The n -tuple $V|_{\mathcal{H}_{V,A}}$ on $\mathcal{H}_{V,A}$ is doubly twisted.

Moreover, $\mathcal{H}_{V,A}$ is maximal, that is, if a closed subspace $\mathcal{K}_{V,A} \subseteq \mathcal{H}$ satisfies the above four conditions, then $\mathcal{K}_{V,A} \subseteq \mathcal{H}_{V,A}$.

Proof. Suppose $A = \emptyset$. Then $\mathcal{H}_{V,\emptyset} = \bigcap_{k \in \mathbb{Z}_+^n} V^k \mathcal{H}$. By part (3) of Lemma 4.8, for each $i \in I_n$, we have $V_i \mathcal{H}_{V,\emptyset} = \mathcal{H}_{V,\emptyset}$ and hence $V_i^* \mathcal{H}_{V,\emptyset} = \mathcal{H}_{V,\emptyset}$. Therefore, $\mathcal{H}_{V,\emptyset}$ reduces V_i and $V_i|_{\mathcal{H}_{V,\emptyset}}$ is a unitary for all $i \in I_n$. Next, suppose $A (\neq \emptyset)$ is a proper subset of I_n . Fix $i \in A$. For each $k \in \mathbb{Z}_+^{|A|}$, there exists a monomial η_k such that $V_i V_A^k = V_A^k V_i \eta_k(U)$ (see Lemma 4.3). By part (2) of Lemma 4.8, we know that $\eta_k(U) \mathcal{W}_A = \mathcal{W}_A$. Therefore

$$\begin{aligned} V_i \mathcal{H}_{V,A} &= V_i \left(\bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \mathcal{W}_A \right) \\ &= \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^{k+e_i} \left(\eta_k(U) \mathcal{W}_A \right) \\ &= \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^{k+e_i} \mathcal{W}_A, \end{aligned}$$

and hence $V_i \mathcal{H}_{V,A} \subseteq \mathcal{H}_{V,A}$. Now we prove that $V_i^* \mathcal{H}_{V,A} \subseteq \mathcal{H}_{V,A}$. Fix $k \in \mathbb{Z}_+^{|A|}$. If $k_i = 0$, then, $V_i^* V_A^k(\xi) = 0$, as $\xi \in \mathcal{W}_A \subseteq \mathcal{E}_A$. If $k_i > 0$, then by Lemma 4.3, there exist monomials η_1 and η_2 such that

$$\begin{aligned} V_i^* V_A^k(\xi) &= V_i^* V_i^{k_i} V_A^{k-k_i e_i} \eta_1(U^*)(\xi) \\ &= V_i^{k_i-1} V_A^{k-k_i e_i} \eta_1(U^*)(\xi) = V_A^{k-e_i} \eta_2(U) \eta_1(U^*)(\xi). \end{aligned}$$

Again, by part (2) of Lemma 4.8, $\eta_2(U) \eta_1(U^*)(\xi) \in \mathcal{W}_A$. This implies $\mathcal{H}_{V,A}$ reduces V_i for all $i \in A$. If $i \notin A$, then part (3) of Lemma 4.8 yields $V_i \mathcal{H}_{V,A} = \mathcal{H}_{V,A}$, which completes the proof of (1).

Next, for each $i \in I_n$, we set $\tilde{V}_i := V_i|_{\mathcal{H}_{V,A}}$ and $\tilde{V} := V|_{\mathcal{H}_{V,A}}$. Clearly, \tilde{V}_i is a shift for all $i \in A$ and unitary for all $i \notin A$. This proves (2) and (3). To check (4), we set

$$\tilde{U}_{ij} := U_{ij}|_{\mathcal{H}_{V,A}} \quad (i \neq j).$$

Clearly, $\tilde{V}_i \tilde{V}_j = \tilde{U}_{ij} \tilde{V}_j \tilde{V}_i$ for all $i \neq j$. Fix $i \neq j$ in I_n . If any one of \tilde{V}_i and \tilde{V}_j is unitary, then evidently $\tilde{V}_i^* \tilde{V}_j = \tilde{U}_{ij}^* \tilde{V}_j \tilde{V}_i^*$. Suppose now that both \tilde{V}_i and \tilde{V}_j are shifts. Of course, in this case $i, j \in A$. Let $V_A^k(\xi) \in \mathcal{H}_{V,A}$ for some $k \in \mathbb{Z}_+^{|A|}$ and $\xi \in \mathcal{W}_A$. If $k_i = 0$, then as $\xi \in \mathcal{W}_A \subseteq \mathcal{E}_A$, it follows that

$$\tilde{V}_i^* \tilde{V}_j (V_A^k \xi) = 0 = \tilde{V}_j \tilde{V}_i^* (V_A^k \xi).$$

If $k_i > 0$ and $i \leq j$, then by Lemma 4.3, there exist monomials η_1, η_2 , and η_3 such that

$$\begin{aligned} \tilde{V}_i^* \tilde{V}_j (V_A^k \xi) &= V_i^* V_A^{k+e_j} \eta_1(U) \xi \\ &= V_i^* V_i^{k_i} V_A^{k-k_i e_i+e_j} \eta_2(U^*) \eta_1(U) \xi \\ &= V_A^{k-e_i+e_j} \eta_3(U) \eta_2(U^*) \eta_1(U) \xi, \end{aligned}$$

and

$$\begin{aligned}
 \tilde{V}_j \tilde{V}_i^* (V_A^k \xi) &= V_j V_i^* V_A^k \xi \\
 &= V_j V_i^* V_i^{k_i} V_A^{k-k_i e_i} \eta_2(U^*) \xi \\
 &= V_j V_A^{k-e_i} \eta_3(U) \eta_2(U^*) \xi \\
 &= V_A^{k-e_i+e_j} (U_{ji}^*) \eta_1(U) \eta_3(U) \eta_2(U^*) \xi \\
 &= U_{ji}^* V_i^* V_j V_A^k \xi \\
 &= \tilde{U}_{ji}^* \tilde{V}_i^* \tilde{V}_j (V_A^k(\xi)),
 \end{aligned}$$

where the last but one equality follows from the fact that $V_j V_A^k = V_A^{k+e_j} \eta_1(U)$, and

$$V_j V_A^{k-e_i} = V_A^{k-e_i+e_j} (U_{ji}^*) \eta_1(U),$$

whenever $i < j$. Similarly, for $i > j$, it follows that $\tilde{V}_i^* \tilde{V}_j = \tilde{U}_{ij}^* \tilde{V}_j \tilde{V}_i^*$, which proves that $V|_{\mathcal{H}_{V,A}}$ is doubly twisted corresponding to the twist $\{\tilde{U}_{ij}\}_{i < j}$.

To prove the maximality of $\mathcal{H}_{V,A}$, consider a closed subspace $\mathcal{K}_{V,A} \subseteq \mathcal{H}$ satisfying all the four conditions. Applying Theorem 3.7 to the doubly twisted isometry $V|_{\mathcal{K}_{V,A}}$ on $\mathcal{K}_{V,A}$, we have

$$\mathcal{K}_{V,A} = \bigoplus_{k \in \mathbb{Z}_+^{|B|}} V_A^k \left(\bigcap_{l \in \mathbb{Z}_+^{n-|A|}} V_{I_n \setminus A}^l \left(\bigcap_{i \in A} \ker V_i^* \cap \mathcal{K}_{V,A} \right) \right).$$

Let $i \in A$, $B \subseteq \{i\}^c$, and suppose $x \in (\bigcap_{i \in A} \ker V_i^*) \cap \mathcal{K}_{V,A}$. Since $V|_{\mathcal{K}_{V,A}}$ is doubly twisted, it follows that $V_i^* V_B^k x = 0$, and hence $x \in \mathcal{E}_A$. Therefore, if $h \in \mathcal{K}_{V,A}$, then for any $l \in \mathbb{Z}_+^{|B|}$ and $B \subseteq A^c$, we can write

$$h = \sum_{k \in \mathbb{Z}_+^{|B|}} V_A^k (V_B^l x_B),$$

for some $x_B \in (\bigcap_{i \in A} \ker V_i^*) \cap \mathcal{K}_{V,A}$. Since $x_B \in \mathcal{E}_A$, it follows that $h \in \mathcal{H}_{V,A}$, which proves that $\mathcal{H}_{V,A}$ is maximal and completes the proof of the proposition. \square

We are finally ready for orthogonal decompositions of twisted isometries.

Theorem 6.2. *Let $V = (V_1, \dots, V_n)$ be a twisted isometry on \mathcal{H} . Then there is a unique orthogonal decomposition*

$$\mathcal{H} = \bigoplus_{A \subseteq I_n} \mathcal{H}_{V,A},$$

where

- (1) $\mathcal{H}_{V,A}$ reduces V for all $A \subseteq I_n$.
- (2) $\mathcal{H}_{V,A}$ is maximal for all $A \subsetneq I_n$ in the sense that $V_i|_{\mathcal{H}_{V,A}}$ is a shift if $i \in A$ and $V_j|_{\mathcal{H}_{V,A}}$ is a unitary if $j \in A^c$.
- (3) $V|_{\mathcal{H}_{V,I_n}}$ is a twisted weak shift.

Proof. For $A \subsetneq I_n$, define $\mathcal{H}_{V,A}$ as in Proposition 6.1 and set $\mathcal{H}_{V,I_n} := \mathcal{H} \ominus \bigoplus_{A \subsetneq I_n} \mathcal{H}_{V,A}$. Then (1) and (2) follow from Proposition 6.1. For the last part we need to prove that

- (a) $V_i|_{\mathcal{E}_{\{i\}^c} \cap \mathcal{H}_{V, I_n}}$ is a shift for all $i \in I_n$, and
- (b) $V_j V_k|_{\mathcal{E}_{\{j, k\}^c} \cap \mathcal{H}_{V, I_n}}$ is a shift for all $j \neq k$.

To this end, fix $i, j, k \in I_n$. Let $\{i\} = I_n \setminus A$ and $\{j, k\} = I_n \setminus B$. Then by the definition of $\mathcal{H}_{V, A}$ and $\mathcal{H}_{V, B}$ (see Proposition 6.1), we have

$$\mathcal{H}_{V, A} = \bigoplus_{k \in \mathbb{Z}_+^{|A|}} V_A^k \left(\bigcap_{m \in \mathbb{Z}_+} V_i^m \mathcal{E}_{\{i\}^c} \right),$$

and

$$\mathcal{H}_{V, B} = \bigoplus_{k \in \mathbb{Z}_+^{|B|}} V_B^k \left(\bigcap_{l \in \mathbb{Z}_+^2} V_{\{j, k\}}^l \mathcal{E}_{\{j, k\}^c} \right).$$

Since $\mathcal{H}_{V, I_n} \perp \mathcal{H}_{V, C}$ for all $C \subsetneq I_n$, in particular, we have

$$\mathcal{H}_{V, I_n} \bigcap \left(\bigcap_{m \in \mathbb{Z}_+} V_i^m \mathcal{E}_{\{i\}^c} \right) = \{0\} = \mathcal{H}_{V, I_n} \bigcap \left(\bigcap_{l \in \mathbb{Z}_+^2} V_{\{j, k\}}^l \mathcal{E}_{\{j, k\}^c} \right),$$

that is, the unitary parts of $V_i|_{\mathcal{E}_{\{i\}^c} \cap \mathcal{H}_{V, I_n}}$ and $V_j V_k|_{\mathcal{E}_{\{j, k\}^c} \cap \mathcal{H}_{V, I_n}}$ are trivial. This proves that $V|_{\mathcal{H}_{V, I_n}}$ is a twisted weak shift. \square

Clearly, in the case of commuting pairs of isometries, our result recovers the Popovici decomposition. It is also worth noting that the extension of Popovici decomposition from pairs of isometries to n -tuples of commuting isometries, $n > 3$, appears to be somewhat less obvious. For instance, the choice of wandering subspaces and the weak shift counterpart for n -tuples of isometries does not directly follow from the case of pairs of isometries.

Acknowledgement: The first author thanks Harish-Chandra Research Institute, Prayagraj, and ISI Bangalore for the Postdoctoral fellowships. The second named author is supported in part by Core Research Grant (CRG/2019/000908), by SERB (DST), Government of India.

REFERENCES

- [1] M. Bożejko and R. Speicher, *An example of a generalized Brownian motion*, Comm. Math. Phys. 137 (1991), 519–531.
- [2] L. A. Coburn, *The C^* -algebra generated by an isometry*, Bull. Amer. Math. Soc. 73 (1967), 722–726.
- [3] D. Gaspar and N. Suciu, *Wold decompositions for commutative families of isometries*, An. Univ. Timișoara Ser. Științ. Mat. 27 (1989), 31–38.
- [4] H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, II, Acta Math. 106 (1961), 175–213.
- [5] M. de Jeu and P. R. Pinto, *The structure of doubly non-commuting isometries*, Adv. Math. 368 (2020), 107–149.
- [6] P. Jørgensen, D. Proskurin and Y. Samoilenko, *On C^* -algebras generated by pairs of q -commuting isometries*, J. Phys. A. 38 (2005), 2669–2680.
- [7] P. Jørgensen, L. Schmitt and R. Werner, *Positive representations of general commutation relations allowing Wick ordering*, J. Funct. Anal. 134 (1995), 33–99.
- [8] M. Kosiek and A. Octavio, *Wold-type extension for N -tuples of commuting contractions*, Studia Math. 137, (1999), 81–91.
- [9] A. Kuzmin, V. Ostrovskiy, D. Proskurin, M. Weber and R. Yakymiv, *On q -tensor products of Cuntz algebras*, Internat. J. Math. 33 (2022), no. 2, Paper No. 2250017, 46 pp.

- [10] P. Masani and N. Wiener, *The prediction theory of multivariate stochastic processes*, Part I.: Acta Math. 98, 111–150: part II. Acta Math. 99, 93–137 (1958).
- [11] A. Miamee and M. Pourahmadi, *Wold decomposition, prediction and parameterization of stationary processes with infinite variance*, Prob. Theory and Related Fields. 79, 145–164 (1988).
- [12] D. Popovici, *A Wold-type decomposition for commuting isometric pairs*, Proc. Amer. Math. Soc. 132(8)(2004), 2303-2314.
- [13] N. Rakshit, J. Sarkar and M. Suryawanshi, *Orthogonal decompositions and twisted isometries*, International Journal of Mathematics. 33(08), 2250058 (2022). [arXiv:2104.07628, 2021].
- [14] J. Sarkar, *Wold decomposition for doubly commuting isometries*, Linear Algebra Appl. 445 (2014), 289-314.
- [15] M. Słociński, *On Wold type decompositions of a pair of commuting isometries*, Ann. Pol. Math. 37 (1980), 255-262.
- [16] I. Suciú, *On the semi-groups of isometries*, Studia Math. 30 (1968), 101–110.
- [17] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North Holland, Amsterdam, 1970.
- [18] T. Omland, *C^* -algebras generated by projective representations of free nilpotent groups*, J. Operator Theory 73 (2015), 3-25.
- [19] J. von Neumann, *Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes*, Math. Nachr. 4 (1951), 258-281.
- [20] M. Weber, *On C^* -algebras generated by isometries with twisted commutation relations*, J. Func. Anal. 264 (2013), 1975-2004.
- [21] H. Wold, *A study in the analysis of stationary time series*, Almqvist and Wiksell, Uppsala, 1938.

NARAYAN RAKSHIT, DEPARTMENT OF MATHEMATICS, VISVESVARAYA NATIONAL INSTITUTE OF TECHNOLOGY, SOUTH AMBAZARI ROAD, AMBAZARI, NAGPUR, MAHARASHTRA 440010, INDIA

Email address: narayan753@gmail.com

J. SARKAR, INDIAN STATISTICAL INSTITUTE, STATISTICS AND MATHEMATICS UNIT, 8TH MILE, MYSORE ROAD, BANGALORE, 560059, INDIA

Email address: jay@isibang.ac.in, jaydeb@gmail.com

MANSI SURYAWANSHI, STATISTICS AND MATHEMATICS UNIT, INDIAN STATISTICAL INSTITUTE, 8TH MILE, MYSORE ROAD, BANGALORE, KARNATAKA - 560059, INDIA

Email address: mansisuryawanshi1@gmail.com