INNER MULTIPLIERS AND RUDIN TYPE INVARIANT SUBSPACES

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Abstract. Let $\mathcal{E}$ be a Hilbert space and $H^2_{\mathcal{E}}(\mathbb{D})$ be the $\mathcal{E}$-valued Hardy space over the unit disc $\mathbb{D}$ in $\mathbb{C}$. The well known Beurling-Lax-Halmos theorem states that every shift invariant subspace of $H^2_{\mathcal{E}}(\mathbb{D})$ other than $\{0\}$ has the form $\Theta H^2_{\mathcal{E}_*}(\mathbb{D})$, where Θ is an operator-valued inner multiplier in $H^\infty_{B(\mathcal{E}_*;\mathcal{E})}(\mathbb{D})$ for some Hilbert space $\mathcal{E}_*$. In this paper we identify $H^2(\mathbb{D}^n)$ with $H^2(\mathbb{D}^{n-1})$-valued Hardy space $H^2_{H^2(\mathbb{D}^{n-1})}(\mathbb{D})$ and classify all such inner multiplier $\Theta \in H^\infty_{\mathcal{B}(H^2(\mathbb{D}^{n-1}))}(\mathbb{D})$ for which $\Theta H^2_{H^2(\mathbb{D}^{n-1})}(\mathbb{D})$ is a Rudin type invariant subspace of $H^2(\mathbb{D}^n)$.

Notation

$\mathcal{E}$	Separable Hilbert space.
$\mathbb{N}$	Set of all natural numbers including 0.
$\mathbb{N}^n$	$\{\boldsymbol{k} = (k_1, \dots, k_n) : k_i \in \mathbb{N}, i = 1, \dots, n\}$.
$\mathbb{C}^n$	Complex n-space.
$\boldsymbol{z}$	$(z_1, \dots, z_n) \in \mathbb{C}^n$.
$\boldsymbol{z}^{\boldsymbol{k}}$	$z_1^{k_1} \cdots z_n^{k_n}$.
$\mathbb{D}^n$	Open unit polydisc $\{\boldsymbol{z} : \lvert z_i\rvert < 1, i = 1, \dots, n\}$.
$\mathbb{T}^n$	$\{\boldsymbol{z} : \lvert z_i\rvert = 1, i = 1, \dots, n\}$- distinguished boundary of $\mathbb{D}^n$.

Throughout this article, we denote by $\mathcal{B}(\mathcal{E}_*;\mathcal{E})$ the space of all bounded linear operators from $\mathcal{E}_*$ to $\mathcal{E}$ and simply write $\mathcal{B}(\mathcal{E})$ when $\mathcal{E} = \mathcal{E}_*$. For a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, $P_{\mathcal{S}}$ denotes the orthogonal projection onto $\mathcal{S}$.

1. Introduction

The $\mathcal{E}$-valued Hardy space over $\mathbb{D}^n$ is denoted by $H^2_{\mathcal{E}}(\mathbb{D}^n)$ and defined by

$$H^2_{\mathcal{E}}(\mathbb{D}^n) := \{f(\boldsymbol{z}) = \sum_{\boldsymbol{k}\in\mathbb{N}^n} \boldsymbol{z}^{\boldsymbol{k}} \eta_{\boldsymbol{k}} : \ \|f\|^2 := \sum_{\boldsymbol{k}\in\mathbb{N}^n} \|\eta_{\boldsymbol{k}}\|^2_{\mathcal{E}} < \infty, \ \boldsymbol{z} \in \mathbb{D}^n\}.$$

A closed subspace $\mathcal{S} \subseteq H^2_{\mathcal{E}}(\mathbb{D}^n)$ is said to be a *shift invariant subspace*, or simply an *invariant subspace*, of $H^2_{\mathcal{E}}(\mathbb{D}^n)$ if $\mathcal{S}$ is invariant under the shift operators $\{M_{z_1}, \dots, M_{z_n}\}$, that is, if $M_{z_i}\mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \dots, n$. Here the tuple of *shift* operators $\{M_{z_1}, \dots, M_{z_n}\}$ on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ is defined by

$$\left(M_{z_i} f\right)(\boldsymbol{w}) = w_i f(\boldsymbol{w}), \qquad (f \in H^2_{\mathcal{E}}(\mathbb{D}^n), \boldsymbol{w} \in \mathbb{D}^n)$$

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for all $i = 1, \ldots, n$. The Banach space of all $\mathcal{B}(\mathcal{E}_*; \mathcal{E})$-valued bounded analytic functions on $\mathbb{D}^n$ is denoted by $H^\infty_{\mathcal{B}(\mathcal{E}_*;\mathcal{E})}(\mathbb{D}^n)$. Each $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*;\mathcal{E})}(\mathbb{D}^n)$ induces a bounded linear map $M_\Theta \in \mathcal{B}(H^2_{\mathcal{E}_*}(\mathbb{D}^n); H^2_{\mathcal{E}}(\mathbb{D}^n))$ defined by

$$(M_\Theta f)(\boldsymbol{w}) = \Theta(\boldsymbol{w}) f(\boldsymbol{w}). \qquad (f \in H^2_{\mathcal{E}_*}(\mathbb{D}^n), \boldsymbol{w} \in \mathbb{D}^n)$$

The elements of $H^\infty_{\mathcal{B}(\mathcal{E}_*;\mathcal{E})}(\mathbb{D}^n)$ are called the *multipliers* and are determined by

$$\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*;\mathcal{E})}(\mathbb{D}^n) \Leftrightarrow M_{z_i} M_\Theta = M_\Theta M_{z_i}, \qquad \forall i = 1, \ldots, n$$

where the shift M_{z_i} on the left hand side and the right hand side act on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ and $H^2_{\mathcal{E}_*}(\mathbb{D}^n)$ respectively. A multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*;\mathcal{E})}(\mathbb{D}^n)$ is said to be *inner* if M_Θ is an isometry, or equivalently, $\Theta(\boldsymbol{z}) \in \mathcal{B}(\mathcal{E}_*; \mathcal{E})$ is an isometry almost everywhere with respect to the Lebesgue measure on $\mathbb{T}^n$.

Inner multipliers are among the most important tools for classifying invariant subspaces of reproducing kernel Hilbert spaces. For instance:

Theorem 1.1. (Beurling-Lax-Halmos [6]) *A non-zero closed subspace $\mathcal{S} \subseteq H^2_{\mathcal{E}}(\mathbb{D})$ is shift invariant if and only if there exists an inner multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*;\mathcal{E})}(\mathbb{D})$ such that*

$$\mathcal{S} = \Theta H^2_{\mathcal{E}_*}(\mathbb{D}),$$

for some Hilbert space $\mathcal{E}_*$.

For the Hardy space $H^2(\mathbb{D}^n)$, $n \geq 2$, Beurling-Lax-Halmos theorem and most of its corollaries turns out to be false in general (see Rudin [8]). In fact, it is shown in [10] that Beurling-Lax-Halmos theorem holds for an invariant subspace of $H^2(\mathbb{D}^n)$ if and only if it is doubly commuting. Recall that a closed shift-invariant subspace $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$ is said to be *doubly commuting* if

$$R_{z_i} R^*_{z_j} = R^*_{z_j} R_{z_i}, \qquad (1 \leq i \neq j \leq n)$$

where

$$R_{z_i} = M_{z_i}|_{\mathcal{S}}. \qquad (i = 1, \ldots, n)$$

Theorem 1.2 ([10]). *Let $\mathcal{S} \neq \{0\}$ be a closed shift-invariant subspace of $H^2(\mathbb{D}^n)$, $n \geq 2$. Then the following are equivalent.*

(i) *$\mathcal{S}$ is a doubly commuting shift-invariant subspace.*
(ii) *$\mathcal{S} = \varphi H^2(\mathbb{D}^n)$ for some inner function φ in $H^2(\mathbb{D}^n)$.*

The analytic structure of invariant subspaces of $H^2(\mathbb{D}^n)$, $n \geq 2$, is more complicated than that of the Hardy space $H^2(\mathbb{D})$ (see [1], [2], [4], [8], [13], [14], [15]).

Now let $n \geq 2$ and $\Theta \in H^\infty_{\mathcal{B}(H^2(\mathbb{D}^{n-1}))}(\mathbb{D})$ be an inner multiplier. Then $\Theta H^2_{H^2(\mathbb{D}^{n-1})}(\mathbb{D}) \subseteq H^2_{H^2(\mathbb{D}^{n-1})}(\mathbb{D})$ and hence by identifying $H^2_{H^2(\mathbb{D}^{n-1})}(\mathbb{D})$ with $H^2(\mathbb{D}^n)$, that $\Theta H^2_{H^2(\mathbb{D}^{n-1})}(\mathbb{D})$ is a closed M_{z_1}-invariant subspace of $H^2(\mathbb{D}^n)$.

Thus, it is natural ask to what extent the structure of inner multipliers determines the structure of invariant subspaces. That is, how to determine inner multiplier $\Theta \in H^\infty_{\mathcal{B}(H^2(\mathbb{D}^{n-1}))}(\mathbb{D})$ such that $\Theta H^2_{H^2(\mathbb{D}^{n-1})}(\mathbb{D})$ is an invariant subspace of $H^2(\mathbb{D}^n)$?

The purpose of this paper is to study the above problem for a special class of inner multipliers (see the definition (2.1) in the next section) and to provide a general recipe for producing invariant subspaces of $H^2(\mathbb{D}^n)$. More precisely, our purpose here is to deduce more detailed structure of invariant subspaces of $H^2(\mathbb{D}^n)$, $n \geq 2$, from Beurling-Lax-Halmos inner multipliers. We refer to [7] and [9] for some closely related constructions of inner multipliers.

The approach that we will take is inspired by the recent work of Y. Yang [17]. However, our results improve and generalize many results proved for the base case $n = 2$ in [17].

The paper is organized as follows. In section 2 we introduce some notations and definitions. Our main results are in Section 3. The last section of the paper, Section 4, is devoted to the study of the unitarily equivalent invariant subspaces of $H^2(\mathbb{D}^n)$.

2. Notations and definitions

We will often identify $H^2(\mathbb{D}^n)$ with the n-fold Hilbert space tensor product $H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$ via the unitary map $H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D}) \ni z^{k_1} \otimes \cdots \otimes z^{k_n} \mapsto z^{\boldsymbol{k}} \in H^2(\mathbb{D}^n)$, $\boldsymbol{k} \in \mathbb{N}^n$. Therefore we can, and do, identify M_{z_i} with

$$I_{H^2(\mathbb{D})} \otimes \cdots \otimes \underbrace{M_z}_{\text{i-th place}} \otimes \cdots \otimes I_{H^2(\mathbb{D})}. \qquad (i = 1, \ldots, n)$$

A sequence of inner functions $\{\varphi_j\}_{j=1}^{\infty}$ in $H^\infty(\mathbb{D}^n)$ is said to be increasing (respectively, decreasing) if $\frac{\varphi_{j+1}}{\varphi_j}$ (respectively, $\frac{\varphi_j}{\varphi_{j+1}}$) is a non-constant inner function for every $j \geq 1$. An inner sequence is a sequence of inner functions $\{\varphi_j\}_{j=1}^{\infty}$ in $H^\infty(\mathbb{D}^n)$ which is either increasing or decreasing.

A sequence of pairwise orthogonal projections $\{P_j\}_{j=1}^{\infty}$ in $\mathcal{B}(H^2(\mathbb{D}^n))$ is said to be a *sequence of orthogonal complementary projections* if

$$\sum_{j=1}^{\infty} P_j = I_{H^2(\mathbb{D}^n)},$$

in strong operator topology. The set of sequences of orthogonal complementary projections in $\mathcal{B}(H^2(\mathbb{D}^n))$ will be denoted by $\mathcal{P}_n$.

From now on, we will assume that $n \geq 2$.

Let $\{P_j\}_{j=1}^{\infty} \in \mathcal{P}_{n-1}$ and $\{\varphi_j\}_{j=1}^{\infty} \subseteq H^\infty(\mathbb{D})$ be an inner sequence. Then

$$\Theta(z) = \sum_{j=1}^{\infty} \varphi_j(z) P_j, \qquad (z \in \mathbb{D}) \tag{2.1}$$

is a $\mathcal{B}(H^2(\mathbb{D}^{n-1}))$-valued analytic function on $\mathbb{D}$.

The following lemma is an immediate consequence of the definition.

Lemma 2.1. *Let Θ be as in* (2.1)*. Then $\Theta \in H^\infty_{\mathcal{B}(H^2(\mathbb{D}^{n-1}))}(\mathbb{D})$ is an inner multiplier.*

The primary goal of this paper is to present a complete characterization of inner multipliers, defined above (depending on $\{P_j\}_{j=1}^{\infty} \in \mathcal{P}_{n-1}$ and inner sequence $\{\varphi_j\}_{j=1}^{\infty}$), for which the corresponding closed subspaces of $H^2(\mathbb{D}^n)$ are shift invariant. Our approach is also related to the study of Rudin type invariant subspaces of $H^2(\mathbb{D}^n)$.

An invariant subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$ is said to be of Rudin type if there exists an integer $1 \leq k < n$, an increasing sequence of inner functions $\{\varphi_j\}_{j=1}^{\infty} \subseteq H^\infty(\mathbb{D}^k)$ and a decreasing sequence of inner functions $\{\psi_j\}_{j=1}^{\infty} \subseteq H^\infty(\mathbb{D}^{n-k})$ such that

$$\mathcal{S} = \bigvee_{j=1}^{\infty} \varphi_j H^2(\mathbb{D}^k) \otimes \psi_j H^2(\mathbb{D}^{n-k}).$$

These invariant subspaces, also known as inner sequence based invariant subspaces of $H^2(\mathbb{D}^n)$, have been studied extensively by various authors in different contexts (see [3], [4], [5], [8], [11], [12]).

3. Inner multipliers and invariant subspaces

In this section, we will prove the main result concerning inner multipliers based shift invariant subspaces of $H^2(\mathbb{D}^n)$. To begin with, we prove a result concerning invariant subspace corresponding to a sequence of orthogonal complementary projections in $H^2(\mathbb{D}^n)$, which will be used to establish our main result.

Lemma 3.1. *Let $\{P_j\}_{j=1}^{\infty} \in \mathcal{P}_n$ and $\mathcal{S}_k := \bigoplus_{j=k}^{\infty} \operatorname{Ran} P_j$ be an invariant subspace of $H^2(\mathbb{D}^n)$ for each $k \geq 1$. Then the following are equivalent*

(i) *$\mathcal{S}_k$ is doubly commuting for all $k \geq 1$.*

(ii) *For all $1 \leq p \neq q \leq n$ and $j \geq 1$,*

$$P_{\mathcal{S}_l} M_{z_p} P_j M_{z_q}^* P_{\mathcal{S}_m} = 0. \quad (l, m \geq j+1)$$

(iii) *For all $1 \leq p \neq q \leq n$ and $j \geq 1$,*

$$P_l M_{z_p} P_j M_{z_q}^* P_m = 0. \quad (l, m \geq j+1)$$

Proof. It is easy to see that (ii) $\Leftrightarrow$ (iii). Therefore, it is enough to prove that (i) $\Leftrightarrow$ (ii).
We first prove that (i) implies (ii). Let $\mathcal{S}_k$, $k \geq 1$, be a doubly commuting subspace . By Theorem 1.2, there is an increasing sequence of inner functions $\{\varphi_j\}_{j=1}^{\infty} \subseteq H^\infty(\mathbb{D}^n)$ such that $\mathcal{S}_k = \varphi_k H^2(\mathbb{D}^n)$, $k \geq 1$. Then

$$\operatorname{Ran} P_j = \varphi_j H^2(\mathbb{D}^n) \ominus \varphi_{j+1} H^2(\mathbb{D}^n),$$

and

$$P_j = M_{\varphi_j} M_{\varphi_j}^* - M_{\varphi_{j+1}} M_{\varphi_{j+1}}^* = M_{\varphi_j}(I - M_{\xi_j} M_{\xi_j}^*) M_{\varphi_j}^*,$$

where $\varphi_{j+1} = \xi_j \varphi_j$ for some inner function $\xi_j \in H^\infty(\mathbb{D}^n)$, $j \geq 1$. Consequently for each $j \geq 1$ and $1 \leq p < q \leq n$, we have

$$\begin{aligned} P_{\mathcal{S}_{j+1}} M_{z_p} P_j M_{z_q}^* P_{\mathcal{S}_{j+1}} &= M_{\varphi_{j+1}}(M_{\varphi_{j+1}}^* M_{\varphi_j} M_{z_p}(I - M_{\xi_j} M_{\xi_j}^*) M_{z_q}^* M_{\varphi_j}^* M_{\varphi_{j+1}}) M_{\varphi_{j+1}}^* \\ &= M_{\varphi_{j+1}}(M_{\xi_j}^* M_{z_p}(I - M_{\xi_j} M_{\xi_j}^*) M_{z_q}^* M_{\xi_j}) M_{\varphi_{j+1}}^* \\ &= M_{\varphi_{j+1}} M_{\xi_j}^* (M_{z_q}^* M_{z_p} - M_{\xi_j} M_{z_p} M_{z_q}^* M_{\xi_j}^*) M_{\xi_j} M_{\varphi_{j+1}}^* \\ &= M_{\varphi_{j+1}}(M_{\xi_j}^* M_{z_q}^* M_{z_p} M_{\xi_j} - M_{z_p} M_{z_q}^*) M_{\varphi_{j+1}}^* \\ &= 0. \end{aligned}$$

Finally, by multiplying the above on the left and right by $P_{\mathcal{S}_l}$ and $P_{\mathcal{S}_m}$ $(l, m > j)$, respectively, we get the desired equality.

We now prove that (ii) implies (i). Let $1 \leq p < q \leq n$ and $k \geq 1$. Then

$$\begin{aligned}
P_{\mathcal{S}_k} M_{z_q}^* M_{z_p}|_{\mathcal{S}_k} &= P_{\mathcal{S}_k} M_{z_p} M_{z_q}^*|_{\mathcal{S}_k} = P_{\mathcal{S}_k} M_{z_p}\left(P_{\mathcal{S}_k} + P_{\mathcal{S}_k^{\perp}}\right) M_{z_q}^*|_{\mathcal{S}_k} \\
&= P_{\mathcal{S}_k} M_{z_p} P_{\mathcal{S}_k} M_{z_q}^*|_{\mathcal{S}_k} + P_{\mathcal{S}_k} M_{z_p}\left(\sum_{j=1}^{k-1} P_j\right) M_{z_q}^*|_{\mathcal{S}_k} \\
&= P_{\mathcal{S}_k} M_{z_p} P_{\mathcal{S}_k} M_{z_q}^*|_{\mathcal{S}_k} + \sum_{j=1}^{k-1} P_{\mathcal{S}_k} M_{z_p} P_j M_{z_q}^* P_{\mathcal{S}_k} \\
&= P_{\mathcal{S}_k} M_{z_p} P_{\mathcal{S}_k} M_{z_q}^*|_{\mathcal{S}_k}, \qquad \text{(by } (ii))
\end{aligned}$$

that is, $(M_{z_q}|_{\mathcal{S}_k})^*(M_{z_p}|_{\mathcal{S}_k}) = (M_{z_p}|_{\mathcal{S}_k})(M_{z_q}|_{\mathcal{S}_k})^*$, or equivalently, $\mathcal{S}_k$ is doubly commuting for all $k \geq 1$. This completes the proof. □

We are now ready to state and prove our main result.

Theorem 3.2. *Let $\{P_j\}_{j=1}^{\infty} \in \mathcal{P}_{n-1}$ and $\{\psi_j\}_{j=1}^{\infty} \subseteq H^{\infty}(\mathbb{D})$ be a decreasing inner sequence. Set $\Theta = \sum_{j=1}^{\infty} \psi_j P_j$, $\mathcal{S} = \Theta H^2_{H^2(\mathbb{D}^{n-1})}(\mathbb{D})$ and $\mathcal{S}_j := \bigoplus_{k=j}^{\infty} \operatorname{Ran} P_k$, $j \geq 1$.*

(a) *$\mathcal{S}$ is an invariant subspace of $H^2(\mathbb{D}^n)$ if and only if $\mathcal{S}_j$ is an invariant subspace of $H^2(\mathbb{D}^{n-1})$ for all $j \geq 1$.*

(b) *The following are equivalent*

(i) *There exists an increasing inner sequence $\{\varphi_j\}_{j=1}^{\infty} \subseteq H^{\infty}(\mathbb{D}^{n-1})$ such that*

$$\mathcal{S} = \bigvee_{j=1}^{\infty} \psi_j H^2(\mathbb{D}) \otimes \varphi_j H^2(\mathbb{D}^{n-1}).$$

(ii) *$\mathcal{S}_j$ is a doubly commuting invariant subspace of $H^2(\mathbb{D}^{n-1})$ for all $j \geq 1$.*

(iii) *For each $j \geq 1$, $\mathcal{S}_j$ is an invariant subspace of $H^2(\mathbb{D}^{n-1})$ and*

$$P_{\mathcal{S}_l} M_{z_p} P_j M_{z_q}^* P_{\mathcal{S}_m} = 0. \quad (l, m > j,\ 1 \leq p < q \leq n-1)$$

(iv) *For each $j \geq 1$, $\mathcal{S}_j$ is an invariant subspace of $H^2(\mathbb{D}^{n-1})$ and*

$$P_l M_{z_p} P_j M_{z_q}^* P_m = 0. \quad (l, m > j,\ 1 \leq p < q \leq n-1)$$

Proof. Set $\psi_0 := 0$. First note that $\mathcal{S} = \bigoplus_{j=1}^{\infty} \psi_j H^2(\mathbb{D}) \otimes \operatorname{Ran} P_j$. Since the inner sequence $\{\psi_j\}_{j=1}^{\infty}$ is decreasing, we have $\psi_j H^2(\mathbb{D}) \subset \psi_{j+1} H^2(\mathbb{D})$ for all $j \geq 1$, and

$$\begin{aligned}
\bigoplus_{j=1}^{\infty}\Big((\psi_j H^2(\mathbb{D}) \ominus \psi_{j-1} H^2(\mathbb{D})) \otimes \mathcal{S}_j\Big) &= \bigoplus_{j=1}^{\infty}\Big((\psi_j H^2(\mathbb{D}) \ominus \psi_{j-1} H^2(\mathbb{D})) \otimes \big(\bigoplus_{k=j}^{\infty} \operatorname{Ran} P_k\big)\Big) \\
&= \bigoplus_{j=1}^{\infty}\Big(\bigoplus_{k=1}^{j}(\psi_k H^2(\mathbb{D}) \ominus \psi_{k-1} H^2(\mathbb{D})) \otimes \operatorname{Ran} P_j\Big) \\
&= \bigoplus_{j=1}^{\infty}\Big(\psi_j H^2(\mathbb{D}) \otimes \operatorname{Ran} P_j\Big),
\end{aligned}$$

where for the last equality we use $\bigoplus_{k=1}^{j}(\psi_k H^2(\mathbb{D}) \ominus \psi_{k-1}H^2(\mathbb{D})) = \psi_j H^2(\mathbb{D}) \quad (j \geq 1)$. Thus

$$\mathcal{S} = \bigoplus_{j=1}^{\infty} \Big((\psi_j H^2(\mathbb{D}) \ominus \psi_{j-1}H^2(\mathbb{D})) \otimes \mathcal{S}_j \Big). \tag{3.1}$$

Proof of part (a): Let $\mathcal{S}$ be an invariant subspace of $H^2(\mathbb{D}^n)$ and $j \geq 1$ be a fixed integer. Let $f \in \psi_j H^2(\mathbb{D}) \ominus \psi_{j-1}H^2(\mathbb{D})$, $g \in \mathcal{S}_j$ and $2 \leq i \leq n$. Since

$$z_i(f \otimes g) = f \otimes z_i g \in \mathcal{S} = \bigoplus_{k=1}^{\infty} \big(\psi_k H^2(\mathbb{D}) \ominus \psi_{k-1}H^2(\mathbb{D})\big) \otimes \mathcal{S}_k,$$

and $f \otimes z_i g \perp (\psi_k H^2(\mathbb{D}) \ominus \psi_{k-1}H^2(\mathbb{D})) \otimes \mathcal{S}_k$ for all $k \neq j$, we have $f \otimes z_i g \in (\psi_j H^2(\mathbb{D}) \ominus \psi_{j-1}H^2(\mathbb{D})) \otimes \mathcal{S}_j$ and hence $z_i g \in \mathcal{S}_j$.
Conversely, let $\mathcal{S}_j$ be an invariant subspace of $H^2(\mathbb{D}^{n-1})$ for all $j \geq 1$. Then by (3.1) it follows that $\mathcal{S}$ is joint $\{M_{z_2}, \cdots, M_{z_n}\}$-invariant. Finally, since $\mathcal{S} = \Theta H^2_{H^2(\mathbb{D}^{n-1})}(\mathbb{D})$, it follows that $\mathcal{S}$ is M_{z_1}-invariant.
Proof of part (b): (ii)$\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follows from Lemma 3.1. Now assume that (i) is true. Then

$$\mathcal{S} = \bigvee_{j=1}^{\infty} \psi_j H^2(\mathbb{D}) \otimes \varphi_j H^2(\mathbb{D}^{n-1}) = \bigoplus_{j=1}^{\infty} \big(\psi_j H^2(\mathbb{D}) \ominus \psi_{j-1}H^2(\mathbb{D})\big) \otimes \varphi_j H^2(\mathbb{D}^{n-1}).$$

Comparing this with (3.1), we have $\mathcal{S}_j = \varphi_j H^2(\mathbb{D}^{n-1})$ for all $j \geq 1$. Then by Theorem 1.2, $\mathcal{S}_j$ is doubly commuting for all $j \geq 1$.
Conversely assume (ii). Then by Theorem 1.2, there exists a sequence of increasing inner functions $\{\varphi_j\}_{j=1}^{\infty} \subseteq H^2(\mathbb{D}^{n-1})$ such that $\mathcal{S}_j = \varphi_j H^2(\mathbb{D}^n)$, $j \geq 1$. Then (i) follows from (3.1). This completes the proof. ☐

One can reformulate the above theorem by replacing the decreasing inner sequence by an increasing one.

Theorem 3.3. *Let $\{P_j\}_{j=1}^{\infty} \in \mathcal{P}_{n-1}$ and $\{\varphi_j\}_{j=1}^{\infty} \subseteq H^\infty(\mathbb{D})$ be an increasing inner sequence. Set $\Theta = \sum_{j=1}^{\infty} \phi_j P_j$, $\mathcal{S} = \Theta H^2_{H^2(\mathbb{D}^{n-1})}(\mathbb{D})$ and $\mathcal{S}_j := \bigoplus_{k=1}^{j} \operatorname{Ran} P_k, j \geq 1$.*
(a) *$\mathcal{S}$ is an invariant subspace of $H^2(\mathbb{D}^n)$ if and only if $\mathcal{S}_j$ is an invariant subspace of $H^2(\mathbb{D}^{n-1})$ for all $j \geq 1$.*
(b) *There exists a decreasing inner sequence $\{\psi_j\}_{j=1}^{\infty} \subseteq H^\infty(\mathbb{D}^{n-1})$ such that*

$$\mathcal{S} = \bigvee_{j=1}^{\infty} \phi_j H^2(\mathbb{D}) \otimes \psi_j H^2(\mathbb{D}^{n-1}),$$

if and only if $\mathcal{S}_j$ is a doubly commuting invariant subspace of $H^2(\mathbb{D}^{n-1})$ for all $j \geq 1$.

Proof. (a) We first note that, under the given assumptions, the subspace $\mathcal{S}$ is given by

$$\mathcal{S} = \bigoplus_{j=1}^{\infty} \varphi_j(z_1)H^2(\mathbb{D}) \otimes \operatorname{Ran} P_j = \bigoplus_{j=1}^{\infty} \big(\varphi_j(z_1)H^2(\mathbb{D}) \ominus \varphi_{j+1}(z_1)H^2(\mathbb{D})\big) \otimes \mathcal{S}_j.$$

By the same argument as in part (a) of Theorem 3.2, it follows that $\mathcal{S}$ is an invariant subspace of $H^2(\mathbb{D}^n)$ if and only if $\mathcal{S}_j$ is an invariant subspace of $H^2(\mathbb{D}^{n-1})$ for all $j \geq 1$.
(b) The proof is identical to the proof of part (b) in Theorem 3.2 except the fact that one obtains a decreasing inner sequence corresponding to the increasing doubly commuting invariant subspaces $\{\mathcal{S}_j\}_{j=1}^{\infty}$ of $H^2(\mathbb{D}^{n-1})$. □

Remark. A modification of our argument yields a similar characterization of invariant subspaces of $H^2(\mathbb{D}^n)$ corresponding to the inner multiplier $\Theta(z_1, \ldots, z_k) = \sum_{j=1}^{\infty} \varphi_j(z_1, \ldots, z_k) P_j$, where $\{\varphi_j\}_{j=1}^{\infty}$ is a decreasing or increasing inner sequence in $H^\infty(\mathbb{D}^k)$ and $\{P_j\}_{j=1}^{\infty} \in \mathcal{P}_{n-k}$.

4. Unitarily equivalent invariant subspaces

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two invariant subspaces of $H^2(\mathbb{D}^n)$. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are said to be unitarily equivalent if there exists a unitary operator $U : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ such that

$$UM_{z_i}|_{\mathcal{S}_1} = M_{z_i}|_{\mathcal{S}_2} U. \quad \text{for} \quad (i = 1, \ldots, n)$$

The unitary equivalence of inner sequence based invariant subspaces and two inner sequences based invariant subspaces of $H^2(\mathbb{D}^2)$ are completely described in [11] and [17], respectively. Here we present a similar result for Rudin type invariant subspaces in n-variables. The proof follows along the same lines as in Theorem 3.1 in [17].

Theorem 4.1. *Let $\{\varphi_j\}_{j=1}^{\infty}, \{\widetilde{\varphi}_j\}_{j=1}^{\infty} \subseteq H^\infty(\mathbb{D})$ be two decreasing inner sequences and $\{\psi_j\}_{j=1}^{\infty}$, $\{\widetilde{\psi}_j\}_{j=1}^{\infty} \subseteq H^\infty(\mathbb{D}^{n-1})$ be two increasing inner sequences with $\psi_1 = 1 = \widetilde{\psi}_1$. Let*

$$\mathcal{S} = \bigvee_{j=1}^{\infty} \varphi_j H^2(\mathbb{D}) \otimes \psi_j H^2(\mathbb{D}^{n-1}), \quad \text{and} \quad \widetilde{\mathcal{S}} = \bigvee_{j=1}^{\infty} \widetilde{\varphi}_j H^2(\mathbb{D}) \otimes \widetilde{\psi}_j H^2(\mathbb{D}^{n-1}).$$

Then $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are unitarily equivalent if and only if there exists an inner function $\eta \in H^\infty(\mathbb{D}^n)$, depending only on the first variable z_1, such that $\mathcal{S} = \eta\widetilde{\mathcal{S}}$.

Proof. It is enough to prove the necessary part. Let $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ be unitarily equivalent. Then $\mathcal{S} = \eta\widetilde{\mathcal{S}}$ for some unimodular function $\eta \in L^\infty(\mathbb{T}^n)$ (see Lemma 1 in [1]). Then both $\eta\widetilde{\varphi}_1(z_1)$ and $\overline{\eta}\varphi_1(z_1)$ are in $H^2(\mathbb{D}^n)$, and therefore η is holomorphic and anti-holomorphic in $z_2, \ldots, z_n$. Thus η depends only on z_1 variable. This completes the proof. □

For more results related to unitarily equivalent invariant subspaces of $H^2(\mathbb{D}^n)$, $n \geq 2$, we refer the readers to [1], [9] and [16].

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