# JORDAN BLOCKS OF $H^2(\mathbb{D}^n)$

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Abstract. We develop a several variables analog of the Jordan blocks of the Hardy space $H^2(\mathbb{D})$. In this consideration, we obtain a complete characterization of the doubly commuting quotient modules of the Hardy module $H^2(\mathbb{D}^n)$. We prove that a quotient module $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$ $(n \geq 2)$ is doubly commuting if and only if

$$\mathcal{Q} = \mathcal{Q}_{\Theta_1} \otimes \cdots \otimes \mathcal{Q}_{\Theta_n},$$

where each $\mathcal{Q}_{\Theta_i}$ is either a one variable Jordan block $H^2(\mathbb{D})/\Theta_i H^2(\mathbb{D})$ for some inner function $\Theta_i$ or the Hardy module $H^2(\mathbb{D})$ on the unit disk for all $i = 1, \ldots, n$. We say that a submodule $\mathcal{S}$ of $H^2(\mathbb{D}^n)$ is co-doubly commuting if the quotient module $H^2(\mathbb{D}^n)/\mathcal{S}$ is doubly commuting. We obtain a Beurling like theorem for the class of co-doubly commuting submodules of $H^2(\mathbb{D}^n)$. We prove that a submodule $\mathcal{S}$ of $H^2(\mathbb{D}^n)$ is co-doubly commuting if and only if

$$\mathcal{S} = \sum_{i=1}^{m} \Theta_i H^2(\mathbb{D}^n),$$

for some integer $m \leq n$ and one variable inner functions $\{\Theta_i\}_{i=1}^m$.

## 1. Introduction

Let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ be the $n$-dimensional complex Euclidean space with $n \geq 1$ and $\mathbb{D}^n = \{(z_1, \ldots, z_n) : |z_i| < 1, i = 1, \ldots, n\}$ be the open unit polydisc. We denote the elements of $\mathbb{C}^n$ by $\boldsymbol{z} = (z_1, \ldots, z_n)$ where $z_i \in \mathbb{C}$ for all $i = 1, \ldots, n$. The *Hardy space* $H^2(\mathbb{D}^n)$ on the polydisc is the Hilbert space of all holomorphic functions $f$ on $\mathbb{D}^n$ such that

$$\|f\|_{H^2(\mathbb{D}^n)} := \left( \sup_{0 \leq r < 1} \int_{\mathbb{T}^n} |f(r\boldsymbol{z})|^2 d\boldsymbol{\theta} \right)^{\frac{1}{2}} < \infty,$$

where $d\boldsymbol{\theta}$ is the normalized Lebesgue measure on the torus $\mathbb{T}^n$, the distinguished boundary of $\mathbb{D}^n$ and $r\boldsymbol{z} := (rz_1, \ldots, rz_n)$ (cf. [14], [7]).

The multiplication operators by the coordinate functions turns $H^2(\mathbb{D}^n)$ into a *Hilbert module* over $\mathbb{C}[\boldsymbol{z}] = \mathbb{C}[z_1, \ldots, z_n]$, the ring of polynomials in $n$ variables with complex coefficients in the following sense:

$$\mathbb{C}[\boldsymbol{z}] \times H^2(\mathbb{D}^n) \to H^2(\mathbb{D}^n), \quad (p, f) \mapsto p(M_{z_1}, \ldots, M_{z_n})f,$$

for all $p \in \mathbb{C}[\boldsymbol{z}]$ and $f \in H^2(\mathbb{D}^n)$ (cf. [6]). We also call the Hilbert module $H^2(\mathbb{D}^n)$ over $\mathbb{C}[\boldsymbol{z}]$ as the *Hardy module*. A closed subspace $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$ is said to be a *submodule* of $H^2(\mathbb{D}^n)$ if

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$M_{z_i}\mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \ldots, n$. A closed subspace $\mathcal{Q} \subseteq H^2(\mathbb{D}^n)$ is said to be a *quotient module* of $H^2(\mathbb{D}^n)$ if $\mathcal{Q}^\perp (\cong H^2(\mathbb{D}^n)/\mathcal{Q})$ is a submodule of $H^2(\mathbb{D}^n)$.

Let $\mathcal{S}$ be a submodule and $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{D}^n)$. Then the module multiplication operators on $\mathcal{S}$ and $\mathcal{Q}$ are given by the restrictions $(R_{z_1}, \ldots, R_{z_n})$ and the compressions $(C_{z_1}, \ldots, C_{z_n})$ of the module multiplications of $H^2(\mathbb{D}^n)$, respectively. That is,

$$R_{z_i} = M_{z_i}|_{\mathcal{S}} \quad \text{and} \quad C_{z_i} = P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}},$$

for all $i = 1, \ldots, n$. Here, for a given closed subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$, we denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$ by $P_{\mathcal{M}}$. Note that

$$R_{z_i}^* = P_{\mathcal{S}} M_{z_i}^*|_{\mathcal{S}} \qquad \text{and} \qquad C_{z_i}^* = M_{z_i}^*|_{\mathcal{Q}},$$

for all $i = 1, \ldots, n$.

Jordan blocks of $H^2(\mathbb{D})$: A closed subspace $\mathcal{Q} \subseteq H^2(\mathbb{D})$ is said to be a *Jordan block* of $H^2(\mathbb{D})$ if $\mathcal{Q}$ is a quotient module and $\mathcal{Q} \neq H^2(\mathbb{D})$ (see [13], [12]). By Beurling's theorem [3], a closed subspace $\mathcal{Q}(\neq H^2(\mathbb{D}))$ is a quotient module of $H^2(\mathbb{D})$ if and only if the submodule $\mathcal{Q}^\perp$ is given by $\mathcal{Q}^\perp = \Theta H^2(\mathbb{D})$ for some inner function $\Theta \in H^\infty(\mathbb{D})$. In other words, the quotient modules and hence the Jordan blocks of $H^2(\mathbb{D})$ are precisely given by

$$\mathcal{Q}_\Theta := H^2(\mathbb{D})/\Theta H^2(\mathbb{D}),$$

for inner functions $\Theta \in H^\infty(\mathbb{D})$.

Jordan blocks of $H^2(\mathbb{D}^n)$ $(n > 1)$: First we note that the Hardy module $H^2(\mathbb{D}^n)$ (with $n > 1$) can be identified with the $n$-fold Hilbert space tensor product of the Hardy space $H^2(\mathbb{D})$ on the disc

$$\underbrace{H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})}_{n \text{ times}},$$

via the unitary map $U : H^2(\mathbb{D}^n) \to H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$, where $U(z_1^{l_1} \cdots z_n^{l_n}) := z^{l_1} \otimes \cdots \otimes z^{l_n}$ for all $l_1, \ldots, l_n \in \mathbb{N}$. Moreover, $H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$ is a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ with the module multiplication operators

$$\{I_{H^2(\mathbb{D})} \otimes \cdots \otimes \underbrace{M_z}_{i^{\text{th}}} \otimes \cdots \otimes I_{H^2(\mathbb{D})}\}_{i=1}^n.$$

Therefore, that $U$ is a module map

$$UM_{z_i} = (I_{H^2(\mathbb{D})} \otimes \cdots \otimes \underbrace{M_z}_{i^{\text{th}}} \otimes \cdots \otimes I_{H^2(\mathbb{D})})U,$$

for all $1 \leq i \leq n$. It is easy to see that

$$M_{z_i} M_{z_j}^* = M_{z_j}^* M_{z_i},$$

for all $i \neq j$.

The above fact is one of the motivations to introduce the following notion and the title of the paper.

DEFINITION 1.1. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{D}^n)$ and $n > 1$. Then $\mathcal{Q}$ is said to be a Jordan block of $H^2(\mathbb{D}^n)$ if $\mathcal{Q}$ is doubly commuting, that is, $C_{z_i} C_{z_j}^* = C_{z_j}^* C_{z_i}$, for all $1 \leq i < j \leq n$ and $\mathcal{Q} \neq H^2(\mathbb{D}^n)$. Also a closed subspace $\mathcal{S}$ of $H^2(\mathbb{D}^n)$ is said to be co-doubly commuting submodule of $H^2(\mathbb{D}^n)$ if $H^2(\mathbb{D}^n)/\mathcal{S}$ is a doubly commuting quotient module.*

However, in most of the following we will simply regard a Jordan block of $H^2(\mathbb{D}^n)$ as a doubly commuting quotient module of $H^2(\mathbb{D}^n)$.

The study of the doubly commuting quotient modules of the Hardy module $H^2(\mathbb{D}^2)$ was initiated by Douglas and Yang in [4] and [5] (also see [2]). Later in [11] Izuchi, Nakazi and Seto obtained a classification of the doubly commuting quotient modules of the Hardy module $H^2(\mathbb{D}^2)$ (see Theorems 2.1 and 3.1 in [11]).

In this paper we completely classify the doubly commuting quotient modules of $H^2(\mathbb{D}^n)$ for any $n \geq 2$. In this consideration, we provide a more refined analysis compared to [11]. More specifically, our method is based on the Hilbert tensor product structure of the Hardy module $H^2(\mathbb{D}^n)$ which also yield new proofs of earlier results by Izuchi, Nakazi and Seto [11] concerning the base case $n = 2$.

A key example of doubly commuting quotient modules over $\mathbb{C}[\boldsymbol{z}]$ is the Hilbert tensor product of $n$ quotient modules of the Hardy module $H^2(\mathbb{D})$. That is, if we consider $n$ quotient modules $\{\mathcal{Q}_i\}_{i=1}^n$ of the Hardy module $H^2(\mathbb{D})$ then

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n,$$

is a doubly commuting quotient module of $H^2(\mathbb{D}^n)$ with the module multiplication operators as

$$\{I_{\mathcal{Q}_1} \otimes \cdots \otimes \underbrace{P_{\mathcal{Q}_i} M_z|_{\mathcal{Q}_i}}_{i^{\text{ th}} \text{ place}} \otimes \cdots \otimes I_{\mathcal{Q}_n}\}_{i=1}^n.$$

We prove that a doubly commuting quotient module of $H^2(\mathbb{D}^n)$ can be also represented by the Hilbert space tensor product of quotient modules of $H^2(\mathbb{D})$ in the above form. This result is then used to prove a Beurling type theorem for the co-doubly commuting submodules.

We now summarize the contents of this paper. In Section 2, we give relevant background for the main results of this paper. In Section 3, we prove that a quotient module $\mathcal{Q}$ of $H^2(\mathbb{D}^n)$ is doubly commuting if and only if $\mathcal{Q}$ is the $n$ times Hilbert tensor product of quotient modules of the Hardy module $H^2(\mathbb{D})$. In Section 4, we characterize the class of co-doubly commuting submodules of $H^2(\mathbb{D}^n)$.

## 2. Preparatory results

In this section, we gather together some concepts and results concerning various aspects of the Hardy modules that are used frequently in the rest of this paper. Some of the results of the present section are of independent interest.

We first recall that the module multiplication of the Hardy module $H^2(\mathbb{D})$ satisfies the following relation

$$M_z M_z^* = I_{H^2(\mathbb{D})} - P_{\mathbb{C}},$$

where $P_{\mathbb{C}}$ denotes the orthogonal projection of $H^2(\mathbb{D})$ onto the space of constant functions. Moreover, if $\mathcal{Q}_\Theta = H^2(\mathbb{D})/\Theta H^2(\mathbb{D})$ is a Jordan block for some inner function $\Theta \in H^\infty(\mathbb{D})$, then we have

$$P_{\mathcal{Q}_\Theta} = I_{H^2(\mathbb{D})} - M_\Theta M_\Theta^* \quad \text{and} \quad P_{\Theta H^2(\mathbb{D})} = M_\Theta M_\Theta^*.$$

We also have

$$I_{\mathcal{Q}} - C_z C_z^* = P_{\mathcal{Q}}(I_{H^2(\mathbb{D})} - M_z M_z^*)|_{\mathcal{Q}} = P_{\mathcal{Q}} P_{\mathbb{C}}|_{\mathcal{Q}},$$

where $\mathcal{Q}$ is a quotient module of $H^2(\mathbb{D})$.

The following lemma is well known.

Lemma 2.1. *Let $\mathcal{Q}_\Theta$ be a Jordan block of $H^2(\mathbb{D})$ for some inner function $\Theta \in H^\infty(\mathbb{D})$. Then*

$$P_{\mathcal{Q}_\Theta} 1 = 1 - \overline{\Theta(0)}\Theta,$$

*and*

$$(P_{\mathcal{Q}_\Theta} P_{\mathbb{C}} P_{\mathcal{Q}_\Theta})1 = (1 - |\Theta(0)|^2)(1 - \overline{\Theta(0)}\Theta).$$

Proof. By virtue of $M_\Theta^* 1 = \overline{\Theta(0)}$ we have

$$P_{\mathcal{Q}_\Theta} 1 = (I_{H^2(\mathbb{D})} - M_\Theta M_\Theta^*)1 = 1 - M_\Theta(M_\Theta^* 1) = 1 - \overline{\Theta(0)}\Theta.$$

For the second equality, we compute

$$(P_{\mathcal{Q}_\Theta} P_{\mathbb{C}} P_{\mathcal{Q}_\Theta})1 = (P_{\mathcal{Q}_\Theta} P_{\mathbb{C}})(1 - \overline{\Theta(0)}\Theta) = P_{\mathcal{Q}_\Theta}(1 - |\Theta(0)|^2) = (1 - |\Theta(0)|^2)(1 - \overline{\Theta(0)}\Theta).$$

This completes the proof. ■

This lemma has the following immediate corollary.

Corollary 2.2. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{D})$. Then*

$$P_{\mathcal{Q}} 1 \in ran(P_{\mathcal{Q}} P_{\mathbb{C}} P_{\mathcal{Q}}).$$

Proof. If $\mathcal{Q} = H^2(\mathbb{D})$ then the result follows trivially. If $\mathcal{Q} \neq H^2(\mathbb{D})$ then $\mathcal{Q}$ is a Jordan block and hence the conclusion follows from Lemma 2.1. ■

The following lemma is a variation on the theme of the isometric dilation theory of contractions.

Lemma 2.3. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{D})$ and $\mathcal{L} = ran(I_{\mathcal{Q}} - C_z C_z^*) = ran(P_{\mathcal{Q}} P_{\mathbb{C}} P_{\mathcal{Q}})$. Then*

$$\mathcal{Q} = \bigvee_{l=0}^{\infty} P_{\mathcal{Q}} M_z^l \mathcal{L}.$$

Proof. The result is trivial if $\mathcal{Q} = \{0\}$. Let $\mathcal{Q} \neq \{0\}$. Notice that

$$\bigvee_{l=0}^{\infty} P_{\mathcal{Q}} M_z^l \mathcal{L} \subseteq \mathcal{Q}.$$

Let now $f \in \mathcal{Q}$ be such that $f \perp \vee_{l=0}^{\infty} P_{\mathcal{Q}} M_z^l \mathcal{L}$. Then for all $l \geq 0$ we have that $f \perp P_{\mathcal{Q}} M_z^l P_{\mathcal{Q}} P_{\mathbb{C}} \mathcal{Q}$, or equivalently, $P_{\mathbb{C}} M_z^{*l} f \in \mathcal{Q}^\perp$. Since $\mathcal{Q}^\perp$ is a proper submodule of $H^2(\mathbb{D})$, it follows that

$$P_{\mathbb{C}} M_z^{*l} f = 0,$$

for all $l \geq 0$. Consequently,

$$f = 0.$$

This conclude the proof. ∎

In the following, we employ the standard multi-index notation that $\mathbb{N}^n = \{(k_1, \dots, k_n) : k_i \in \mathbb{N}, i = 1, \dots, n\}$ and for any $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ we denote $\boldsymbol{z}^{\boldsymbol{k}} = z_1^{k_1} \cdots z_n^{k_n}$ and $M_{\boldsymbol{z}}^{\boldsymbol{k}} = M_{z_1}^{k_1} \cdots M_{z_n}^{k_n}$.

We now present a characterization of $M_{z_1}$-reducing subspace of $H^2(\mathbb{D}^n)$. However the technique used here seems to be well known in the study of the reducing subspaces.

Proposition 2.4. *Let $n > 1$ and $\mathcal{S}$ be a closed subspace of $H^2(\mathbb{D}^n)$. Then $\mathcal{S}$ is a $(M_{z_2}, \dots, M_n)$-reducing subspace of $H^2(\mathbb{D}^n)$ if and only if*

$$\mathcal{S} = \mathcal{S}_1 \otimes \underbrace{H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})}_{(n-1)\,\text{times}},$$

*for some closed subspace $\mathcal{S}_1$ of $H^2(\mathbb{D})$.*

Proof. Let $\mathcal{S}$ be a $(M_{z_2}, \dots, M_{z_n})$-reducing closed subspace of $H^2(\mathbb{D}^n)$, that is, for all $2 \le i \le n$ we have

$$M_{z_i} P_{\mathcal{S}} = P_{\mathcal{S}} M_{z_i}.$$

Following Agler's hereditary functional calculus (cf. [1])

$$\begin{aligned}(\prod_{i=2}^{n}(1 - z_i\bar{w}_i))(\boldsymbol{M_z}, \boldsymbol{M_z}) &= \sum_{\substack{0 \le i_1 < \ldots < i_l \le n \\ i_1, i_2 \neq 1}} (-1)^l (z_{i_1} \cdots z_{i_l} \bar{w}_{i_1} \cdots \bar{w}_{i_l})(\boldsymbol{M_z}, \boldsymbol{M_z}) \\ &= \sum_{\substack{0 \le i_1 < \ldots < i_l \le n \\ i_1, i_2 \neq 1}} (-1)^l M_{z_{i_1}} \cdots M_{z_{i_l}} M_{z_{i_1}}^* \cdots M_{z_{i_l}}^* \\ &= P_{H^2(\mathbb{D})} \otimes P_{\mathbb{C}} \otimes \cdots \otimes P_{\mathbb{C}},\end{aligned}$$

where $\boldsymbol{M_z} = (M_{z_1}, \dots, M_{z_n})$. Consequently,

$$(P_{H^2(\mathbb{D})} \otimes P_{\mathbb{C}} \otimes \cdots \otimes P_{\mathbb{C}}) P_{\mathcal{S}} = P_{\mathcal{S}} (P_{H^2(\mathbb{D})} \otimes P_{\mathbb{C}} \otimes \cdots \otimes P_{\mathbb{C}}),$$

which yields that $P_{\mathcal{S}} (P_{H^2(\mathbb{D})} \otimes P_{\mathbb{C}} \otimes \cdots \otimes P_{\mathbb{C}})$ is an orthogonal projection and

$$P_{\mathcal{S}} (P_{H^2(\mathbb{D})} \otimes P_{\mathbb{C}} \otimes \cdots \otimes P_{\mathbb{C}}) = (P_{H^2(\mathbb{D})} \otimes P_{\mathbb{C}} \otimes \cdots \otimes P_{\mathbb{C}}) P_{\mathcal{S}} = P_{\tilde{\mathcal{S}}_1},$$

where $\tilde{\mathcal{S}}_1 := (H^2(\mathbb{D}) \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C}) \cap \mathcal{S}$. Let

$$\tilde{\mathcal{S}}_1 = \mathcal{S}_1 \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C},$$

for some closed subspace $\mathcal{S}_1$ of $H^2(\mathbb{D})$. We claim that

$$\mathcal{S} = \overline{\text{span}}\{M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} \tilde{\mathcal{S}}_1 : l_2, \dots, l_n \in \mathbb{N}\} = \mathcal{S}_1 \otimes H^2(\mathbb{D}^{n-1}).$$

Now for any

$$f = \sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}} \in \mathcal{S},$$

we have

$$f = P_{\mathcal{S}} f = P_{\mathcal{S}}(\sum_{\boldsymbol{k} \in \mathbb{N}^n} M_{\boldsymbol{z}}^{\boldsymbol{k}} a_{\boldsymbol{k}}) = \sum_{\boldsymbol{k} \in \mathbb{N}^n} M_{z_2}^{k_2} \cdots M_{z_n}^{k_n} P_{\mathcal{S}}(a_{\boldsymbol{k}} z_1^{k_1}),$$

where $a_{\boldsymbol{k}} \in \mathbb{C}$ for all $\boldsymbol{k} \in \mathbb{N}^n$. But $P_{\mathcal{S}}(a_{\boldsymbol{k}} z_1^{k_1}) = P_{\mathcal{S}}(P_{H^2(\mathbb{D})} \otimes P_{\mathbb{C}} \otimes \cdots \otimes P_{\mathbb{C}})(a_{\boldsymbol{k}} z_1^{k_1}) \in \tilde{\mathcal{S}}_1$ and hence $f \in \mathcal{S}_1 \otimes H^2(\mathbb{D}^{n-1})$. That is, $\mathcal{S} \subseteq \mathcal{S}_1 \otimes H^2(\mathbb{D}^{n-1})$. On the other hand, since $\tilde{\mathcal{S}}_1 \subseteq \mathcal{S}$ and that $\mathcal{S}$ is a $(M_{z_2}, \ldots, M_{z_n})$-reducing subspace, we see that $\mathcal{S} = \mathcal{S}_1 \otimes H^2(\mathbb{D}^{n-1})$.
The converse part is immediate. This concludes the proof of the proposition. ∎

The following result will be used in the final section.

Lemma 2.5. *Let $\{P_i\}_{i=1}^n$ be a collection of commuting orthogonal projections on a Hilbert space $\mathcal{H}$. Then*

$$\mathcal{L} := \sum_{i=1}^n ranP_i,$$

*is closed and the orthogonal projection of $\mathcal{H}$ onto $\mathcal{L}$ is given by*

$$\begin{aligned} P_{\mathcal{L}} &= P_1(I-P_2)\cdots(I-P_n) + P_2(I-P_3)\cdots(I-P_n) + \cdots + P_{n-1}(I-P_n) + P_n \\ &= P_n(I-P_{n-1})\cdots(I-P_1) + P_{n-1}(I-P_{n-2})\cdots(I-P_1) + \cdots + P_2(I-P_1) + P_1. \end{aligned}$$

*Moreover,*

$$P_{\mathcal{L}} = I - \prod_{i=1}^n (I - P_i).$$

Proof. We let

$$O_i = P_i(I-P_{i+1})\cdots(I-P_{n-1})(I-P_n),$$

so that

$$O_i = \prod_{j=i+1}^n (I-P_j) - \prod_{j=i}^n (I-P_j),$$

for all $i = 1, \ldots, n-1$ and $O_n = P_n$. By the assumptions, $\{O_i\}_{i=1}^n$ is a family of orthogonal projections with orthogonal ranges. We claim that

$$\mathcal{L} = \text{ran}O_1 \oplus \cdots \oplus \text{ran}O_n.$$

From the definition we see that $\mathcal{L} \supseteq \text{ran}O_1 \oplus \cdots \oplus \text{ran}O_n$. To prove the reverse inclusion, first we observe that

$$\sum_{i=1}^n O_i = I - \prod_{i=1}^n (I - P_i).$$

Now let $f = f_1 + \cdots + f_n \in \mathcal{L}$ where $f_i \in \text{ran}P_i$ for all $i = 1, \ldots, n$. Then

$$\begin{aligned} (\sum_{i=1}^n O_i)f &= (I - \prod_{i=1}^n (I-P_i))f = f - \prod_{i=1}^n (I-P_i)f \\ &= f - \sum_{j=1}^n \prod_{i=1}^n (I-P_i)f_j = f - \sum_{j=1}^n 0 \\ &= f, \end{aligned}$$

and hence the equality follows. This implies that $\mathcal{L}$ is a closed subspace and

$$P_{\mathcal{L}} = \sum_{i=1}^n O_i = I - \prod_{i=1}^n (I - P_i).$$

This completes the proof of the lemma. ∎

## 3. Quotient Modules

In this section we prove the central result of this paper that a doubly commuting quotient module of $H^2(\mathbb{D}^n)$ is precisely the Hilbert tensor product of $n$ quotient modules of the Hardy module $H^2(\mathbb{D})$.

We begin by generalizing the fact that a closed subspace $\mathcal{M}$ of $H^2(\mathbb{D}^n)$ is $M_{z_1}$-reducing if and only if

$$\mathcal{M} = H^2(\mathbb{D}) \otimes \mathcal{E},$$

for some closed subspace $\mathcal{E} \subseteq H^2(\mathbb{D}^{n-1})$.

Proposition 3.1. *Let $\mathcal{Q}_1$ be a quotient module of $H^2(\mathbb{D})$ and $\mathcal{M}$ be a closed subspace of*

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \underbrace{H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})}_{(n-1) \text{ times}} \subseteq H^2(\mathbb{D}^n).$$

*Then $\mathcal{M}$ is a $P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}$-reducing subspace of $\mathcal{Q}$ if and only if*

$$\mathcal{M} = \mathcal{Q}_1 \otimes \mathcal{E},$$

*for some closed subspace $\mathcal{E}$ of $H^2(\mathbb{D}^{n-1})$.*

Proof. Let $\mathcal{M}$ be a $P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}$-reducing subspace of $\mathcal{Q}$. Then

$$(P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}) P_{\mathcal{M}} = P_{\mathcal{M}} (P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}), \tag{3.1}$$

or equivalently,

$$(P_{\mathcal{Q}_1} M_z|_{\mathcal{Q}_1} \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}) P_{\mathcal{M}} = P_{\mathcal{M}} (P_{\mathcal{Q}_1} M_z|_{\mathcal{Q}_1} \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}).$$

Now

$$I_{\mathcal{Q}} - (P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}})(P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}})^* = (P_{\mathcal{Q}_1} P_{\mathbb{C}}|_{\mathcal{Q}_1}) \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}.$$

Further (3.1) yields

$$\begin{aligned} P_{\mathcal{M}} ((P_{\mathcal{Q}_1} P_{\mathbb{C}}|_{\mathcal{Q}_1}) \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}) \\ = ((P_{\mathcal{Q}_1} P_{\mathbb{C}}|_{\mathcal{Q}_1}) \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}) P_{\mathcal{M}}. \end{aligned}$$

Consider

$$\begin{aligned} \mathcal{L} :&= \mathcal{M} \cap \text{ran } ((P_{\mathcal{Q}_1} P_{\mathbb{C}}|_{\mathcal{Q}_1}) \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}) \\ &= \mathcal{M} \cap (\mathcal{L}_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})), \end{aligned}$$

where

$$\mathcal{L}_1 = \text{ran } (P_{\mathcal{Q}_1} P_{\mathbb{C}}|_{\mathcal{Q}_1}) \subseteq \mathcal{Q}_1.$$

Since

$$\mathcal{L} \subseteq \mathcal{L}_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D}),$$

and $\dim \mathcal{L}_1 = 1$ (otherwise, by Lemma 2.3 that $\mathcal{L}_1 = \{0\}$ is equivalent to $\mathcal{Q}_1 = \{0\}$) we obtain

$$\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{E},$$

for some closed subspace $\mathcal{E} \subseteq H^2(\mathbb{D}^{n-1})$. We claim that

$$\mathcal{M} = \bigvee_{l=0}^{\infty} P_{\mathcal{Q}} M_{z_1}^l \mathcal{L}.$$

Since $\mathcal{M}$ is $P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}$-reducing subspace and $\mathcal{M} \supseteq \mathcal{L}$, it follows that

$$\mathcal{M} \supseteq \bigvee_{l=0}^{\infty} P_{\mathcal{Q}} M_{z_1}^l \mathcal{L}.$$

To prove the reverse inclusion, we let $f \in \mathcal{M}$ and $f = \sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}$, where $a_{\boldsymbol{k}} \in \mathbb{C}$ for all $\boldsymbol{k} \in \mathbb{N}^n$. Then

$$f = P_{\mathcal{M}} P_{\mathcal{Q}} f = P_{\mathcal{M}} P_{\mathcal{Q}} \sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}.$$

Observe now that for all $\boldsymbol{k} \in \mathbb{N}^n$,

$$\begin{aligned} P_{\mathcal{M}} P_{\mathcal{Q}} \boldsymbol{z}^{\boldsymbol{k}} &= P_{\mathcal{M}}((P_{\mathcal{Q}_1} z_1^{k_1})(z_2^{k_2} \cdots z_n^{k_n})) \\ &= P_{\mathcal{M}}((P_{\mathcal{Q}_1} M_{z_1}^{k_1} P_{\mathcal{Q}_1} 1)(z_2^{k_2} \cdots z_n^{k_n})) \\ &= (P_{\mathcal{M}} P_{\mathcal{Q}} M_{z_1}^{k_1} P_{\mathcal{Q}})(P_{\mathcal{Q}_1} 1 \otimes z_2^{k_2} \cdots z_n^{k_n}) \\ &= P_{\mathcal{Q}} M_{z_1}^{k_1} (P_{\mathcal{M}}(P_{\mathcal{Q}_1} 1 \otimes z_2^{k_2} \cdots z_n^{k_n})), \end{aligned}$$

by (3.1), where the second equality follows from

$$\langle z_1^{k_1}, f \rangle = \langle 1, (M_{z_1}^{k_1})^* f \rangle = \langle P_{\mathcal{Q}_1} M_{z_1}^{k_1} P_{\mathcal{Q}_1} 1, f \rangle,$$

for all $f \in \mathcal{Q}_1$. Now by applying Corollary 2.2, we obtain that $P_{\mathcal{Q}_1} 1 \in \mathcal{L}_1$ and hence

$$P_{\mathcal{M}}(P_{\mathcal{Q}_1} 1 \otimes z_2^{k_2} \cdots z_n^{k_n}) \in \mathcal{L}.$$

Therefore, we infer

$$P_{\mathcal{M}} P_{\mathcal{Q}} \boldsymbol{z}^{\boldsymbol{k}} \in \bigvee_{l=0}^{\infty} P_{\mathcal{Q}_1} M_{z_1}^l \mathcal{L},$$

for all $\boldsymbol{k} \in \mathbb{N}^n$ and hence $f \in \vee_{l=0}^{\infty} P_{\mathcal{Q}} M_{z_1}^l \mathcal{L}$. Thus we get

$$\mathcal{M} = \vee_{l=0}^{\infty} P_{\mathcal{Q}} M_{z_1}^l \mathcal{L}.$$

Finally, $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{E}$ yields

$$\mathcal{M} = \bigvee_{l=0}^{\infty} P_{\mathcal{Q}} M_{z_1}^l \mathcal{L} = (\bigvee_{l=0}^{\infty} P_{\mathcal{Q}_1} M_{z_1}^l \mathcal{L}_1) \otimes \mathcal{E},$$

and therefore by Lemma 2.3,

$$\mathcal{M} = \mathcal{Q}_1 \otimes \mathcal{E}.$$

The converse part is trivial. This finishes the proof. ■

We are now ready to prove the main result of this section.

Theorem 3.2. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{D}^n)$. Then $\mathcal{Q}$ is doubly commuting if and only if there exists quotient modules $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ of $H^2(\mathbb{D})$ such that*

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n.$$

Proof. Let $\mathcal{Q}$ be a doubly commuting quotient module of $H^2(\mathbb{D}^n)$. Define

$$\tilde{\mathcal{Q}}_1 = \overline{\text{span}}\{z_2^{l_2} \cdots z_n^{l_n}\mathcal{Q} : l_2, \ldots, l_n \in \mathbb{N}\}.$$

Then $\tilde{\mathcal{Q}}_1$ is a joint $(M_{z_2}, \ldots, M_{z_n})$-reducing subspace of $H^2(\mathbb{D}^n)$. But Proposition 2.4 now allows us to conclude that

$$\tilde{\mathcal{Q}}_1 = \mathcal{Q}_1 \otimes \underbrace{H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})}_{(n-1)\text{ times}},$$

for some closed subspace $\mathcal{Q}_1$ of $H^2(\mathbb{D})$. Since $\tilde{\mathcal{Q}}_1$ is $M_{z_1}^*$-invariant subspace, that $\mathcal{Q}_1$ is a $M_z^*$-invariant subspace of $H^2(\mathbb{D})$, that is, $\mathcal{Q}_1$ is a quotient module of $H^2(\mathbb{D})$.
Note that $\mathcal{Q} \subseteq \tilde{\mathcal{Q}}_1$. We claim that $\mathcal{Q}$ is a $M_{z_1}^*|_{\tilde{\mathcal{Q}}_1}$-reducing subspace of $\tilde{\mathcal{Q}}_1$, that is,

$$P_{\mathcal{Q}}(M_{z_1}^*|_{\tilde{\mathcal{Q}}_1}) = (M_{z_1}^*|_{\tilde{\mathcal{Q}}_1})P_{\mathcal{Q}}.$$

In order to prove the claim we first observe that for all $l \geq 0$ and $2 \leq i \leq n$,

$$C_{z_1}^* C_{z_i}^l = C_{z_i}^l C_{z_1}^*,$$

and hence

$$C_{z_1}^* C_{z_2}^{l_2} \cdots C_{z_n}^{l_n} = C_{z_2}^{l_2} \cdots C_{z_n}^{l_n} C_{z_1}^*,$$

for all $l_2, \ldots, l_n \geq 0$, that is,

$$M_{z_1}^* P_{\mathcal{Q}} M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_{\mathcal{Q}} = P_{\mathcal{Q}} M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} M_{z_1}^* P_{\mathcal{Q}},$$

or,

$$M_{z_1}^* P_{\mathcal{Q}} M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_{\mathcal{Q}} = P_{\mathcal{Q}} M_{z_1}^* M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_{\mathcal{Q}}.$$

From this it follows that for all $f \in \mathcal{Q}$ and $l_2, \ldots, l_n \geq 0$,

$$\begin{aligned}(P_{\mathcal{Q}} M_{z_1}^*|_{\tilde{\mathcal{Q}}_1})(z_2^{l_2} \cdots z_n^{l_n} f) &= P_{\mathcal{Q}} M_{z_1}^*(z_2^{l_2} \cdots z_n^{l_n} f) = (P_{\mathcal{Q}} M_{z_1}^* M_{z_2}^{l_2} \cdots M_{z_n}^{l_n}) f \\ &= (P_{\mathcal{Q}} M_{z_1}^* M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_{\mathcal{Q}}) f = M_{z_1}^* P_{\mathcal{Q}} M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_{\mathcal{Q}} f \\ &= M_{z_1}^* P_{\mathcal{Q}} M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} f = (M_{z_1}^* P_{\mathcal{Q}})(z_2^{l_2} \cdots z_n^{l_n} f).\end{aligned}$$

Also by $P_{\mathcal{Q}}\tilde{\mathcal{Q}}_1 \subseteq \tilde{\mathcal{Q}}_1$ we have

$$P_{\mathcal{Q}} P_{\tilde{\mathcal{Q}}_1} = P_{\tilde{\mathcal{Q}}_1} P_{\mathcal{Q}} P_{\tilde{\mathcal{Q}}_1}.$$

This yields

$$\begin{aligned}(P_{\mathcal{Q}} M_{z_1}^*|_{\tilde{\mathcal{Q}}_1})(z_2^{l_2} \cdots z_n^{l_n} f) &= (M_{z_1}^* P_{\mathcal{Q}})(z_2^{l_2} \cdots z_n^{l_n} f) \\ &= M_{z_1}^* P_{\mathcal{Q}} P_{\tilde{\mathcal{Q}}_1}(z_2^{l_2} \cdots z_n^{l_n} f) \\ &= M_{z_1}^* P_{\tilde{\mathcal{Q}}_1} P_{\mathcal{Q}} P_{\tilde{\mathcal{Q}}_1}(z_2^{l_2} \cdots z_n^{l_n} f) \\ &= (M_{z_1}^*|_{\tilde{\mathcal{Q}}_1} P_{\mathcal{Q}})(z_2^{l_2} \cdots z_n^{l_n} f),\end{aligned}$$

for all $f \in \mathcal{Q}$ and $l_2, \ldots, l_n \geq 0$, and therefore

$$P_{\mathcal{Q}}(M_{z_1}^*|_{\tilde{\mathcal{Q}}_1}) = (M_{z_1}^*|_{\tilde{\mathcal{Q}}_1})P_{\mathcal{Q}}.$$

Hence $\mathcal{Q}$ is a $M_{z_1}^*|_{\tilde{\mathcal{Q}}_1}$-reducing subspace of $\tilde{\mathcal{Q}}_1 = \mathcal{Q}_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$. Applying Proposition 3.1, we obtain a closed subspace $\mathcal{E}_1$ of $H^2(\mathbb{D}^{n-1})$ such that

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \mathcal{E}_1.$$

Moreover, since

$$\bigvee_{l=0}^{\infty} z_1^l \mathcal{Q} = \bigvee_{l=0}^{\infty} z_1^l (\mathcal{Q}_1 \otimes \mathcal{E}_1) = H^2(\mathbb{D}) \otimes \mathcal{E}_1,$$

and $\vee_{l=0}^{\infty} z_1^l \mathcal{Q}$ is a doubly commuting quotient module of $H^2(\mathbb{D}^n)$, we have that $\mathcal{E}_1 \subseteq H^2(\mathbb{D}^{n-1})$ a doubly commuting quotient module of $H^2(\mathbb{D}^{n-1})$.
By the same argument as above, we conclude that

$$\mathcal{E}_1 = \mathcal{Q}_2 \otimes \mathcal{E}_2,$$

for some doubly commuting quotient module of $H^2(\mathbb{D}^{n-2})$. Continuing this process, we have

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n,$$

where $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ are quotient modules of $H^2(\mathbb{D})$.
The converse implication follows from the fact that the module multiplication operators on $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ are given by

$$\{I_{\mathcal{Q}_1} \otimes \cdots \otimes \underbrace{P_{\mathcal{Q}_i} M_z|_{\mathcal{Q}_i}}_{i^{\text{th}}\text{ place}} \otimes \cdots \otimes I_{\mathcal{Q}_n}\}_{i=1}^n,$$

which is certainly doubly commuting. This completes the proof. ■

As a corollary of the above model, we have the following result.

Corollary 3.3. *Let $\mathcal{Q}$ be a closed subspace of $H^2(\mathbb{D}^n)$. Then $\mathcal{Q}$ is doubly commuting quotient module if and only if there exists $\{\Theta_i\}_{i=1}^n \subseteq H^\infty(\mathbb{D})$ such that each $\Theta_i$ is either inner or the zero function for all $1 \leq i \leq n$ and*

$$\mathcal{Q} = \mathcal{Q}_{\Theta_1} \otimes \cdots \otimes \mathcal{Q}_{\Theta_n}.$$

Proof. Let $\mathcal{Q}$ be a doubly commuting quotient module of $H^2(\mathbb{D}^n)$. By Theorem 3.2, we know that

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n,$$

where $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ are quotient modules of $H^2(\mathbb{D})$. For each $i \in \{1, \ldots, n\}$, if $\mathcal{Q}_i \subsetneq H^2(\mathbb{D})$ then

$$\mathcal{Q}_i = \mathcal{Q}_{\Theta_i} = H^2(\mathbb{D})/\Theta_i H^2(\mathbb{D}),$$

for some inner function $\Theta_i \in H^\infty(\mathbb{D})$. Otherwise, $\mathcal{Q}_i = H^2(\mathbb{D})$ and we define $\Theta_i \equiv 0$ on $\mathbb{D}$ so that

$$\mathcal{Q}_i = H^2(\mathbb{D}) = \mathcal{Q}_{\Theta_i} = H^2(\mathbb{D})/(0 \cdot H^2(\mathbb{D})).$$

The converse part again follows from Theorem 3.2, and the corollary is proved. ■

This result was obtained by Izuchi, Nakazi and Seto in [11] for the base case $n = 2$ (also see [10]).

We conclude this section by recording the uniqueness of the tensor product representations of the doubly commuting quotient modules in Theorem 3.2. The same conclusion holds for a more general framework. Here, we provide a proof using the Hardy space method.

Let $\mathcal{Q}$ be a doubly commuting quotient module of $H^2(\mathbb{D}^n)$ and

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n = \mathcal{R}_1 \otimes \ldots \otimes \mathcal{R}_n,$$

for quotient modules $\{\mathcal{Q}_i\}_{i=1}^n$ and $\{\mathcal{R}_i\}_{i=1}^n$ of $H^2(\mathbb{D})$. We claim that $\mathcal{Q}_i = \mathcal{R}_i$ for all $i$. In fact,

$$\tilde{\mathcal{Q}}_1 := \bigvee_{l_2,\ldots,l_n \geq 0}^{\infty} z_2^{l_2} \cdots z_n^{l_n} \mathcal{Q} = \mathcal{Q}_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D}) = \mathcal{R}_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D}),$$

and

$$\bigcap_{i=2}^n \ker M_{z_i}^*|_{\tilde{\mathcal{Q}}_1} = \mathcal{Q}_1 \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C} = \mathcal{R}_1 \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C}.$$

Consequently, $\mathcal{Q}_1 = \mathcal{R}_1$ and similarly, for all other $i = 2, \ldots, n$.

## 4. SUBMODULES

In this section we relate the Hilbert tensor product structure of the doubly commuting quotient modules to the Beurling like representations of the corresponding co-doubly commuting submodules.

To proceed further we introduce one more piece of notation.
*Let $\Theta_i \in H^\infty(\mathbb{D})$ be a given function indexed by $i \in \{1, \ldots, n\}$. In what follows by $\tilde{\Theta}_i \in H^\infty(\mathbb{D}^n)$ we denote the extension function defined by*

$$\tilde{\Theta}_i(\boldsymbol{z}) = \Theta_i(z_i),$$

*for all $\boldsymbol{z} \in \mathbb{D}^n$.*

The following provides an explicit correspondence between the doubly commuting quotient modules and the co-doubly commuting submodules of $H^2(\mathbb{D}^n)$.

THEOREM 4.1. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{D}^n)$ and $\mathcal{Q} \neq H^2(\mathbb{D}^n)$. Then $\mathcal{Q}$ is doubly commuting if and only if there exists inner functions $\Theta_{i_j} \in H^\infty(\mathbb{D})$ for $1 \leq i_1 < \ldots < i_m \leq n$ for some integer $m \in \{1, \ldots, n\}$ such that*

$$\mathcal{Q} = H^2(\mathbb{D}^n)/[\tilde{\Theta}_{i_1} H^2(\mathbb{D}^n) + \cdots + \tilde{\Theta}_{i_m} H^2(\mathbb{D}^n)],$$

*where $\tilde{\Theta}_{i_j}(\boldsymbol{z}) = \Theta_{i_j}(z_{i_j})$ for all $\boldsymbol{z} \in \mathbb{D}^n$.*

Proof. Let $\mathcal{Q}$ be a doubly commuting quotient module of $H^2(\mathbb{D}^n)$. Then by Theorem 3.2 we have

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n,$$

where for each $1 \leq i \leq n$, $\mathcal{Q}_i$ is a submodule of $H^2(\mathbb{D})$. Choose $1 \leq m \leq n$ such that

$$\mathcal{Q}_{i_j} \neq H^2(\mathbb{D}),$$

for $1 \leq i_1 < \ldots < i_m \leq n$. Then

$$\mathcal{Q} = H^2(\mathbb{D}) \otimes \cdots \otimes \mathcal{Q}_{i_1} \otimes \cdots \otimes \mathcal{Q}_{i_m} \otimes \cdots \otimes H^2(\mathbb{D}),$$

where $\mathcal{Q}_{i_j} \subsetneq H^2(\mathbb{D})$ for all $1 \leq i_1 < \ldots < i_m \leq n$. Let

$$\mathcal{Q}_{i_j} = \mathcal{Q}_{\Theta_{i_j}} = (\text{ran} M_{\Theta_{i_j}})^\perp = \text{ran}(I_{H^2(\mathbb{D})} - M_{\Theta_{i_j}} M_{\Theta_{i_j}}^*),$$

for some inner function $\Theta_{i_j} \in H^\infty(\mathbb{D})$ for all $j = 1, \ldots, m$. Let $\tilde{\Theta}_{i_j}$ be the extension of $\Theta_{i_j}$ to $H^\infty(\mathbb{D}^n)$, that is, as a multiplier,

$$M_{\tilde{\Theta}_{i_j}} = I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})} \otimes M_{\Theta_{i_j}} \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})}.$$

Hence,

$$I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}} = I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})} \otimes (I_{H^2(\mathbb{D})} - M_{\Theta_{i_j}} M^*_{\Theta_{i_j}}) \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})},$$

so that

$$\begin{aligned}&\prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}) \\ &= I_{H^2(\mathbb{D})} \otimes \cdots \otimes (I_{H^2(\mathbb{D})} - M_{\Theta_{i_1}} M^*_{\Theta_{i_1}}) \otimes \cdots \otimes (I_{H^2(\mathbb{D})} - M_{\Theta_{i_m}} M^*_{\Theta_{i_m}}) \otimes I_{H^2(\mathbb{D})} \otimes \cdots \otimes I_{H^2(\mathbb{D})},\end{aligned}$$

Taking into account that $\mathcal{Q}$ is the range of the right hand side operator, that is,

$$\begin{aligned}\mathcal{Q} &= H^2(\mathbb{D}) \otimes \cdots \otimes \mathcal{Q}_{i_1} \otimes \cdots \otimes \mathcal{Q}_{i_m} \otimes \cdots \otimes H^2(\mathbb{D}) \\ &= \text{ran}[\prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}})],\end{aligned}$$

we deduce readily that

$$P_{\mathcal{Q}^\perp} = I_{H^2(\mathbb{D}^n)} - \prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}).$$

Consequently, by Lemma 2.5 we have

$$\mathcal{Q}^\perp = \tilde{\Theta}_{i_1} H^2(\mathbb{D}^n) + \cdots + \tilde{\Theta}_{i_m} H^2(\mathbb{D}^n),$$

or

$$\mathcal{Q} = H^2(\mathbb{D}^n)/[\tilde{\Theta}_{i_1} H^2(\mathbb{D}^n) + \cdots + \tilde{\Theta}_{i_m} H^2(\mathbb{D}^n)].$$

Conversely, let

$$\mathcal{Q} = H^2(\mathbb{D}^n)/[\tilde{\Theta}_{i_1} H^2(\mathbb{D}^n) + \cdots + \tilde{\Theta}_{i_m} H^2(\mathbb{D}^n)].$$

Then

$$P_{\mathcal{Q}} = \prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}).$$

Also for all $s \neq t$,

$$\begin{aligned}P_{\mathcal{Q}} M_{z_s} M^*_{z_t} P_{\mathcal{Q}} &= \prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}) M^*_{z_t} M_{z_s} \prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}) \\ &= P_{\mathcal{Q}} M^*_{z_t} [\prod_{\substack{1 \leq i_1 < \ldots < i_m \leq n \\ i_j \neq t}} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}})] \\ &\quad [\prod_{\substack{1 \leq i_1 < \ldots < i_m \leq n \\ i_j \neq s}} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}})] M_{z_s} P_{\mathcal{Q}} \\ &= P_{\mathcal{Q}} M^*_{z_t} \prod_{1 \leq i_1 < \ldots < i_m \leq n} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}) M_{z_s} P_{\mathcal{Q}} \\ &= P_{\mathcal{Q}} M^*_{z_t} P_{\mathcal{Q}} M_{z_s} P_{\mathcal{Q}}.\end{aligned}$$

Consequently, for all $s \neq t$

$$C_{z_s} C^*_{z_t} = P_{\mathcal{Q}} M_{z_s} M^*_{z_t}|_{\mathcal{Q}} = P_{\mathcal{Q}} M^*_{z_t} P_{\mathcal{Q}} M_{z_s}|_{\mathcal{Q}} = C^*_{z_t} C_{z_s},$$

and hence $\mathcal{Q}$ is doubly commuting. This concludes the proof. ∎

This result is a generalization of Theorem 3.1 of [11] by Izuchi, Nakazi and Seto on the base case $n = 2$.

To complete this section, we present the following result concerning the orthogonal projection formulae of the co-doubly commuting submodules and the doubly commuting quotient modules of $H^2(\mathbb{D}^n)$.

COROLLARY 4.2. *Let $\mathcal{Q}$ be a doubly commuting submodule of $H^2(\mathbb{D}^n)$. Then there exists an integer $m \in \{1, \ldots, n\}$ and inner functions $\Theta_{i_j} \in H^\infty(\mathbb{D})$ such that*

$$\mathcal{Q}^\perp = \sum_{1 \leq i_1 < \ldots < i_m \leq n} \tilde{\Theta}_{i_j} H^2(\mathbb{D}^n),$$

*where $\tilde{\Theta}_i(\boldsymbol{z}) = \Theta_{i_j}(z_{i_j})$ for all $\boldsymbol{z} \in \mathbb{D}^n$. Moreover,*

$$P_{\mathcal{Q}} = I_{H^2(\mathbb{D}^n)} - \prod_{j=1}^{m} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}),$$

*and*

$$P_{\mathcal{Q}^\perp} = \prod_{j=1}^{m} (I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}).$$

The above result is the co-doubly commuting submodules analogue of Beurling's theorem on submodules of $H^2(\mathbb{D})$.

We finally point out that the earlier classifications of the doubly commuting quotient modules by Izuchi, Nakazi and Seto [11] has many deep applications in the study of the submodules and the quotient modules of the Hardy module over the bidisc (cf. [8, 9]). Some of these extensions in $n$-variables ($n \geq 2$) will be addressed in future work. However, the issue of essential doubly commutativity of the co-doubly commuting submodules of $H^2(\mathbb{D}^n)$ will be discussed in the forthcoming paper [15].

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