AN INVARIANT SUBSPACE THEOREM AND INVARIANT SUBSPACES OF ANALYTIC REPRODUCING KERNEL HILBERT SPACES - II

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ABSTRACT. This paper is a follow-up contribution to our work [20] where we discussed some invariant subspace results for contractions on Hilbert spaces. Here we extend the results of [20] to the context of n-tuples of bounded linear operators on Hilbert spaces. Let $T = (T_1, \ldots, T_n)$ be a pure commuting co-spherically contractive n-tuple of operators on a Hilbert space $\mathcal{H}$ and $\mathcal{S}$ be a non-trivial closed subspace of $\mathcal{H}$. One of our main results states that: $\mathcal{S}$ is a joint T-invariant subspace if and only if there exists a partially isometric operator $\Pi \in \mathcal{B}(H^2_n(\mathcal{E}), \mathcal{H})$ such that $\mathcal{S} = \Pi H^2_n(\mathcal{E})$, where H^2_n is the Drury-Arveson space and $\mathcal{E}$ is a coefficient Hilbert space and $T_i \Pi = \Pi M_{z_i}$, $i = 1, \ldots, n$. In particular, it follows that a shift invariant subspace of a "nice" reproducing kernel Hilbert space over the unit ball in $\mathbb{C}^n$ is the range of a "multiplier" with closed range. Our work addresses the case of joint shift invariant subspaces of the Hardy space and the weighted Bergman spaces over the unit ball in $\mathbb{C}^n$.

1. Introduction

Let T be a bounded linear operator on a separable Hilbert space $\mathcal{H}$. Furthermore, assume that T is a contraction (that is, $\|Tf\| \leq \|f\|$ for all $f \in \mathcal{H}$) and $T^{*m} \to 0$ as $m \to 0$, in the strong operator topology. Examples of such $C_{\cdot 0}$-contractions include the multiplication operator M_z on $H^2_{\mathcal{E}}(\mathbb{D})$, where $\mathcal{E}$ is a separable Hilbert space and $H^2_{\mathcal{E}}(\mathbb{D})$ is the $\mathcal{E}$-valued Hardy space over the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

One of the cornerstones of operator theory and function theory is that a non-trivial closed M_z-invariant subspace of $H^2_{\mathcal{E}}(\mathbb{D})$ is the range of a partially isometric multiplier [19]. More precisely, let $\mathcal{S}$ be a non-trivial closed subspace of $H^2_{\mathcal{E}}(\mathbb{D})$. Then $\mathcal{S}$ is a M_z-invariant subspace of $H^2_{\mathcal{E}}(\mathbb{D})$ if and only if there exists a Hilbert space $\mathcal{F}$ and a partially isometric multiplier M_Θ with symbol $\Theta \in H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D})$ such that

$$\mathcal{S} = M_\Theta H^2_{\mathcal{F}}(\mathbb{D}) = \Theta H^2_{\mathcal{F}}(\mathbb{D}).$$

An equivalent formulation is that

$$P_{\mathcal{S}} = M_\Theta M_\Theta^*,$$

where $P_{\mathcal{S}}$ denotes the orthogonal projection of $H^2_{\mathcal{E}}(\mathbb{D})$ onto $\mathcal{S}$. This is the celebrated Beurling-Lax-Halmos theorem, due to A. Beurling [5], P. Lax [13] and P. Halmos [12].

In previous work [20], we have shown that the invariant subspaces of a $C_{\cdot 0}$ contraction T on $\mathcal{H}$ are given by the image of those partially isometric operators $\Pi : H^2_{\mathcal{F}}(\mathbb{D}) \to \mathcal{H}$ which

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intertwines T and M_z, that is, $\Pi M_z = T\Pi$. It also follows that the shift invariant subspaces of a large class of reproducing kernel Hilbert spaces also can be parameterized by the set of all partially isometric multipliers. These results provide a unifying framework for numerous invariant subspace theorems, including in particular the Beurling-Lax-Halmos theorem (see [17], [21], [9]) and shift-invariant subspace theorem for weighted Bergman spaces over the unit disc $\mathbb{D}$ (see [1], [6], [7], [14], [18]6).

With this motivation, in this paper, we consider a generalization of the classification of invariant subspaces of $C_{\cdot 0}$-contractions to pure co-spherically contractive n-tuple of commuting bounded linear operators.

Our present approach is an attempt to understand the joint invariant subspaces of tuples of commuting operators in the study of operator theory and function theory in several complex variables. The proofs of the results in this paper exploit systematically the well known properties of dilation theory and multiplier spaces of reproducing kernel Hilbert spaces [3], so they become simple and clear. Although our methods of proof are similar to those used in [20], the present paper extends significantly the class of multiplication operators tuples on reproducing kernel Hilbert spaces for which classification of invariant subspaces is known to hold (cf. [15], [10]). In particular, we obtain a complete characterization of shift invariant subspaces of the Hardy space, the Bergman space and the weighted Bergman spaces over the unit ball in $\mathbb{C}^n$.

Finally it is worth noting that, among other things, Theorem 4.4 address the following basic question: Let $\mathcal{H}$ denote the Drury-Arveson space, or the Hardy space, or the Bergman space, or a weighted Bergman space over $\mathbb{B}^n$ and $\mathcal{S}$ be a non-trivial closed joint $(M_{z_1}, \ldots, M_{z_n})$-invariant subspace of $\mathcal{H}$. Does there exists a "multiplier" with closed range such that $\mathcal{S}$ is the range of the multiplier?

However, in the Drury-Arveson space case, this issue was addressed by McCullough and Trent [15] (see also [10]).

The paper is organized as follows. Section 2 below contains the background material on commuting tuples of operators on Hilbert spaces. Section 3 contains a characterization of invariant subspaces of pure commuting co-spherically contractive tuples, and Section 4 presents the invariant subspace theorem for reproducing kernel Hilbert spaces.

List of symbols:

(1) All Hilbert spaces considered in this paper are separable and over $\mathbb{C}$. We denote the set of natural numbers including zero by $\mathbb{N}$.
(2) Let $\mathcal{H}$ be a Hilbert space and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. The orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$ is denoted by $P_{\mathcal{S}}$.
(3) Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}$ be Hilbert spaces. We denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ and $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$.
(4) Let $n \geq 1$ and $n \in \mathbb{N}$. The set of multi-indices will be denoted by $\mathbb{N}^n$. That is, $\mathbb{N}^n = \{\boldsymbol{k} = (k_1, \ldots, k_n) : k_i \in \mathbb{N}\}$.
(5) For $\{z_i\}_{i=1}^n \subseteq \mathbb{C}$, we denote $(z_1, \ldots, z_n) \in \mathbb{C}^n$ by $\boldsymbol{z}$.
(6) For each $\boldsymbol{k} \in \mathbb{N}^n$, define $z^{\boldsymbol{k}} = z_1^{k_1} \cdots z_n^{k_n}$.

(7) $\mathbb{B}^n = \{\boldsymbol{z} \in \mathbb{C}^n : \|\boldsymbol{z}\|_{\mathbb{C}^n} < 1\}$.

2. Preliminaries

A commuting n-tuple ($n \geq 1$) of bounded linear operators $T = (T_1, \ldots, T_n)$ is said to be *co-spherically contractive*, or define a *row contraction*, if

$$\|\sum_{i=1}^n T_i h_i\|^2 \leq \sum_{i=1}^n \|h_i\|^2, \quad (h_1, \ldots, h_n \in \mathcal{H}),$$

or, equivalently, if

$$\sum_{i=1}^n T_i T_i^* \leq I_{\mathcal{H}}.$$

Define the *defect operator* and the *defect space* of $T = (T_1, \ldots, T_n)$ as $D = (I_{\mathcal{H}} - \sum_{i=1}^n T_i T_i^*)^{\frac{1}{2}} \in \mathcal{B}(\mathcal{H})$ and $\mathcal{D} = \overline{\text{ran}} D$, respectively.

Natural examples of commuting co-spherically contractive tuples are the multiplication operator tuples $(M_{z_1}, \ldots, M_{z_n})$ on the Drury-Arveson space [4], the Hardy space, the Bergman space and the weighted Bergman spaces (see [2], [11], [22], or Proposition 4.1) all defined over $\mathbb{B}^n$. Recall that the Drury-Arveson space, denoted by H^2_n, is determined by the kernel function

$$K_1(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^n z_i \bar{w}_i)^{-1}. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n)$$

More generally, for each $\lambda \geq 1$, define the positive definite function $K_\lambda : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ by

$$K_\lambda(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^n z_i \bar{w}_i)^{-\lambda}. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n)$$

Then the Hardy space $H^2(\mathbb{B}^n)$, the Bergman space $L^2_a(\mathbb{B}^n)$, and the weighted Bergman spaces $L^2_{a,\alpha}(\mathbb{B}^n)$, with $\alpha > 0$, are reproducing kernel Hilbert spaces with kernel K_λ for $\lambda = n$, $n+1$ and $n + 1 + \alpha$, respectively.

Let $\mathcal{E}$ be a Hilbert space. We identify the Hilbert tensor product $H^2_n \otimes \mathcal{E}$ with the $\mathcal{E}$-valued H^2_n space $H^2_n(\mathcal{E})$, or the $\mathcal{B}(\mathcal{E})$-valued reproducing kernel Hilbert space with kernel function

$$(\boldsymbol{z}, \boldsymbol{w}) \mapsto K_1(\boldsymbol{z}, \boldsymbol{w}) I_{\mathcal{E}}. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n)$$

Then

$$H^2_n(\mathcal{E}) = \{f \in \mathcal{O}(\mathbb{B}^n, \mathcal{E}) : f(z) = \sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} z^{\boldsymbol{k}}, a_{\boldsymbol{k}} \in \mathcal{E}, \|f\|^2 := \sum_{\boldsymbol{k} \in \mathbb{N}^n} \frac{\|a_{\boldsymbol{k}}\|^2}{\gamma_{\boldsymbol{k}}} < \infty\},$$

where $\gamma_{\boldsymbol{k}} = \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!}$ are the multinomial coefficients and $\boldsymbol{k} \in \mathbb{N}^n$ (see [4], [21]).

Given a co-spherically contractive tuple $T = (T_1, \ldots, T_n)$ on $\mathcal{H}$, define the completely positive map $P_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$P_T(X) = \sum_{i=1}^n T_i X T_i^*. \qquad (X \in \mathcal{B}(\mathcal{H}))$$

Note that

$$\sum_{i=1}^{n} T_i T_i^* = P_T(I_{\mathcal{H}}) \leq I_{\mathcal{H}},$$

implies

$$I_{\mathcal{H}} \geq P_T(I_{\mathcal{H}}) \geq P_T^2(I_{\mathcal{H}}) \geq \cdots \geq P_T^m(I_{\mathcal{H}}) \geq \cdots \geq 0.$$

It then follows that,

$$P_\infty(T) := \text{SOT} - \lim_{m\to\infty} P_T^m(I_{\mathcal{H}})$$

exists and $0 \leq P_\infty(T) \leq I_{\mathcal{H}}$. A co-spherically contractive T is said to be *pure* (cf. [4], [16]) if

$$P_\infty(T) = 0.$$

3. Invariant subspaces of co-spherically contractive tuples

It is well known that a pure co-spherically contractive tuple T on $\mathcal{H}$ is jointly unitarily equivalent to the compressed multiplication operator tuple

$$P_{\mathcal{S}} M_z|_{\mathcal{Q}} := (P_{\mathcal{S}} M_{z_1}|_{\mathcal{Q}}, \ldots, P_{\mathcal{S}} M_{z_n}|_{\mathcal{Q}}),$$

for some joint $(M_{z_1}^*, \ldots, M_{z_n}^*)$-invariant subspace $\mathcal{Q}$ of $H_n^2(\mathcal{E})$ and a coefficient Hilbert space $\mathcal{E}$ (cf. [2], [4], [16], [21]). We include a proof of this fact for the sake of completeness in a relevant form regarding our purposes.

Theorem 3.1. *Let T be a pure commuting co-spherically contractive tuple on $\mathcal{H}$. Then the map $\Pi \in \mathcal{B}(H_n^2(\mathcal{D}), \mathcal{H})$ defined by*

$$\Pi(K_1(\cdot, \boldsymbol{w})\eta) = (I_{\mathcal{H}} - \sum_{i=1}^{n} \bar{w}_i T_i)^{-1} D\eta, \qquad (\boldsymbol{w} \in \mathbb{B}^n,\ \eta \in \mathcal{D})$$

is co-isometric and

$$\Pi M_{z_i} = T_i \Pi. \qquad (i = 1, \ldots, n)$$

Moreover

$$(\Pi^* h)(\boldsymbol{w}) = D(I_{\mathcal{H}} - \sum_{i=1}^{n} w_i T_i^*)^{-1} h, \qquad (\boldsymbol{w} \in \mathbb{B}^n,\ h \in \mathcal{H}),$$

and

$$H_n^2(\mathcal{D}) = \overline{span}\{z^{\boldsymbol{k}}(\Pi^*\mathcal{H}) : \boldsymbol{k} \in \mathbb{N}^n\}.$$

Proof. First, for $\boldsymbol{w} \in \mathbb{B}^n$ define $(w_1 I_{\mathcal{H}}, \ldots, w_n I_{\mathcal{H}}) \in \mathcal{B}(\mathcal{H}^n, \mathcal{H})$ by

$$(w_1 I_{\mathcal{H}}, \ldots, w_n I_{\mathcal{H}})(h_1, \ldots, h_n) = \sum_{i=1}^{n} w_i h_i. \qquad (h_1, \ldots, h_n \in \mathcal{H})$$

Since

$$\|(w_1 I_{\mathcal{H}}, \ldots, w_n I_{\mathcal{H}})\| = (\sum_{i=1}^{n} |w_i|^2)^{\frac{1}{2}} = \|\boldsymbol{w}\|_{\mathbb{C}^n},$$

it follows that

$$\|\sum_{i=1}^n w_i T_i^*\| = \|(w_1 I_{\mathcal{H}}, \ldots, w_n I_{\mathcal{H}})^*(T_1, \ldots, T_n)\| \leq \|(w_1 I_{\mathcal{H}}, \ldots, w_n I_{\mathcal{H}})^*\| \|(T_1, \ldots, T_n)\|$$

$$= (\sum_{i=1}^n |w_i|^2)^{\frac{1}{2}} \|\sum_{i=1}^n T_i T_i^*\|^{\frac{1}{2}} = \|\boldsymbol{w}\|_{\mathbb{C}^n} \|\sum_{i=1}^n T_i T_i^*\|^{\frac{1}{2}} < 1.$$

We now define $\Pi^* \in \mathcal{B}(\mathcal{H}, H^2_n(\mathcal{D}))$ by

$$(\Pi^* h)(\boldsymbol{z}) := D(I_{\mathcal{H}} - \sum_{i=1}^n z_i T_i^*)^{-1} h = \sum_{\boldsymbol{k} \in \mathbb{N}^n} (\gamma_{\boldsymbol{k}} D T^{*\boldsymbol{k}} h) z^{\boldsymbol{k}},$$

for $h \in \mathcal{H}$ and $\boldsymbol{z} \in \mathbb{B}^n$. Since for all $m \geq 1$,

$$P_T^m(D^2) = P_T^m(I_{\mathcal{H}} - P_T(I_{\mathcal{H}})) = P_T^m(I_{\mathcal{H}}) - P_T^{m+1}(I_{\mathcal{H}}),$$

and since $\{P_T^m(I_{\mathcal{H}})\}$ forms a telescoping series, it follows that

$$\|\Pi^* h\|^2 = \|\sum_{\boldsymbol{k} \in \mathbb{N}^n} (\gamma_{\boldsymbol{k}} D T^{*\boldsymbol{k}} h) z^{\boldsymbol{k}}\|^2 = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \gamma_{\boldsymbol{k}}^2 \|D T^{*\boldsymbol{k}} h\|^2 \|z^{\boldsymbol{k}}\|^2 = \sum_{\boldsymbol{k} \in \mathbb{N}^n} \gamma_{\boldsymbol{k}}^2 \|D T^{*\boldsymbol{k}} h\|^2 \frac{1}{\gamma_{\boldsymbol{k}}}$$

$$= \sum_{\boldsymbol{k} \in \mathbb{N}^n} \gamma_{\boldsymbol{k}} \|D T^{*\boldsymbol{k}} h\|^2 = \sum_{m=0}^\infty \sum_{|\boldsymbol{k}|=m} \gamma_{\boldsymbol{k}} \|D T^{*\boldsymbol{k}} h\|^2 = \sum_{m=0}^\infty \sum_{|\boldsymbol{k}|=m} \gamma_{\boldsymbol{k}} \langle T^{\boldsymbol{k}} D^2 T^{*\boldsymbol{k}} h, h\rangle$$

$$= \sum_{m=0}^\infty \langle \sum_{|\boldsymbol{k}|=m} \gamma_{\boldsymbol{k}} T^{\boldsymbol{k}} D^2 T^{*\boldsymbol{k}} h, h\rangle = \sum_{m=0}^\infty \langle P_T^m(D^2) h, h\rangle$$

$$= \sum_{m=0}^\infty (\langle P_T^m(I_{\mathcal{H}}) h, h\rangle - \langle P_T^{m+1}(I_{\mathcal{H}}) h, h\rangle)$$

$$= \|h\|^2 - \langle \lim_{m \to 0} P_T^m(I_{\mathcal{H}}) h, h\rangle,$$

for all $h \in \mathcal{H}$. Then, by applying $P_\infty(T) = \lim_{l \to \infty} P_T^l(I_{\mathcal{H}}) = 0$, we obtain

$$\|\Pi^* h\| = \|h\|. \qquad (h \in \mathcal{H})$$

In other words, Π is a co-isometry. Moreover, for $h \in \mathcal{H}$ and $\boldsymbol{w} \in \mathbb{B}^n$ and $\eta \in \mathcal{D}$, we have

$$\begin{aligned}
\langle \Pi(K_1(\cdot, \boldsymbol{w})\eta), h\rangle_{\mathcal{H}} &= \langle K_1(\cdot, \boldsymbol{w})\eta, D(I_{\mathcal{H}} - \sum_{i=1}^n w_i T_i^*)^{-1} h\rangle_{H^2_n(\mathcal{D})} \\
&= \langle \sum_{\boldsymbol{k} \in \mathbb{N}^n} (\gamma_{\boldsymbol{k}} \bar{w}^{\boldsymbol{k}} \eta) z^{\boldsymbol{k}}, \sum_{\boldsymbol{k} \in \mathbb{N}^n} (\gamma_{\boldsymbol{k}} D T^{*\boldsymbol{k}} h) z^{\boldsymbol{k}}\rangle_{H^2_n(\mathcal{D})} \\
&= \sum_{\boldsymbol{k} \in \mathbb{N}^n} \gamma_{\boldsymbol{k}} \bar{w}^{\boldsymbol{k}} \langle T^{\boldsymbol{k}} D\eta, h\rangle_{\mathcal{H}} \\
&= \langle (I_{\mathcal{H}} - \sum_{i=1}^n \bar{w}_i T_i)^{-1} D\eta, h\rangle_{\mathcal{H}},
\end{aligned}$$

which implies that

$$\Pi(K_1(\cdot, \boldsymbol{w})\eta) = (I_{\mathcal{H}} - \sum_{i=1}^{n} \bar{w}_i T_i)^{-1} D\eta.$$

Next, it follows easily that

$$\langle \Pi(z^{\boldsymbol{l}}\eta), h\rangle = \langle z^{\boldsymbol{l}}\eta, \sum_{\boldsymbol{k}\in\mathbb{N}^n} (\gamma_{\boldsymbol{k}} D T^{*\boldsymbol{k}} h) z^{\boldsymbol{k}}\rangle = \gamma_{\boldsymbol{l}} \|z^{\boldsymbol{l}}\|^2 \langle \eta, D T^{*\boldsymbol{l}} h\rangle = \langle T^{\boldsymbol{l}} D\eta, h\rangle,$$

where $\eta \in \mathcal{D}$ and $\boldsymbol{l} \in \mathbb{N}^n$, and hence

$$\Pi(z^{\boldsymbol{l}}\eta) = T^{\boldsymbol{l}} D\eta. \qquad (\boldsymbol{l} \in \mathbb{N}^n, \eta \in \mathcal{D})$$

Therefore, we have

$$\Pi M_z(z^{\boldsymbol{k}}\eta) = \Pi(z^{\boldsymbol{k}+e_i}\eta) = T^{\boldsymbol{k}+e_i} D\eta = T_i(T^{\boldsymbol{k}} D\eta) = T_i \Pi(z^{\boldsymbol{k}}\eta),$$

for $\boldsymbol{l} \in \mathbb{N}^n$ and $\eta \in \mathcal{D}$, proving $\Pi M_{z_i} = T_i \Pi$ for $i = 1, \ldots, n$.
Finally, since $\overline{\text{span}}\{z^{\boldsymbol{k}}(\Pi^* \mathcal{H}) : \boldsymbol{k} \in \mathbb{N}^n\}$ is a joint $(M_{z_1}, \ldots, M_{z_n})$-reducing subspace of $H^2_n(\mathcal{D})$, we have

$$H^2_n(\mathcal{E}) = \overline{\text{span}}\{z^{\boldsymbol{k}} \Pi^* \mathcal{H} : \boldsymbol{k} \in \mathbb{N}^n\},$$

for some closed subspace $\mathcal{E} \subseteq \mathcal{D}$. On the other hand,

$$(I_{H^2_n(\mathcal{E})} - \sum_{i=1}^{n} M_{z_i} M_{z_i}^*) = P_{\mathcal{E}},$$

yields

$$Dh = (\Pi^* h)(0) = P_{\mathcal{E}}(\Pi^* h) = (I_{H^2_n(\mathcal{E})} - \sum_{i=1}^{n} M_{z_i} M_{z_i}^*)(\Pi^* h). \qquad (h \in \mathcal{H})$$

Therefore, $\mathcal{D} \subseteq \mathcal{E}$ and hence $\mathcal{D} = \mathcal{E}$. This completes the proof. ■

Now we present the main theorem of this section.

THEOREM 3.2. *Let $T = (T_1, \ldots, T_n)$ be a pure commuting co-spherically contractive tuple on $\mathcal{H}$ and $\mathcal{S}$ be a non-trivial closed subspace of $\mathcal{H}$. Then $\mathcal{S}$ is a joint T-invariant subspace of $\mathcal{H}$ if and only if there exists a Hilbert space $\mathcal{D}$ and a partially isometric operator $\Pi \in \mathcal{B}(H^2_n(\mathcal{E}), \mathcal{H})$ such that*

$$\Pi M_{z_i} = T_i \Pi,$$

and that

$$\mathcal{S} = \Pi(H^2_n(\mathcal{E})).$$

Proof. Let $\mathcal{S}$ be a non-trivial joint T-invariant closed subspace of $\mathcal{H}$. We denote by $T|_{\mathcal{S}} = (T_1|_{\mathcal{S}}, \ldots, T_n|_{\mathcal{S}})$ the n tuple of operators on $\mathcal{S}$. Note that $T|_{\mathcal{S}}$ is a commuting tuple and

$$\|\sum_{i=1}^{n} T_i|_{\mathcal{S}} h_i\|^2 = \|\sum_{i=1}^{n} T_i h_i\|^2. \qquad (h_1, \ldots, h_n \in \mathcal{S})$$

This clearly implies the row contractivity of $T|_{\mathcal{S}}$. Using the identity

$$P^m_{T|_{\mathcal{S}}}(I_{\mathcal{S}}) = \sum_{|\boldsymbol{k}|=m} \gamma_{\boldsymbol{k}} (T|_{\mathcal{S}})^{\boldsymbol{k}} (T|_{\mathcal{S}})^{*\boldsymbol{k}} = \sum_{|\boldsymbol{k}|=m} \gamma_{\boldsymbol{k}} P_{\mathcal{S}} T^{\boldsymbol{k}} P_{\mathcal{S}} T^{*\boldsymbol{k}}|_{\mathcal{S}} = \sum_{|\boldsymbol{k}|=m} \gamma_{\boldsymbol{k}} T^{\boldsymbol{k}} P_{\mathcal{S}} T^{*\boldsymbol{k}}|_{\mathcal{S}},$$

for $m \in \mathbb{N}$, we have

$$\begin{aligned}\langle P^m_{T|_{\mathcal{S}}}(I_{\mathcal{S}})h, h\rangle &= \sum_{|\boldsymbol{k}|=m} \gamma_{\boldsymbol{k}} \langle T^{\boldsymbol{k}} P_{\mathcal{S}} T^{*\boldsymbol{k}} h, h\rangle = \sum_{|\boldsymbol{k}|=m} \gamma_{\boldsymbol{k}} \|P_{\mathcal{S}} T^{*\boldsymbol{k}} h\|^2 \leq \sum_{|\boldsymbol{k}|=m} \gamma_{\boldsymbol{k}} \|T^{*\boldsymbol{k}} h\|^2 \\ &= \langle P^m_T(I_{\mathcal{H}})h, h\rangle. \qquad (h \in \mathcal{S})\end{aligned}$$

From this and the fact that $P^m_T(I_{\mathcal{H}}) \to 0$, in the strong operator topology, it readily follows that

$$P^m_{T|_{\mathcal{S}}}(I_{\mathcal{S}}) \to 0,$$

in the strong operator topology. By Theorem 3.1 applied to the pure co-spherically contractive tuple $T|_{\mathcal{S}}$, there exists a Hilbert space $\mathcal{E}$ and a co-isometric map $\Pi_{\mathcal{S}} : H^2_n(\mathcal{E}) \to \mathcal{S}$ such that

$$\Pi_{\mathcal{S}} M_{z_i} = T_i|_{\mathcal{S}} \Pi_{\mathcal{S}}. \qquad (i = 1, \ldots, n)$$

Now, consider the inclusion map $i_{\mathcal{S}} : \mathcal{S} \to \mathcal{H}$. The properties of the inclusion map imply immediately that $i_{\mathcal{S}}$ is an isometry and

$$i_{\mathcal{S}} T_j|_{\mathcal{S}} = T_j i_{\mathcal{S}}. \qquad (j = 1, \ldots, n)$$

Define $\Pi : H^2_n(\mathcal{E}) \to \mathcal{H}$ by

$$\Pi = i_{\mathcal{S}} \Pi_{\mathcal{S}}.$$

It follows that

$$\Pi M_{z_j} = i_{\mathcal{S}}\, \Pi_{\mathcal{S}} M_{z_j} = i_{\mathcal{S}} T_j|_{\mathcal{S}} \Pi_{\mathcal{S}} = T_j i_{\mathcal{S}} \Pi_{\mathcal{S}} = T_j \Pi,$$

for $j = 1, \ldots, n$, and

$$\Pi \Pi^* = (i_{\mathcal{S}}\, \Pi_{\mathcal{S}})(\Pi^*_{\mathcal{S}}\, i^*_{\mathcal{S}}) = i_{\mathcal{S}} i^*_{\mathcal{S}} = P_{\mathcal{S}}.$$

Thus Π is partially isometric and $\operatorname{ran} \Pi = \mathcal{S}$. This proves the necessary part.
The sufficient part follows easily from the intertwining property $T_i \Pi = \Pi M_{z_i}$, for all $i = 1, \ldots, n$, and the fact that $\mathcal{S} = \Pi(H^2_n(\mathcal{E}))$. This completes the proof. ∎

Also, the joint invariant subspaces of pure co-spherically contractive tuples can be characterized by the following corollary.

Corollary 3.3. *Let $T = (T_1, \ldots, T_n)$ be a pure commuting co-spherically contractive tuple on $\mathcal{H}$ and $\mathcal{S}$ be a non-trivial closed subspace of $\mathcal{H}$. Then $\mathcal{S}$ is a joint T-invariant subspace of $\mathcal{H}$ if and only if there exists a Hilbert space $\mathcal{E}$ and a bounded linear operator $\Pi \in \mathcal{B}(H^2_n(\mathcal{E}), \mathcal{H})$ such that $\Pi M_{z_i} = T_i \Pi$, for $i = 1, \ldots, n$, and*

$$P_{\mathcal{S}} = \Pi \Pi^*.$$

4. Invariant subspaces of analtytic Hilbert spaces

In this section we classify the joint shift invariant subspaces of a large class of reproducing kernel Hilbert spaces over $\mathbb{B}^n$ by applying the reasonings from the previous section. We begin by formulating the notion of analytic Hilbert spaces.

Let $K : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ be a positive definite kernel such that $K(\boldsymbol{z}, \boldsymbol{w})$ is holomorphic in the $\boldsymbol{z}$ variables and anti-holomorphic in $\boldsymbol{w}$ variables. Then the reproducing kernel Hilbert space $\mathcal{H}_K$, corresponding to the kernel function K, is a Hilbert space of holomorphic functions

on $\mathbb{B}^n$ (cf. [3], [21]). We say that $\mathcal{H}_K$ is an *analytic Hilbert space* over $\mathbb{B}^n$ if the following conditions are satisfied:
(i) the multiplication operators by the coordinate functions, denoted by $\{M_{z_1}, \ldots, M_{z_n}\}$ and defined by

$$(M_{z_i} f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}), \qquad (i = 1, \ldots, n)$$

are bounded, and
(ii) the tuple n-tuple $(M_{z_1}, \ldots, M_{z_n})$ on $\mathcal{H}_K$ is a pure co-spherically contractive tuple on $\mathcal{H}_K$, that is,

$$\sum_{i=1}^{n} M_{z_i} M_{z_i}^* \leq I_{\mathcal{H}_K},$$

and

$$P_\infty(M_z) = 0.$$

Common and important examples of analytic Hilbert spaces include the Drury-Arveson space H_n^2. We also give some typical examples of analytic Hilbert spaces.

Proposition 4.1. *Let $\lambda \geq 1$ and $K_\lambda : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ be the positive definite kernel defined as*

$$K_\lambda(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-\lambda}. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n)$$

Then $\mathcal{H}_{K_\lambda}$ is analytic.

Proof. If $\lambda = 1$, then $\mathcal{H}_K = H_n^2$, and hence the result holds trivially. So assume $\lambda > 1$. Notice that

$$K_\lambda(\boldsymbol{z}, \boldsymbol{w}) = K_1(\boldsymbol{z}, \boldsymbol{w}) K_{\lambda-1}(\boldsymbol{z}, \boldsymbol{w}), \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n)$$

and $K_{\lambda-1} : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ is positive a definite kernel on $\mathbb{B}^n$. By Theorem 2 of [8], there exists a coefficient Hilbert space $\mathcal{E}$ and a joint $(M_{z_1}^* \otimes I_{\mathcal{E}}, \ldots, M_{z_n}^* \otimes I_{\mathcal{E}})$-invariant subspace $\mathcal{Q}$ of $H_n^2(\mathcal{E})$ such that

$$(M_{z_1}, \ldots, M_{z_n}) \cong P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}} := P_{\mathcal{Q}}(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_n} \otimes I_{\mathcal{E}})|_{\mathcal{Q}}.$$

For $m \in \mathbb{N}$ we have

$$\begin{aligned} P^m_{P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}}}(I_{\mathcal{Q}}) &= \sum_{|\boldsymbol{k}|=m} \gamma_{\boldsymbol{k}} \big(P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}}\big)^{\boldsymbol{k}} \big(P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}}\big)^{*\boldsymbol{k}} \\ &= \sum_{|\boldsymbol{k}|=m} \gamma_{\boldsymbol{k}} P_{\mathcal{Q}} (M_z \otimes I_{\mathcal{E}})^{\boldsymbol{k}} (M_z \otimes I_{\mathcal{E}})^{*\boldsymbol{k}}|_{\mathcal{Q}}. \end{aligned}$$

This and the fact that H_n^2 is analytic readily implies that $P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}}$ is co-spherically contractive and

$$P_\infty(P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}}) = SOT - \lim_{m \to 0} P^m_{P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}}}(I_{\mathcal{Q}}) = 0.$$

Therefore $\mathcal{H}_K$ is analytic. This completes the proof. ∎

Let $\mathcal{H}_{K_1}$ and $\mathcal{H}_{K_2}$ be two analytic Hilbert spaces corresponding to the kernel functions K_1 and K_2 on $\mathbb{B}^n$ and $\mathcal{E}_1$ and $\mathcal{E}_2$ be two coefficient Hilbert spaces. An operator-valued map $\Theta : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ is said to be a *multiplier* from $\mathcal{H}_{K_1} \otimes \mathcal{E}_1$ to $\mathcal{H}_{K_2} \otimes \mathcal{E}_2$ if

$$\Theta f \in \mathcal{H}_{K_2} \otimes \mathcal{E}_2. \qquad (f \in \mathcal{H}_{K_1} \otimes \mathcal{E}_1)$$

The set of all multipliers from $\mathcal{H}_{K_1} \otimes \mathcal{E}_1$ to $\mathcal{H}_{K_2} \otimes \mathcal{E}_2$ is denoted by $\mathcal{M}(\mathcal{H}_{K_1} \otimes \mathcal{E}_1, \mathcal{H}_{K_2} \otimes \mathcal{E}_2)$. If $\Theta \in \mathcal{M}(\mathcal{H}_{K_1} \otimes \mathcal{E}_1, \mathcal{H}_{K_2} \otimes \mathcal{E}_2)$, then the multiplication operator $M_\Theta : \mathcal{H}_{K_1} \otimes \mathcal{E}_1 \to \mathcal{H}_{K_2} \otimes \mathcal{E}_2$ defined by

$$(M_\Theta f)(\boldsymbol{w}) = (\Theta f)(\boldsymbol{w}) = \Theta(\boldsymbol{w}) f(\boldsymbol{w}), \qquad (f \in \mathcal{H}_{K_1} \otimes \mathcal{E}_1, \boldsymbol{w} \in \mathbb{B}^n)$$

is bounded. This fact follows readily from the closed graph theorem.

The following provides a characterization of intertwining maps between analytic Hilbert spaces.

PROPOSITION 4.2. *Let $\mathcal{H}_{K_1}$ and $\mathcal{H}_{K_2}$ be two analytic Hilbert spaces over $\mathbb{B}^n$ such that*

$$\bigcap_{i=1}^{n} ker(M_{z_i} - w_i I_{\mathcal{H}_{K_1}})^* = \mathbb{C} K_1(\cdot, \boldsymbol{w}), \qquad (\boldsymbol{w} \in \mathbb{B}^n)$$

and, let $X \in \mathcal{B}(\mathcal{H}_{K_1} \otimes \mathcal{E}_1, \mathcal{H}_{K_2} \otimes \mathcal{E}_2)$. Then

$$X(M_{z_i} \otimes I_{\mathcal{E}_1}) = (M_{z_i} \otimes I_{\mathcal{E}_2}) X, \qquad (i = 1, \ldots, n)$$

if and only if $X = M_\Theta$ for some $\Theta \in \mathcal{M}(\mathcal{H}_{K_1} \otimes \mathcal{E}_1, \mathcal{H}_{K_2} \otimes \mathcal{E}_2)$.

Proof. Let $X \in \mathcal{B}(\mathcal{H}_{K_1} \otimes \mathcal{E}_1, \mathcal{H}_{K_2} \otimes \mathcal{E}_2)$ and $X(M_{z_i} \otimes I_{\mathcal{E}_1}) = (M_{z_i} \otimes I_{\mathcal{E}_2})X$, for all $i = 1, \ldots, n$. If $i = 1, \ldots, n$, $\zeta \in \mathcal{E}_2$ and $\boldsymbol{w} \in \mathbb{B}^n$, then

$$\begin{aligned}(M_{z_i} \otimes I_{\mathcal{E}_1})^*[X^*(K_2(\cdot, \boldsymbol{w}) \otimes \zeta)] &= X^*(M_{z_i} \otimes I_{\mathcal{E}_2})^*(K_2(\cdot, \boldsymbol{w}) \otimes \zeta) \\ &= \bar{w}_i [X^*(K_2(\cdot, \boldsymbol{w}) \otimes \zeta)].\end{aligned}$$

Thus

$$X^*(K_2(\cdot, \boldsymbol{w}) \otimes \zeta) \in \bigcap_{i=1}^{n} \ker\big((M_{z_i} \otimes I_{\mathcal{E}_1}) - w_i\big)^*.$$

Using this with the fact that

$$\bigcap_{i=1}^{n} \ker\Big(M_{z_i} - w_i I_{\mathcal{H}_{K_1}}\Big)^* = \mathbb{C} K_1(\cdot, \boldsymbol{w}),$$

we have

$$X^*(K_2(\cdot, \boldsymbol{w}) \otimes \zeta) = K_1(\cdot, \boldsymbol{w}) \otimes X(\boldsymbol{w})\zeta, \qquad (\zeta \in \mathcal{E}_2)$$

for some linear map $X(\boldsymbol{w}) : \mathcal{E}_2 \to \mathcal{E}_1$, and for all $\boldsymbol{w} \in \mathbb{B}^n$. Moreover,

$$\|X(\boldsymbol{w})\zeta\|_{\mathcal{E}_1} = \frac{1}{\|K_1(\cdot, \boldsymbol{w})\|_{\mathcal{H}_{K_1}}} \|X^*(K_2(\cdot, \boldsymbol{w}) \otimes \zeta)\|_{\mathcal{H}_{K_1} \otimes \mathcal{E}_1} \leq \frac{\|K_2(\cdot, \boldsymbol{w})\|_{\mathcal{H}_{K_2}}}{\|K_1(\cdot, \boldsymbol{w})\|_{\mathcal{H}_{K_1}}} \|X\| \|\zeta\|_{\mathcal{E}_2},$$

for all $\boldsymbol{w} \in \mathbb{B}^n$ and $\zeta \in \mathcal{E}_2$. Therefore $X(\boldsymbol{w})$ is bounded and $\Theta(\boldsymbol{w}) := X(\boldsymbol{w})^* \in \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ for each $\boldsymbol{w} \in \mathbb{B}^n$. Thus

$$X^*(K_2(\cdot, \boldsymbol{w}) \otimes \zeta) = K_1(\cdot, \boldsymbol{w}) \otimes \Theta(\boldsymbol{w})^* \zeta. \qquad (\boldsymbol{w} \in \mathbb{B}^n, \, \zeta \in \mathcal{E}_2)$$

In order to prove that $\Theta(\boldsymbol{w})$ is holomorphic we compute

$$\begin{aligned}\langle \Theta(\boldsymbol{w})\eta, \zeta\rangle_{\mathcal{E}_2} &= \langle \eta, \Theta(\boldsymbol{w})^*\zeta\rangle_{\mathcal{E}_1} = \langle K_1(\cdot,0)\otimes\eta, K_1(\cdot,\boldsymbol{w})\otimes\Theta(\boldsymbol{w})^*\zeta\rangle_{\mathcal{H}_{K_1}\otimes\mathcal{E}_1}\\ &= \langle X(K_1(\cdot,0)\otimes\eta), K_2(\cdot,\boldsymbol{w})\otimes\zeta\rangle_{\mathcal{H}_{K_2}\otimes\mathcal{E}_2}. \qquad (\eta\in\mathcal{E}_1,\ \zeta\in\mathcal{E}_2)\end{aligned}$$

Since $\boldsymbol{w}\mapsto K_2(\cdot,\boldsymbol{w})$ is anti-holomorphic, we conclude that $\boldsymbol{w}\mapsto\Theta(\boldsymbol{w})$ is holomorphic. Hence $\Theta\in\mathcal{M}(\mathcal{H}_{K_1}\otimes\mathcal{E}_1,\mathcal{H}_{K_2}\otimes\mathcal{E}_2)$.
If $\eta\in\mathcal{E}_1$, $\zeta\in\mathcal{E}_2$ and $\boldsymbol{z},\boldsymbol{w}\in\mathbb{B}^n$, then

$$\begin{aligned}\langle X(K_1(\cdot,\boldsymbol{w})\otimes\eta), K_2(\cdot,\boldsymbol{z})\otimes\zeta\rangle_{\mathcal{H}_{K_2}\otimes\mathcal{E}_2} &= \langle (K_1(\cdot,\boldsymbol{w})\otimes\eta), X^*(K_2(\cdot,\boldsymbol{z})\otimes\zeta)\rangle_{\mathcal{H}_{K_1}\otimes\mathcal{E}_1}\\ &= \langle (K_1(\cdot,\boldsymbol{w})\otimes\eta), K_1(\cdot,\boldsymbol{z})\otimes\Theta(\boldsymbol{z})^*\zeta\rangle_{\mathcal{H}_{K_1}\otimes\mathcal{E}_1}\\ &= K_1(\boldsymbol{z},\boldsymbol{w})\langle\eta,\Theta(\boldsymbol{z})^*\zeta\rangle_{\mathcal{E}_1}\\ &= K_1(\boldsymbol{z},\boldsymbol{w})\langle\Theta(\boldsymbol{z})\eta,\zeta\rangle_{\mathcal{E}_2}\\ &= \langle (M_\Theta(K_1(\cdot,\boldsymbol{w})\otimes\eta))(\boldsymbol{z}),\zeta\rangle_{\mathcal{E}_2}\\ &= \langle M_\Theta(K_1(\cdot,\boldsymbol{w})\otimes\eta), K_2(\cdot,\boldsymbol{z})\otimes\zeta\rangle_{\mathcal{H}_{K_2}\otimes\mathcal{E}_2}.\end{aligned}$$

Thus $X = M_\Theta$.
Conversely, let $\Theta\in\mathcal{M}(\mathcal{H}_{K_1}\otimes\mathcal{E}_1,\mathcal{H}_{K_2}\otimes\mathcal{E}_2)$. If $f\in\mathcal{H}_{K_1}\otimes\mathcal{E}_1$ and $\boldsymbol{w}\in\mathbb{B}^n$, then

$$(z_i\Theta f)(\boldsymbol{w}) = w_i\Theta(\boldsymbol{w})f(\boldsymbol{w}) = \Theta(\boldsymbol{w})w_i f(\boldsymbol{w}) = (\Theta z_i f)(\boldsymbol{w}),$$

for all $i=1,\ldots,n$. This completes the proof. ∎

The following corollary is a straightforward consequence of Proposition 4.2 and the fact that, for H^2_n,

$$\bigcap_{i=1}^n \ker(M_{z_i} - w_i I_{H^2_n})^* = \mathbb{C}K_1(\cdot,\boldsymbol{w}). \qquad (\boldsymbol{w}\in\mathbb{B}^n)$$

Corollary 4.3. *Let $\mathcal{H}_K$ be an analytic Hilbert space over $\mathbb{B}^n$ and $\mathcal{E}$ and $\mathcal{E}_*$ be two coefficient Hilbert spaces. Let X be in $\mathcal{B}(H^2_n\otimes\mathcal{E},\mathcal{H}_K\otimes\mathcal{E}_*)$. Then*

$$X(M_{z_i}\otimes I_{\mathcal{E}_1}) = (M_{z_i}\otimes I_{\mathcal{E}_2})X, \qquad (i=1,\ldots,n)$$

if and only if $X = M_\Theta$ for some $\Theta\in\mathcal{M}(H^2_n\otimes\mathcal{E},\mathcal{H}_K\otimes\mathcal{E}_*)$.

From the previous corollary and Theorem 3.2 we readily obtain the main result of this section.

Theorem 4.4. *Let $\mathcal{H}_K$ be an analytic Hilbert space over $\mathbb{B}^n$ and $\mathcal{E}_*$ be a coefficient Hilbert space. Let $\mathcal{S}$ be a non-trivial closed subspace of $\mathcal{H}_K\otimes\mathcal{E}_*$. Then $\mathcal{S}$ is a joint $(M_{z_1}\otimes I_{\mathcal{E}_*},\ldots,M_{z_n}\otimes I_{\mathcal{E}_*})$-invariant subspace of $\mathcal{H}_K\otimes\mathcal{E}_*$ if and only if there exists a Hilbert space $\mathcal{E}$ and a partially isometric multiplier $\Theta\in\mathcal{M}(H^2_n\otimes\mathcal{E},\mathcal{H}_K\otimes\mathcal{E}_*)$ such that*

$$\mathcal{S} = \Theta H^2_n(\mathcal{E}),$$

or equivalently,

$$P_{\mathcal{S}} = M_\Theta M_\Theta^*.$$

As a particular case of this theorem, we recover the following results of McCullough and Trent on a generalization of the Beurling-Lax Halmos theorem in the context of shift invariant subspaces of vector-valued Drury-Arveson space [15] (see also [10]).

COROLLARY 4.5. *Let $\mathcal{E}_*$ be a Hilbert space and $\mathcal{S}$ be a non-trivial closed subspace of $H^2_n \otimes \mathcal{E}_*$. Then $\mathcal{S}$ is a joint $(M_{z_1} \otimes I_{\mathcal{E}_*}, \ldots, M_{z_n} \otimes I_{\mathcal{E}_*})$-invariant subspace of $H^2_n \otimes \mathcal{E}_*$ if and only if there exists a Hilbert space $\mathcal{E}$ and a partially isometric multiplier $\Theta \in \mathcal{M}(H^2_n \otimes \mathcal{E}, H^2_n \otimes \mathcal{E}_*)$ such that $\mathcal{S} = \Theta(H^2_n \otimes \mathcal{E})$.*

The classification result, Theorem 4.4, is completely new even for the case of Hardy space and for the case of weighted Bergman spaces over $\mathbb{B}^n$.

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