TOEPLITZ AND ASYMPTOTIC TOEPLITZ OPERATORS ON $H^2(\mathbb{D}^n)$

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ABSTRACT. We initiate a study of Toeplitz operators and asymptotic Toeplitz operators on the Hardy space $H^2(\mathbb{D}^n)$ (over the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$). Our main results on Toeplitz and asymptotic Toeplitz operators can be stated as follows: Let T_{z_i} denote the multiplication operator on $H^2(\mathbb{D}^n)$ by the i^{th} coordinate function z_i, $i = 1, \ldots, n$, and let T be a bounded linear operator on $H^2(\mathbb{D}^n)$. Then the following hold:

(i) T is a Toeplitz operator (that is, $T = P_{H^2(\mathbb{D}^n)} M_\varphi|_{H^2(\mathbb{D}^n)}$, where M_φ is the Laurent operator on $L^2(\mathbb{T}^n)$ for some $\varphi \in L^\infty(\mathbb{T}^n)$) if and only if $T_{z_i}^* T T_{z_i} = T$ for all $i = 1, \ldots, n$.

(ii) T is an asymptotic Toeplitz operator if and only if $T =$ Toeplitz $+$ compact.

The case $n = 1$ is the well known results of Brown and Halmos, and Feintuch, respectively. We also present related results in the setting of vector-valued Hardy spaces over the unit disc.

1. INTRODUCTION

Although concrete bounded linear operators on Hilbert spaces exist in great variety and can exhibit interesting properties, one of the main concerns of function theory and operator theory has generally been the study of operators which are connected with the spaces of holomorphic and integrable functions. The class of Toeplitz and analytic Toeplitz operators have turned out to be one of the most important classes of concrete operators from this point of view.

Toeplitz operators on the Hardy space (or, on the l^2 space) were first studied by O. Toeplitz (and then by P. Hartman and A. Wintner in [16]). However, a systematic study of Toeplitz operators was triggered by the seminal paper of Brown and Halmos [4] on algebraic properties of Toeplitz operators on $H^2(\mathbb{D})$. Here $H^2(\mathbb{D})$ denote the Hardy space over the open unit disc $\mathbb{D}$ in $\mathbb{C}$. The study of Toeplitz operators on Hilbert spaces of holomorphic functions, like the Hardy space, the Bergman space and the weighted Bergman spaces, on domains in $\mathbb{C}^n$ is also one of the very active area of current research that brings together several areas of mathematics. For more information on this direction of research, we refer the reader to [8], [9], [10], [11], [17], [21] and the references therein.

Recall that a bounded linear operator T on $H^2(\mathbb{D})$ is said to be a *Toeplitz operator* if $T = P_{H^2(\mathbb{D})} M_\varphi|_{H^2(\mathbb{D})}$, where M_φ is the Laurent operator on $L^2(\mathbb{T})$ for some $\varphi \in L^\infty(\mathbb{T})$. Here $P_{H^2(\mathbb{D})}$ denotes the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$. The well-known Brown-Halmos theorem characterizes Toeplitz operators on $H^2(\mathbb{D})$ as follows (see the matricial characterization, Theorem 6 in [4]): Let T be a bounded linear operator on $H^2(\mathbb{D})$. Then T is a

2010 *Mathematics Subject Classification.* 47B35, 47A13, 47A45, 30H10, 30H50, 47A20, 47B07, 15B05.

Key words and phrases. Toeplitz operators, Hardy space over the polydisc, vector-valued Hardy spaces, compact operators, quotient spaces, model spaces.

Toeplitz operator if and only if

$$T_z^* T T_z = T.$$

One of the main results of this paper is the following generalization of Brown-Halmos theorem (see Theorem 3.1): A bounded linear operator T on $H^2(\mathbb{D}^n)$ is a Toeplitz operator if and only if

$$T_{z_j}^* T T_{z_j} = T,$$

for all $j = 1, \ldots, n$ (see Section 2 for notation and background definitions).

The notion of Toeplitzness was extended to more general settings by Barría and Halmos [2] and Feintuch [12]. Also see Popescu [13] for Toeplitzness in the non-commutative setting. Accordingly, following Feintuch (and Barría and Halmos [2]) we shall say that a bounded linear operator T on $H^2(\mathbb{D})$ is (uniformly) *asymptotically Toeplitz* if $\{T_z^{*m} T T_z^m\}_{m\geq 1}$ converges in operator norm. The following theorem due to Feintuch [12] gives a remarkable characterization of asymptotically Toeplitz operators: A bounded linear operator T on $H^2(\mathbb{D})$ is asymptotically Toeplitz if and only if $T =$ Toeplitz $+$ compact.

After a preliminary section (Section 2) on the Hardy space over unit polydisc, in Section 3, we introduce the asymptotic Toeplitz operators in polydisc setting (see Definition 3.3). In Theorem 3.4, we prove the following generalization of Feintuch's theorem: A bounded linear operator T on $H^2(\mathbb{D}^n)$ is asymptotically Toeplitz if and only if $T =$ Toeplitz $+$ compact.

In Section 4, we investigate Toeplitzness and asymptotic Toeplitzness of compressions of the n-tuple of multiplication operators $(T_{z_1}, \ldots, T_{z_n})$ to Beurling type quotient spaces of $H^2(\mathbb{D}^n)$. More specifically, let $\theta \in H^\infty(\mathbb{D}^n)$ be an inner function, that is, $|\theta| = 1$ on the distinguished boundary $\mathbb{T}^n$ of $\mathbb{D}^n$. Set

$$\mathcal{Q}_\theta = H^2(\mathbb{D}^n) \ominus \theta H^2(\mathbb{D}^n),$$

and

$$C_{z_i} = P_{\mathcal{Q}_\theta} T_{z_i}|_{\mathcal{Q}_\theta},$$

where $P_{\mathcal{Q}_\theta}$ denotes the orthogonal projection from $H^2(\mathbb{D}^n)$ onto $\mathcal{Q}_\theta$. A basic question is now to characterize those $T \in \mathcal{B}(\mathcal{Q}_\theta)$ for which

$$C_{z_i}^* T C_{z_i} = T.$$

Similarly, characterize those $T \in \mathcal{B}(\mathcal{Q}_\theta)$ for which

$$C_{z_i}^{*m} T C_{z_i}^m \to A,$$

in norm, for some $A \in \mathcal{B}(\mathcal{Q}_\theta)$ and for all $i = 1, \ldots, n$ (given a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$). In this general setting, to remedy the subtlety of the product domain $\mathbb{D}^n$, we modify the above condition by adding another natural condition. The main content of Section 4 is the following: Let $T, A \in \mathcal{B}(\mathcal{Q}_\theta)$. Then $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \ldots, n$, if and only if $A = 0$. Moreover, the following are equivalent:

(i) $C_{z_i}^{*m} T C_{z_i}^m \to A$ and $C_{z_i}^{*m}(T - A) C_{z_j}^m \to 0$ in norm for all $i, j = 1, \ldots, n$;

(ii) $C_{z_i}^{*m} T C_{z_i}^m \to 0$ in norm for all $i = 1, \ldots, n$;

(iii) T is compact.

In Section 5, we study the above questions in the vector-valued Hardy space over the unit disc setting. To be precise, let $\mathcal{E}$ be a Hilbert space, and let $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ be an inner multiplier [18]. Then the *model space* and the *model operator* are defined by $\mathcal{Q}_\Theta = H^2_{\mathcal{E}}(\mathbb{D}) \ominus \Theta H^2_{\mathcal{E}}(\mathbb{D})$ and $S_\Theta = P_{\mathcal{Q}_\Theta} T_z|_{\mathcal{Q}_\Theta}$, respectively. We prove that for every $T \in \mathcal{B}(\mathcal{Q}_\Theta)$, the following holds: (i) $S_\Theta^* T S_\Theta = T$ if and only if $T = 0$, and (ii) $\{S_\Theta^{*m} T S_\Theta^m\}_{m\geq 1}$ converges in norm if and only if T is compact.

2. Preliminaries

Let $n \geq 1$ and $\mathbb{D}^n$ be the open unit polydisc in $\mathbb{C}^n$. In the sequel, $\boldsymbol{z}$ will always denote a vector $\boldsymbol{z} = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$. The *Hardy space* $H^2(\mathbb{D}^n)$ over $\mathbb{D}^n$ is the Hilbert space of all holomorphic functions f on $\mathbb{D}^n$ such that

$$\|f\|_{H^2(\mathbb{D}^n)} := \left(\sup_{0 \leq r < 1} \int_{\mathbb{T}^n} |f(re^{i\theta_1}, \ldots, re^{i\theta_n})|^2 \, d\boldsymbol{\theta} \right)^{\frac{1}{2}} < \infty,$$

where $d\boldsymbol{\theta}$ is the normalized Lebesgue measure on the torus $\mathbb{T}^n$, the distinguished boundary of $\mathbb{D}^n$. Let $(T_{z_1}, \ldots, T_{z_n})$ denote the n-tuple of multiplication operators by the coordinate functions $\{z_i\}_{i=1}^n$, that is,

$$(T_{z_i} f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}),$$

for all $\boldsymbol{w} \in \mathbb{D}^n$ and $i = 1, \ldots, n$. We will often identify $H^2(\mathbb{D}^n)$ with the n-fold Hilbert space tensor product of one variable Hardy space as $H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$. In this identification, T_{z_i} can be represented as

$$I_{H^2(\mathbb{D})} \otimes \cdots \otimes \underbrace{T_z}_{i^{th}\ place} \otimes \cdots \otimes I_{H^2(\mathbb{D})},$$

for all $i = 1, \ldots, n$. Also one can identify the Hardy space (via the radial limits of functions in $H^2(\mathbb{D}^n)$) with the closed subspace of $L^2(\mathbb{T}^n)$ in the following sense: Let $\{e_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{Z}^n\}$ be the orthonormal basis of $L^2(\mathbb{T}^n)$, where $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $e_{\boldsymbol{k}} = e^{i\theta_1 k_1} \cdots e^{i\theta_n k_n}$. Then a function

$$\sum_{\boldsymbol{k} \in \mathbb{Z}^n} a_{\boldsymbol{k}} e_{\boldsymbol{k}} \in L^2(\mathbb{T}^n),$$

is the radial limit function of some function in $H^2(\mathbb{D}^n)$ if and only if $a_{\boldsymbol{k}} = 0$ whenever at least one of the k_j, $j = 1, \ldots, n$, is negative. In particular, the set of all monomials $\{\boldsymbol{z}^{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{Z}_+^n\}$ form an orthonormal basis for $H^2(\mathbb{D}^n)$, where $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ and $\boldsymbol{z}^{\boldsymbol{k}} = z_1^{k_1} \cdots z_n^{k_n}$ (cf. [1], [20]). We use $P_{H^2(\mathbb{D}^n)}$ to denote the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$, that is,

$$P_{H^2(\mathbb{D}^n)}\Big(\sum_{\boldsymbol{k} \in \mathbb{Z}^n} a_{\boldsymbol{k}} e_{\boldsymbol{k}}\Big) = \sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} a_{\boldsymbol{k}} e_{\boldsymbol{k}},$$

for all $\sum_{\boldsymbol{k} \in \mathbb{Z}^n} a_{\boldsymbol{k}} e_{\boldsymbol{k}}$ in $L^2(\mathbb{T}^n)$.

For $\varphi \in L^\infty(\mathbb{T}^n)$, the *Toeplitz operator* with symbol φ is the operator $T_\varphi \in \mathcal{B}(H^2(\mathbb{D}^n))$ defined by

$$T_\varphi f = P_{H^2(\mathbb{D}^n)}(M_\varphi f) \qquad (f \in H^2(\mathbb{D}^n)),$$

where M_φ is the Laurent operator on $L^2(\mathbb{T}^n)$ defined by $M_\varphi g = \varphi g$ for all $g \in L^2(\mathbb{T}^n)$. Therefore

$$T_\varphi = P_{H^2(\mathbb{D}^n)} M_\varphi|_{H^2(\mathbb{D}^n)}.$$

For the relevant results on Toeplitz operators on $H^2(\mathbb{D}^n)$ we refer the reader to [3, 6, 9, 17, 19] and references therein.

The following lemma will prove useful in what follows.

Lemma 2.1. *Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ be a compact operator. If R is a contraction on $\mathcal{H}$, and if $R^{*m} \to 0$ in strong operator topology, then $R^{*m} A \to 0$ in norm.*

Proof. This is a particular case of ([3], 1.3 (d), page 3). □

In what follows, for each $\boldsymbol{k} \in \mathbb{Z}_+^n$ and $\boldsymbol{l} \in \mathbb{Z}^n$, we write $T_{\boldsymbol{z}}^{\boldsymbol{k}} = T_{z_1}^{k_1} \cdots T_{z_n}^{k_n}$, $M_{e^{i\theta}}^{\boldsymbol{l}} = M_{e^{i\theta_1}}^{l_1} \cdots M_{e^{i\theta_n}}^{l_n}$, $T_{\boldsymbol{z}}^{*\boldsymbol{k}} = T_{z_1}^{*k_1} \cdots T_{z_n}^{*k_n}$ and $M_{e^{i\theta}}^{*\boldsymbol{l}} = M_{e^{i\theta_1}}^{*l_1} \cdots M_{e^{i\theta_n}}^{*l_n}$.

3. Toeplitz operators in several variables

In the following we prove a generalization of Brown and Halmos characterization [4] of Toeplitz operators on $H^2(\mathbb{D})$. This result should be compared with the algebraic characterization of Guo and Wang [15] which states that T in $\mathcal{B}(H^2(\mathbb{D}^n))$ is a Toeplitz operator if and only if $T_\varphi^* T T_\varphi = T$ for all inner function $\varphi \in H^\infty(\mathbb{D}^n)$.

Theorem 3.1. *Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$. Then T is a Toeplitz operator if and only if $T_{z_j}^* T T_{z_j} = T$ for all $j = 1, \ldots n$.*

Proof. For each $k \in \mathbb{Z}_+$, define $\boldsymbol{k}_d \in \mathbb{Z}_+^n$ by $\boldsymbol{k}_d = (k, \ldots, k)$. From $T_{z_j}^* T T_{z_j} = T$, $j = 1, \ldots n$, we obtain that

$$T_{\boldsymbol{z}}^{*\boldsymbol{k}_d} T T_{\boldsymbol{z}}^{\boldsymbol{k}_d} = T,$$

which implies that

$$\begin{aligned}\langle T e_{\boldsymbol{i}+\boldsymbol{k}_d}, e_{\boldsymbol{j}+\boldsymbol{k}_d} \rangle &= \langle T T_{\boldsymbol{z}}^{\boldsymbol{k}_d} e_{\boldsymbol{i}}, T_{\boldsymbol{z}}^{\boldsymbol{k}_d} e_{\boldsymbol{j}} \rangle \\ &= \langle T e_{\boldsymbol{i}}, e_{\boldsymbol{j}} \rangle,\end{aligned}$$

for all $k \in \mathbb{Z}_+$ and $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}_+^n$. Now for each $\boldsymbol{l}, \boldsymbol{m} \in \mathbb{Z}^n$, there exists $\boldsymbol{t} = (t_1, \ldots t_n) \in \mathbb{Z}_+^n$ such that $\boldsymbol{l} + \boldsymbol{k}_d, \boldsymbol{m} + \boldsymbol{k}_d \in \mathbb{Z}_+^n$ for all $\boldsymbol{k}_d \geq \boldsymbol{t}$ (that is, $k \geq t_j$ for all $j = 1, \ldots, n$). Hence setting

$$A_k = M_{e^{i\theta}}^{*\boldsymbol{k}_d} T P_{H^2(\mathbb{D}^n)} M_{e^{i\theta}}^{\boldsymbol{k}_d},$$

for each $k \geq 1$, we have

$$\begin{aligned}\langle A_k e_{\boldsymbol{l}}, e_{\boldsymbol{m}} \rangle_{L^2(\mathbb{T}^n)} &= \langle T P_{H^2(\mathbb{D}^n)} M_{e^{i\theta}}^{\boldsymbol{k}_d} e_{\boldsymbol{l}}, M_{e^{i\theta}}^{\boldsymbol{k}_d} e_{\boldsymbol{m}} \rangle_{L^2(\mathbb{T}^n)} \\ &= \langle T P_{H^2(\mathbb{D}^n)} e_{\boldsymbol{l}+\boldsymbol{k}_d}, e_{\boldsymbol{m}+\boldsymbol{k}_d} \rangle_{L^2(\mathbb{T}^n)},\end{aligned}$$

and therefore, for all $\boldsymbol{k}_d \geq \boldsymbol{t}$, we have that

$$\begin{aligned}\langle A_k e_{\boldsymbol{l}}, e_{\boldsymbol{m}} \rangle_{L^2(\mathbb{T}^n)} &= \langle T e_{\boldsymbol{l}+\boldsymbol{k}_d}, e_{\boldsymbol{m}+\boldsymbol{k}_d} \rangle_{H^2(\mathbb{D}^n)} \\ &= \langle T e_{\boldsymbol{l}+\boldsymbol{t}}, e_{\boldsymbol{m}+\boldsymbol{t}} \rangle_{H^2(\mathbb{D}^n)}.\end{aligned}$$

This implies in particular that

$$\langle A_k e_{\boldsymbol{l}}, e_{\boldsymbol{m}} \rangle \to \langle T e_{\boldsymbol{l}+\boldsymbol{t}}, e_{\boldsymbol{m}+\boldsymbol{t}} \rangle \text{ as } k \to \infty.$$

Let the bilinear form η on the linear span of $\{e_{\boldsymbol{s}} : \boldsymbol{s} \in \mathbb{Z}^n\}$ be defined by

$$\eta(e_{\boldsymbol{l}}, e_{\boldsymbol{m}}) = \lim_{k\to\infty} \langle A_k e_{\boldsymbol{l}}, e_{\boldsymbol{m}}\rangle,$$

for all $\boldsymbol{l}, \boldsymbol{m} \in \mathbb{Z}^n$. Since $\|A_k\| \leq \|T\|$, $k \geq 1$, it follows that η is a bounded bilinear form. Therefore, η can be extended to a bounded bilinear form (again denoted by η) on all of $L^2(\mathbb{T}^n)$, and hence there exists a unique bounded linear operator A_∞ on $L^2(\mathbb{T}^n)$ such that

$$\eta(f, g) = \langle A_\infty f, g\rangle = \lim_{k\to\infty} \langle A_k f, g\rangle,$$

for all $f, g \in L^2(\mathbb{T}^n)$. Now let $j \in \{1, \ldots, n\}$, $\boldsymbol{l}, \boldsymbol{m} \in \mathbb{Z}^n$ and set

$$\epsilon_j = (0, \ldots, \underbrace{1}_{j^{th}\ place}, \ldots, 0).$$

Then for all k sufficiently large (depending on $\boldsymbol{l}, \boldsymbol{m}$ and j), we have

$$\begin{aligned}
\langle (M^{*\boldsymbol{k}_d}_{e^{i\theta}} T P_{H^2(\mathbb{D}^n)} M^{\boldsymbol{k}_d}_{e^{i\theta}}) e_{\boldsymbol{l}+\epsilon_j}, e_{\boldsymbol{m}+\epsilon_j}\rangle_{L^2(\mathbb{T}^n)} &= \langle T P_{H^2(\mathbb{D}^n)} e_{\boldsymbol{l}+\boldsymbol{k}_d+\epsilon_j}, e_{\boldsymbol{m}+\boldsymbol{k}_d+\epsilon_j}\rangle_{L^2(\mathbb{T}^n)} \\
&= \langle T e_{\boldsymbol{l}+\boldsymbol{k}_d+\epsilon_j}, e_{\boldsymbol{m}+\boldsymbol{k}_d+\epsilon_j}\rangle_{H^2(\mathbb{D}^n)} \\
&= \langle T^*_{z_j} T T_{z_j} e_{\boldsymbol{l}+\boldsymbol{k}_d}, e_{\boldsymbol{m}+\boldsymbol{k}_d}\rangle_{H^2(\mathbb{D}^n)} \\
&= \langle T e_{\boldsymbol{l}+\boldsymbol{k}_d}, e_{\boldsymbol{m}+\boldsymbol{k}_d}\rangle_{H^2(\mathbb{D}^n)} \\
&= \langle A_k e_{\boldsymbol{l}}, e_{\boldsymbol{m}}\rangle_{L^2(\mathbb{T}^n)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle A_\infty e_{\boldsymbol{l}+\epsilon_j}, e_{\boldsymbol{m}+\epsilon_j}\rangle_{L^2(\mathbb{T}^n)} &= \lim_{k\to\infty} \langle (M^{*\boldsymbol{k}_d}_{e^{i\theta}} T P_{H^2(\mathbb{D}^n)} M^{\boldsymbol{k}_d}_{e^{i\theta}}) e_{\boldsymbol{l}+\epsilon_j}, e_{\boldsymbol{m}+\epsilon_j}\rangle_{L^2(\mathbb{T}^n)} \\
&= \langle A_\infty e_{\boldsymbol{l}}, e_{\boldsymbol{m}}\rangle_{L^2(\mathbb{T}^n)},
\end{aligned}$$

and consequently $M^*_{e^{i\theta_j}} A_\infty M_{e^{i\theta_j}} = A_\infty$, that is, $A_\infty M_{e^{i\theta_j}} = M_{e^{i\theta_j}} A_\infty$. Hence there exists φ in $L^\infty(\mathbb{T}^n)$ such that $A_\infty = M_\varphi$ [18]. Finally, we note that for $f, g \in H^2(\mathbb{D}^n)$,

$$\begin{aligned}
\langle A_\infty f, g\rangle_{L^2(\mathbb{T}^n)} &= \lim_{k\to\infty} \langle M^{*\boldsymbol{k}_d}_{e^{i\theta}} T P_{H^2(\mathbb{D}^n)} M^{\boldsymbol{k}_d}_{e^{i\theta}} f, g\rangle_{L^2(\mathbb{T}^n)} \\
&= \lim_{k\to\infty} \langle T^{*\boldsymbol{k}_d}_{\boldsymbol{z}} T T^{\boldsymbol{k}_d}_{\boldsymbol{z}} f, g\rangle_{H^2(\mathbb{D}^n)},
\end{aligned}$$

that is,

$$\langle A_\infty f, g\rangle_{L^2(\mathbb{T}^n)} = \langle T f, g\rangle_{H^2(\mathbb{D}^n)},$$

and hence

$$\begin{aligned}
\langle P_{H^2(\mathbb{D}^n)} A_\infty f, g\rangle_{H^2(\mathbb{D}^n)} &= \langle A_\infty f, g\rangle_{L^2(\mathbb{T}^n)} \\
&= \langle T f, g\rangle_{H^2(\mathbb{D}^n)}.
\end{aligned}$$

Therefore, $T = P_{H^2(\mathbb{D}^n)} A_\infty|_{H^2(\mathbb{D}^n)} = P_{H^2(\mathbb{D}^n)} M_\varphi|_{H^2(\mathbb{D}^n)}$, that is, T is a Toeplitz operator.

Conversely, let $\varphi \in L^\infty(\mathbb{T}^n)$ and $T = P_{H^2(\mathbb{D}^n)} M_\varphi|_{H^2(\mathbb{D}^n)}$. Then for $f, g \in H^2(\mathbb{D}^n)$ and $j = 1, \ldots n$, we have

$$\begin{aligned}
\langle (T^*_{z_j} T T_{z_j}) f, g\rangle_{H^2(\mathbb{D}^n)} &= \langle \varphi e^{i\theta_j} f, e^{i\theta_j} g\rangle_{L^2(\mathbb{T}^n)} \\
&= \langle \varphi f, g\rangle_{L^2(\mathbb{T}^n)},
\end{aligned}$$

that is,

$$\langle (T_{z_j}^* T T_{z_j}) f, g\rangle_{H^2(\mathbb{D}^n)} = \langle P_{H^2(\mathbb{D}^n)} M_\varphi|_{H^2(\mathbb{D}^n)} f, g\rangle_{H^2(\mathbb{D}^n)},$$

and therefore $T_{z_j}^* T T_{z_j} = T$ for all $j = 1, \ldots n$, as desired. □

We now characterize compact operators on $H^2(\mathbb{D}^n)$ in terms of the multiplication operators $\{T_{z_1}, \ldots, T_{z_n}\}$. This characterization was proved by Feintuch [12] in the case of $n = 1$.

Theorem 3.2. *A bounded linear map T on $H^2(\mathbb{D}^n)$ is compact if and only if $T_{z_i}^{*m} T T_{z_j}^m \to 0$ in norm for all $i, j \in \{1, ..., n\}$.*

Proof. Let T on $H^2(\mathbb{D}^n)$ be a bounded operator. First observe that for each $m \geq 1$, we have

$$T_z^m T_z^{*m} = I_{H^2(\mathbb{D})} - P_{\mathcal{F}_m},$$

where

$$\mathcal{F}_m = \mathbb{C} \oplus z\mathbb{C} \oplus \cdots \oplus z^{m-1}\mathbb{C},$$

is an m-dimensional subspace of $H^2(\mathbb{D})$. For each $m \geq 1$, set

$$F_m = \prod_{i=1}^n (I_{H^2(\mathbb{D}^n)} - T_{z_i}^m T_{z_i}^{*m}).$$

Then

$$\begin{aligned} F_m &= \prod_{i=1}^n (I_{H^2(\mathbb{D})} \otimes \cdots \otimes \underbrace{(I_{H^2(\mathbb{D})} - T_z^m T_z^{*m})}_{i^{th} place} \otimes \cdots \otimes I_{H^2(\mathbb{D})}) \\ &= \prod_{i=1}^n (I_{H^2(\mathbb{D})} \otimes \cdots \otimes \underbrace{P_{\mathcal{F}_m}}_{i^{th} place} \otimes \cdots \otimes I_{H^2(\mathbb{D})}) \\ &= P_{\mathcal{F}_m} \otimes \cdots \otimes P_{\mathcal{F}_m}, \end{aligned}$$

which gives that F_m is a finite rank operator and hence

$$\tilde{F}_m = TF_m + F_m T - F_m T F_m,$$

is a finite rank operator, $m \geq 1$. Moreover

$$\begin{aligned} T - \tilde{F}_m &= T - (TF_m + F_m T - F_m T F_m) \\ &= (I_{H^2(\mathbb{D}^n)} - F_m) T (I_{H^2(\mathbb{D}^n)} - F_m). \end{aligned}$$

Finally, observe that

$$\begin{aligned} I_{H^2(\mathbb{D}^n)} - F_m &= \sum_{1 \leq i_1 < \cdots < i_l \leq n} (-1)^{l+1} T_{z_{i_1}}^m \cdots T_{z_{i_l}}^m T_{z_{i_1}}^{*m} \cdots T_{z_{i_l}}^{*m} \\ &= \sum_{1 \leq i_1 < \cdots < i_l \leq n} (-1)^{l+1} (T_{z_{i_1}} \cdots T_{z_{i_l}})^m (T_{z_{i_1}} \cdots T_{z_{i_l}})^{*m}, \end{aligned}$$

for all $m \geq 1$. Hence, by hypothesis and the triangle inequality we have

$$\|T - \tilde{F}_m\| = \|(I_{H^2(\mathbb{D}^n)} - F_m) T (I_{H^2(\mathbb{D}^n)} - F_m)\| \to 0,$$

as $m \to \infty$, that is, T is a compact operator.
The converse follows from Lemma 2.1. This completes the proof. $\square$

In view of the preceding theorem, it seems reasonable to define asymptotic Toeplitz operators as follows (compare this with Feintuch [12] and Barría and Halmos [2]):

Definition 3.3. A bounded linear operator T on $H^2(\mathbb{D}^n)$ is said to be an asymptotic Toeplitz operator if there exists $A \in \mathcal{B}(H^2(\mathbb{D}^n))$ such that $T_{z_i}^{*m} T T_{z_i}^m \to A$ and $T_{z_i}^{*m}(T-A)T_{z_j}^m \to 0$ as $m \to \infty$ in norm, $1 \leq i, j \leq n$.

We close this section by characterizing asymptotic Toeplitz operators on $H^2(\mathbb{D}^n)$ as analogous characterization of asymptotic Toeplitz operators on $H^2(\mathbb{D})$ (see [12] and also Theorem 5.4 in Section 5).

Theorem 3.4. *Let T be a bounded linear operator on $H^2(\mathbb{D}^n)$. Then T is an asymptotic Toeplitz operator if and only if T is a compact perturbation of Toeplitz operator.*

Proof. Let $A \in \mathcal{B}(H^2(\mathbb{D}^n))$, $T_{z_i}^{*m} T T_{z_i}^m \to A$ and $T_{z_i}^{*m}(T-A)T_{z_j}^m \to 0$ in norm, as $m \to \infty$, and $1 \leq i, j \leq n$. Then for all $m \geq 1$,

$$\begin{aligned} \|A - T_{z_j}^* A T_{z_j}\| &\leq \|A - T_{z_j}^{*(m+1)} T T_{z_j}^{m+1}\| + \|T_{z_j}^{*(m+1)} T T_{z_j}^{m+1} - T_{z_j}^* A T_{z_j}\| \\ &\leq \|A - T_{z_j}^{*(m+1)} T T_{z_j}^{m+1}\| + \|T_{z_j}^{*m} T T_{z_j}^m - A\|, \end{aligned}$$

yields $T_{z_j}^* A T_{z_j} = A$ for all $j = 1, \ldots, n$. Also by Theorem 3.2, $T - A$ is compact on $H^2(\mathbb{D}^n)$. The converse follows from Lemma 2.1 and Theorem 3.1. This completes the proof. $\square$

The more interesting question now is to describe bounded linear operators T on $H^2(\mathbb{D}^n)$ (in terms of Toeplitz and Hankel operators) such that $T_{z_i}^{*m} T T_{z_i}^m \to A$ and $T_{z_i}^{*m}(T-A)T_{z_j}^m \to 0$ for some $A \in \mathcal{B}(H^2(\mathbb{D}^n))$ and as $m \to \infty$, $1 \leq i, j \leq n$, in the weak or strong operator topology.

4. Quotient spaces of $H^2(\mathbb{D}^n)$

The purpose of this section is to extend some of the results of Section 3 in the case when the ambient operator is the compression of $(T_{z_1}, \ldots, T_{z_n})$ to a quotient space of $H^2(\mathbb{D}^n)$, that is, a joint $(T_{z_1}^*, \ldots, T_{z_n}^*)$-invariant closed subspace of $H^2(\mathbb{D}^n)$. Note that a rich source of n-tuples of commuting contractions comes from quotient Hilbert spaces of $H^2(\mathbb{D}^n)$.

Let $\mathcal{Q}$ be a joint $(T_{z_1}^*, \ldots, T_{z_n}^*)$-invariant subspace of $H^2(\mathbb{D}^n)$. Set

$$C_{z_i} = P_{\mathcal{Q}} T_{z_i}|_{\mathcal{Q}},$$

for all $i = 1, \ldots, n$. Note that $\mathcal{Q}^\perp$ is a joint invariant subspace for $(T_{z_1}, \ldots, T_{z_n})$ and so

$$C_{z_i}^* = T_{z_i}^*|_{\mathcal{Q}} \in \mathcal{B}(\mathcal{Q}).$$

In the case $n = 1$, C_z is called a *Jordan block* [18]. In the several variables quotient space setting, we have the following analogue of Theorem 3.4.

Theorem 4.1. *Let $T, A \in \mathcal{B}(\mathcal{Q})$. Then $C_{z_i}^{*m} T C_{z_i}^m \to A$ and $C_{z_i}^{*m}(T-A)C_{z_j}^m \to 0$ in norm for all $i, j = 1, \ldots, n$ if and only if $T = A + K$, where $K \in \mathcal{B}(\mathcal{Q})$ is a compact operator and $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \ldots n$.*

Proof. We first note that, as in the proof of Theorem 3.4, the assumption $C_{z_i}^{*m} T C_{z_i}^m \to A$ as $m \to \infty$ implies that

$$C_{z_i}^* A C_{z_i} = A,$$

for all $i = 1, \dots n$. Now it follows from the definition of C_{z_i} that

$$C_{z_i}^{*m} = T_{z_i}^{*m}|_{\mathcal{Q}},$$

and hence

$$C_{z_i}^{*m}(T - A)C_{z_j}^m = T_{z_i}^{*m}(T - A)P_{\mathcal{Q}}T_{z_j}^m|_{\mathcal{Q}},$$

for all $i, j = 1, \dots, n$ and $m \geq 1$. By once again using the fact that

$$P_{\mathcal{Q}}T_{z_j}^m P_{\mathcal{Q}} = P_{\mathcal{Q}}T_{z_j}^m,$$

one sees that

$$T_{z_i}^{*m}(T - A)P_{\mathcal{Q}}T_{z_j}^m = T_{z_i}^{*m}(T - A)P_{\mathcal{Q}}T_{z_j}^m P_{\mathcal{Q}}.$$

Hence $C_{z_i}^{*m}(T - A)C_{z_j}^m \to 0$ in $\mathcal{B}(\mathcal{Q})$ if and only if $T_{z_i}^{*m}(T - A)P_{\mathcal{Q}}T_{z_j}^m \to 0$ in $\mathcal{B}(H^2(\mathbb{D}^n))$ as $m \to \infty$.

Therefore, if $C_{z_i}^{*m}(T - A)C_{z_j}^m \to 0$ as $m \to \infty$ in norm for all $i, j = 1, \dots, n$, then $T_{z_i}^{*m}(T - A)P_{\mathcal{Q}}T_{z_j}^m \to 0$ in $\mathcal{B}(H^2(\mathbb{D}^n))$ as $m \to \infty$, and consequently by Theorem 3.2, $(T - A)|_{\mathcal{Q}}$ is a compact operator on $H^2(\mathbb{D}^n)$. Therefore

$$(T - A) = (T - A)|_{\mathcal{Q}},$$

is a compact operator on $\mathcal{Q}$, which proves the necessary part.

Conversely, let $T - A$ be a compact operator on $\mathcal{Q}$ and $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \dots n$. Since $C_{z_i}^{*m} \to 0$ as $m \to \infty$ in the strong operator topology, Lemma 2.1 implies that

$$C_{z_i}^{*m}(T - A)C_{z_j}^m \to 0,$$

as $m \to \infty$. In particular, for all $i = 1, \dots n$

$$C_{z_i}^{*m} T C_{z_i}^m \to C_{z_i}^{*m} A C_{z_i}^m.$$

But $C_{z_i}^* A C_{z_i} = A$, $i = 1, \dots n$, yields us

$$C_{z_i}^{*m} T C_{z_i}^m \to A.$$

This completes the proof. $\square$

Considering the particular case $\mathcal{Q}_\theta = H^2(\mathbb{D}^n) \ominus \theta H^2(\mathbb{D}^n)$, the so called Beurling type quotient space of $H^2(\mathbb{D}^n)$, where $\theta \in H^\infty(\mathbb{D}^n)$ is an inner function, we get the following result.

Theorem 4.2. *Let $\theta \in H^\infty(\mathbb{D}^n)$ be an inner function and $\mathcal{Q}_\theta = H^2(\mathbb{D}^n) \ominus \theta H^2(\mathbb{D}^n)$ and $A \in \mathcal{B}(\mathcal{Q}_\theta)$. Then $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \dots n$, if and only if $A = 0$.*

Proof. Let $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \dots, n$. Since

$$\mathcal{Q}_\theta^\perp = \theta H^2(\mathbb{D}^n),$$

is a joint invariant subspace for $(T_{z_1}, \dots, T_{z_n})$, it follows that

$$P_{\mathcal{Q}_\theta} T_{z_i}^*|_{\mathcal{Q}_\theta} = T_{z_i}^* P_{\mathcal{Q}_\theta},$$

and hence

$$\begin{aligned} AP_{\mathcal{Q}_\theta} &= (C_{z_i}^* A C_{z_i}) P_{\mathcal{Q}_\theta} \\ &= (P_{\mathcal{Q}_\theta} T_{z_i}^*|_{\mathcal{Q}_\theta} A P_{\mathcal{Q}_\theta} T_{z_i}|_{\mathcal{Q}_\theta}) P_{\mathcal{Q}_\theta} \\ &= T_{z_i}^* A P_{\mathcal{Q}_\theta} T_{z_i} P_{\mathcal{Q}_\theta} \\ &= T_{z_i}^* A P_{\mathcal{Q}_\theta} T_{z_i} \\ &= T_{z_i}^* (A P_{\mathcal{Q}_\theta}) T_{z_i}. \end{aligned}$$

for all $i = 1, \dots, n$. This and Theorem 3.1 implies that $AP_{\mathcal{Q}_\theta}$ is a Toeplitz operator. Consequently, there exists $\psi \in L^\infty(\mathbb{T}^n)$ such that

$$AP_{\mathcal{Q}_\theta} = T_\psi.$$

On the other hand, since T_θ is an analytic Toeplitz operator, it follows that

$$AP_{\mathcal{Q}} T_\theta = 0.$$

Hence, using [Theorem 1, C. Gu [14]], we conclude that

$$\begin{aligned} T_{\psi\theta} &= T_\psi T_\theta \\ &= A P_{\mathcal{Q}_\theta} T_\theta \\ &= 0. \end{aligned}$$

This completes the proof of the theorem. □

Summing up the above two results and Lemma 2.1, we have the following generalization of Theorem 1.2 in [5].

Theorem 4.3. *For an inner function $\theta \in H^\infty(\mathbb{D}^n)$ and bounded linear operators T and A on $\mathcal{Q}_\theta = H^2(\mathbb{D}^n) \ominus \theta H^2(\mathbb{D}^n)$, the following are equivalent:*
(i) $C_{z_i}^{*m} T C_{z_i}^m \to A$ and $C_{z_i}^{*m}(T - A) C_{z_j}^m \to 0$ in norm for all $i, j = 1, \dots, n$;
(ii) $C_{z_i}^{*m} T C_{z_i}^m \to 0$ in norm for all $i = 1, \dots, n$;
(iii) T is compact.

For asymptotic Toeplitzness of composition operators on the Hardy space of the unit sphere in $\mathbb{C}^n$ we refer the reader to Nazarov and Shapiro [19], and Cuckovic and Le [7].

5. Asymptotic Toeplitz operators on $H^2_{\mathcal{E}}(\mathbb{D})$

The main purpose of this section is to characterize the compact operators on the model space $H^2_{\mathbb{C}^p}(\mathbb{D}) \ominus \Theta H^2_{\mathbb{C}^p}(\mathbb{D})$, where $\Theta \in H^\infty_{\mathcal{B}(\mathbb{C}^p)}(\mathbb{D})$ is an inner function. We note that this result for $p = 1$ case can be found in [5]. Moreover, our proof seems more shorter and conceptually different (for instance, compare Theorem 5.5 with Proposition 2.10 in [5]).

We begin with the definition of a Toeplitz operator with operator-valued symbol.

Definition 5.1. Let $\mathcal{E}$ be a Hilbert space. A bounded linear operator T on $H^2_{\mathcal{E}}(\mathbb{D})$ is said to be Toeplitz if there exists an operator-valued function Φ in $L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ such that $T = P_{H^2_{\mathcal{E}}(\mathbb{D})} M_\Phi |_{H^2_{\mathcal{E}}(\mathbb{D})}$.

Here let us observe, before we proceed further, the following characterization of Toeplitz operators on a vector-valued Hardy space. The result is probably known to the experts but we were not able to find a reference in the literature. Since the result follows from concepts and techniques used in the proof of Theorem 3.1, we give a sketch of the proof.

Theorem 5.2. *Let $\mathcal{E}$ be a Hilbert space and $T \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}))$. Then T is a Toeplitz operator if and only if $T_z^* T T_z = T$.*

Proof. Note first that $\{e_m \eta : m \in \mathbb{Z}, \eta \in \mathcal{E}\}$ is a total set in $L^2_{\mathcal{E}}(\mathbb{T})$, where $e_m = e^{im\theta}$, $m \in \mathbb{Z}$. For each $k \geq 1$, set

$$A_k = M^{*k}_{e^{i\theta}} T P M^k_{e^{i\theta}},$$

where $M_{e^{i\theta}}$ is the bilateral shift on $L^2_{\mathcal{E}}(\mathbb{T})$ and P is the orthogonal projection from $L^2_{\mathcal{E}}(\mathbb{T})$ onto $H^2_{\mathcal{E}}(\mathbb{D})$. If $T_z^* T T_z = T$ and $k \in \mathbb{Z}_+$, then

$$\langle T e_{i+k}\eta, e_{j+k}\zeta\rangle = \langle T e_i \eta, e_j \zeta\rangle,$$

for all $i, j \geq 0$. Then for each $l, m \in \mathbb{Z}$, as in the proof of Theorem 3.1, there exists $t \geq 0$ such that $l + k, m + k \geq 0$ for all $k \geq t$, and so

$$\langle A_k e_l \eta, e_m \zeta\rangle \to \langle T e_{l+t}\eta, e_{m+t}\zeta\rangle,$$

as $k \to \infty$. Then

$$(e_l\eta, e_m\zeta) \mapsto \lim_{k\to\infty} \langle A_k e_l \eta, e_m \zeta\rangle,$$

defines a bounded bilinear form on the span of $\{e_l \eta : l \in \mathbb{Z}, \eta \in \mathcal{E}\}$. Therefore, there exists (again, following the proof of Theorem 3.1) $A_\infty \in \mathcal{B}(L^2_{\mathcal{E}}(\mathbb{T}))$ such that

$$\langle A_\infty f, g\rangle = \lim_{k\to\infty} \langle A_k f, g\rangle,$$

for all $f, g \in L^2_{\mathcal{E}}(\mathbb{T})$. This yields

$$A_\infty M_{e^{i\theta}} = M_{e^{i\theta}} A_\infty.$$

Hence there exists a $\Phi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ such that

$$A_\infty = M_\Phi,$$

and hence

$$T = P_{H^2_{\mathcal{E}}(\mathbb{D})} M_\Phi |_{H^2_{\mathcal{E}}(\mathbb{D})} .$$

The proof of the converse part proceeds verbatim as that of Theorem 3.1. This completes the proof of the theorem. □

Following Feintuch [12] we now define an asymptotic Toeplitz operator on a vector-valued Hardy space.

Definition 5.3. Let $\mathcal{E}$ be a Hilbert space. A bounded linear operator T on $H^2_{\mathcal{E}}(\mathbb{D})$ is said to be an asymptotic Toeplitz operator if there exists $A \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}))$ such that $T_z^{*m} T T_z^m \to A$ as $m \to \infty$ in norm.

In the theorem below, we generalize the Feintuch's characterization [12] (see also Theorem F, page 195, [19]) of asymptotic Toeplitz operators on Hardy space to asymptotic Toeplitz operators on $\mathbb{C}^p$-valued Hardy space. However, the method of proof here is adapted from the original proof by Feintuch.

Theorem 5.4. *Let $T, A \in \mathcal{B}(H^2_{\mathbb{C}^p}(\mathbb{D}))$. Then $T_z^{*m} T T_z^m \to A$ in norm if and only if A is a Toeplitz operator and $(T - A)$ is compact.*

Proof. Suppose that $T_z^{*m} T T_z^m \to A$ in norm. It follows that

$$\|T_z^{*(m+1)} T T_z^{m+1} - T_z^* A T_z\| \leq \|T_z^{*m} T T_z^m - A\| \to 0$$

as $m \to \infty$. This and the triangle inequality yields $A = T_z^* A T_z$. Now let $R_m = T_z^m T_z^{*m}$ and

$$Q_m = I - R_m.$$

Further, let $P_{\mathbb{C}^p}$ denote the orthogonal projection of $H^2_{\mathbb{C}^p}(\mathbb{D})$ onto the space of ($\mathbb{C}^p$-valued) constant functions. Since $T_z T_z^* = I_{H^2_{\mathbb{C}^p}(\mathbb{D})} - P_{\mathbb{C}^p}$, it follows that

$$Q_m = \sum_{k=0}^{m-1} T_z^k P_{\mathbb{C}^p} T_z^{*k} \qquad (m \geq 1).$$

Then Q_m, $m \geq 1$, is a finite rank operator, and therefore

$$F_m = (T - A)Q_m + Q_m(T - A) - Q_m(T - A)Q_m \qquad (m \geq 1),$$

is also a finite rank operator. Moreover

$$(T - A) - F_m = R_m(T - A)R_m \qquad (m \geq 1),$$

yields

$$\|(T - A) - F_m\| = \|R_m(T - A)R_m\| \leq \|T_z^{*m} T T_z^m - A\| \to 0,$$

as $m \to \infty$. So $T - A$ is compact as desired.
The converse follows from Lemma 2.1. This completes the proof. □

Given a Hilbert space $\mathcal{E}$ and an inner multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$, the model space $\mathcal{Q}_\Theta$ and the model operator S_Θ are defined by

$$\mathcal{Q}_\Theta = H^2_{\mathcal{E}}(\mathbb{D}) \ominus \Theta H^2_{\mathcal{E}}(\mathbb{D}),$$

and

$$S_\Theta = P_{\mathcal{Q}_\Theta} M_z|_{\mathcal{Q}_\Theta},$$

respectively. Model spaces (and hence model operators) represent a wide and very important class of bounded linear operators [18]. We have the following result in the model space setting.

Proposition 5.5. *Let $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ be an inner multiplier and $T \in \mathcal{B}(\mathcal{Q}_\Theta)$. Assume that $\Theta(e^{i\theta})$ is invertible a.e. Then $S_\Theta^* T S_\Theta = T$ if and only if $T = 0$.*

Proof. The proof goes exactly along the same lines as the proof of Theorem 4.2. Since

$$TP_{\mathcal{Q}_\Theta} = T_z^*(TP_{\mathcal{Q}_\Theta})T_z,$$

it follows from Theorem 5.2 that $TP_{\mathcal{Q}}$ is a Toeplitz operator. Consequently, there exists $\Psi \in L^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{T})$ [18] such that

$$TP_{\mathcal{Q}_\Theta} = T_\Psi.$$

Since T_Θ is an analytic Toeplitz operator, again as in the proof of Theorem 4.2, it follows that

$$T_{\Psi\Theta} = 0,$$

and hence

$$\Psi\Theta = 0.$$

Since Θ is invertible a.e., it follows that $\Psi = 0$ a.e. and hence $T = 0$. This completes the proof. $\square$

Not only is this proposition a considerable generalization of Proposition 2.10 of [5], but our proof is much simpler. The principal tool is the identity $S_\Theta^* = T_z^*|_{\mathcal{Q}_\Theta}$.

We have the following characterization which generalizes the characterization of compact operators on $\mathcal{Q}_\Theta$ for $p = 1$ (see the implication (i) and (iii) in Theorem 1.2 in [5]).

Theorem 5.6. *Let $\Theta \in H^\infty_{\mathcal{B}(\mathbb{C}^p)}(\mathbb{D})$ be an inner multiplier and $T \in \mathcal{B}(\mathcal{Q}_\Theta)$. Then T is compact if and only if $\{S_\Theta^{*m} T S_\Theta^m\}_{m\geq 1}$ converges in norm.*

Proof. If T is compact on $\mathcal{Q}_\Theta$, then by Lemma 2.1, $\|S_\Theta^{*m} T S_\Theta^m\| \to 0$ as $m \to \infty$. To prove the converse, let $A \in \mathcal{B}(\mathcal{Q}_\Theta)$ and $S_\Theta^{*m} T S_\Theta^m \to A$, as $m \to \infty$, in norm. Then by the same argument used in the proof of Theorem 4.1, we have $S_\Theta^* A S_\Theta = A$. It now follows from Proposition 5.5 that $A = 0$ and therefore $T_z^{*m} T P_{\mathcal{Q}_\Theta} T_z^m \to 0$ as $m \to \infty$. Now Theorem 5.4 implies that $TP_{\mathcal{Q}_\Theta}$ is a compact operator on $H^2_{\mathbb{C}^p}(\mathbb{D})$. Therefore $T = TP_{\mathcal{Q}_\Theta}$ is a compact operator on $\mathcal{Q}_\Theta$. This completes the proof. $\square$

Theorem 5.6 and Lemma 2.1 give us the following generalization of Theorem 1.2 in [5].

Theorem 5.7. *Let $\Theta \in H^\infty_{\mathcal{B}(\mathbb{C}^p)}(\mathbb{D})$ be an inner multiplier and $T \in \mathcal{B}(\mathcal{Q}_\Theta)$. Then the following are equivalent:*

(i) $\{S_\Theta^{*m} T S_\Theta^m\}_{m\geq 1}$ converges in norm;
(ii) $S_\Theta^{*m} T S_\Theta^m \to 0$ in norm;
(iii) T is a compact operator.

Acknowledgement: We are grateful to Professor A. Bottcher for pointing out that our Lemma 2.1 is a special case of 1.3 (d) in the monograph by Bottcher and Silbermann [3]. The first author's research work is supported by NBHM Post Doctoral Fellowship No. 2/40(50)/2015/ R & D - II/11569. The second author is supported in part by (1) National Board of Higher Mathematics (NBHM), India, grant NBHM/R.P.64/2014, and (2) Mathematical Research Impact Centric Support (MATRICS) grant, File No : MTR/2017/000522, by the Science and Engineering Research Board (SERB), Department of Science & Technology (DST), Government of India.

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