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CHARACTERIZATION OF INVARIANT SUBSPACES IN THE POLYDISC

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ABSTRACT. We give a complete characterization of invariant subspaces for $(M_{z_1}, \dots, M_{z_n})$ on the Hardy space $H^2(\mathbb{D}^n)$ over the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$, $n > 1$. In particular, this yields a complete set of unitary invariants for invariant subspaces for $(M_{z_1}, \dots, M_{z_n})$ on $H^2(\mathbb{D}^n)$. As a consequence, we classify a large class of n-tuples of commuting isometries. All of our results hold for vector-valued Hardy spaces over $\mathbb{D}^n$, $n > 1$.

KEYWORDS: *Invariant subspaces, commuting isometries, Hardy space, polydisc, bounded analytic functions*

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1. Introduction

An important problem in multivariable operator theory and function theory of several complex variables is the question of a Beurling type representations of joint invariant subspaces for the n-tuple of multiplication operators $(M_{z_1}, \dots, M_{z_n})$ on $H^2(\mathbb{D}^n)$, $n > 1$. Here $H^2(\mathbb{D}^n)$ denotes the Hardy space over the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$ (see Section 2 for notation and definitions). The main obstacle here seems to be the subtleties of the theory of holomorphic functions in several complex variables. This problem is compounded by another difficulty associated with the complex (and mostly unknown) structure of n-tuples, $n > 1$, of commuting isometries on Hilbert spaces.

In this paper, we answer the above question by providing a complete list of natural conditions on closed subspaces of $H^2(\mathbb{D}^n)$. Here we use the analytic representations of shift invariant subspaces, representations of Toeplitz operators on the unit disc, geometry of tensor product of Hilbert spaces and identification of bounded linear operators under unitary equivalence to overcome such difficulties.

As motivation, recall that if $n = 1$, then the celebrated Beurling theorem [3] says that a non-zero closed subspace $\mathcal{S}$ of $H^2(\mathbb{D})$ is invariant for M_z if and only if there exists an inner function $\theta \in H^\infty(\mathbb{D})$ such that

$$\mathcal{S} = \theta H^2(\mathbb{D}).$$

Note also that it follows (or the other way around) in particular from the above representation of $\mathcal{S}$ that

$$\mathcal{S} \ominus z\mathcal{S} = \theta\mathbb{C},$$

and so

$$\mathcal{S} = \bigoplus_{m=0}^{\infty} z^m(\mathcal{S} \ominus z\mathcal{S}).$$

One may now ask whether an analogous characterization holds for invariant subspaces for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2(\mathbb{D}^n)$, $n > 1$. However, Rudin's pathological examples (see Rudin [20], page 70) indicates that the above Beurling type properties does not hold in general for invariant subspaces for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2(\mathbb{D}^n)$, $n > 1$: There exist invariant subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ for (M_{z_1}, M_{z_2}) on $H^2(\mathbb{D}^2)$ such that

(1) $\mathcal{S}_1$ is not finitely generated, and

(2) $\mathcal{S}_2 \cap H^\infty(\mathbb{D}^2) = \{0\}$.

In fact, Beurling type invariant subspaces for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2(\mathbb{D}^n)$, $n > 1$, are rare. They are closely connected with the tensor product structure of the Hardy space (or the product domain $\mathbb{D}^n$).

Therefore, the structure of invariant subspaces for

$$(M_{z_1}, \ldots, M_{z_n}) \text{ on } H^2(\mathbb{D}^n),\ n > 1,$$

is quite complicated. The list of important works in this area include the papers by Agrawal, Clark, and Douglas [1], Ahern and Clark [2], Douglas and Yan [6], Douglas, Paulsen, Sah and Yan [5], Guo [9, 10], Fang [7], Guo, Sun, Zheng and Zhong [11], Rudin [21], Guo and Yang [12], Izuchi [14], Mandrekar [17] etc. (also see the references therein).

In this paper, first, we represent $H^2(\mathbb{D}^{n+1})$, $n \geq 1$, by the $H^2(\mathbb{D}^n)$-valued Hardy space $H^2_{H^2(\mathbb{D}^n)}(\mathbb{D})$. Under this identification, we prove that

$$(M_{z_1}, M_{z_2}, \ldots, M_{z_{n+1}}) \text{ on } H^2(\mathbb{D}^{n+1}),$$

corresponds to

$$(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n}) \text{ on } H^2_{H^2(\mathbb{D}^n)}(\mathbb{D}),$$

where $\kappa_i \in H^\infty_{\mathcal{B}(H^2(\mathbb{D}^n))}(\mathbb{D})$, $i = 1, \ldots, n$, is a constant as well as simple and explicit $\mathcal{B}(H^2(\mathbb{D}^n))$-valued analytic function (see Theorem 3.1, or part (i) of Theorem 1.1 below). Then we prove that a closed subspace $\mathcal{S} \subseteq H^2_{H^2(\mathbb{D}^n)}(\mathbb{D})$ is invariant for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ if and only if $\mathcal{S}$ is of Beurling [3], Lax [15] and Halmos [13] type and the corresponding Beurling, Lax and Halmos inner function solves, in an appropriate sense, n operator equations explicitly and uniquely (see Theorem 3.2, or part (ii) of Theorem 1.1 below, and Theorem 5.2).

Recall that two m-tuples, $m \geq 1$, of commuting operators $(A_1, \ldots, A_m)$ on $\mathcal{H}$ and $(B_1, \ldots, B_m)$ on $\mathcal{K}$ are said to be *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that $UA_i = B_i U$ for all $i = 1, \ldots, m$.

We now summarize the main contents, namely, Theorems 3.1 and 3.2 restricted to the scalar-valued Hardy space case, of this paper in the following statement.

Theorem 1.1. *Let n be a natural number, and let $H_n = H^2(\mathbb{D}^n)$. Let $\kappa_i \in H^\infty_{\mathcal{B}(H_n)}(\mathbb{D})$ denote the $\mathcal{B}(H_n)$-valued constant function on $\mathbb{D}$ defined by*

$$\kappa_i(w) = M_{z_i} \in \mathcal{B}(H_n),$$

for all $w \in \mathbb{D}$, and let M_{κ_i} denote the multiplication operator on $H^2_{H_n}(\mathbb{D})$ defined by

$$M_{\kappa_i} f = \kappa_i f,$$

for all $f \in H^2_{H_n}(\mathbb{D})$ and $i = 1, \ldots, n$. Then the following statements hold true:

(i) $(M_{z_1}, M_{z_2} \ldots, M_{z_{n+1}})$ on $H^2(\mathbb{D}^{n+1})$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{H_n}(\mathbb{D})$ are unitarily equivalent.

(ii) Let $\mathcal{S}$ be a closed subspace of $H^2_{H_n}(\mathbb{D})$, and let $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$. Then $\mathcal{S}$ is invariant for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ if and only if $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is an n-tuple of commuting shifts on $H^2_{\mathcal{W}}(\mathbb{D})$ and there exists an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}, H_n)}(\mathbb{D})$ such that

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$

and

$$\kappa_i \Theta = \Theta \Phi_i,$$

where

$$\Phi_i(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - w P_{\mathcal{S}} M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$

The representation of $\mathcal{S}$, in terms of $\mathcal{W}$, Θ and $\{M_{\Phi_i}\}_{i=1}^n$, in part (ii) above is unique in an appropriate sense (see Theorem 5.2). Furthermore, the multiplier Φ_i can be represented as

$$\Phi_i(w) = P_{\mathcal{W}} M_\Theta (I_{H^2_{\mathcal{W}}(\mathbb{D})} - w M_z^*)^{-1} M_\Theta^* M_{\kappa_i}|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$. For a more detailed discussion on the analytic functions $\{\Phi_i\}_{i=1}^n$ on $\mathbb{D}$ we refer to Remarks 3.3 and 3.5.

As an immediate application of Theorem 1.1 we have (see Corollary 3.6): If $\mathcal{S} \subseteq H^2_{H_n}(\mathbb{D})$ is a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$, then the tuples $(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \ldots, M_{\kappa_n}|_{\mathcal{S}})$ on $\mathcal{S}$ and $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent, where $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$ and

$$\Phi_i(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - w P_{\mathcal{S}} M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$. Our approach also yields a complete set of unitary invariants for invariant subspaces: The n-tuples of commuting shifts $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ is a complete set of unitary invariants for invariant subspaces for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{H_n}(\mathbb{D})$ (see Theorem 6.1 for more details).

We also contribute to the classification problem of commuting tuples of isometries on Hilbert spaces. On the one hand, n-tuples of commuting isometries play a central role in multivariable operator theory and function theory, whereas, on the other hand, the structure of n-tuples, $n > 1$, of commuting isometries on Hilbert spaces is complicated. In Corollary 3.6, as a byproduct of our analysis, we completely classify n-tuples of commuting isometries of the form $(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \ldots, M_{\kappa_n}|_{\mathcal{S}})$ on $\mathcal{S}$, where $\mathcal{S}$ is a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$.

This paper is organized as follows. In Section 2 we give various background definitions and results on the Hardy space over the unit polydisc. In Section 3, we prove the central result of this paper - representations of invariant subspaces of vector-valued Hardy spaces over polydisc. In chapter 4 we study and analyze the model tuples of commuting isometries. Section 5 complements the main results on representations of invariant subspaces and deals with the uniqueness part. In Section 6 we give some applications related to the main theorems. The final section of this paper is devoted to an appendix on a dimension inequality which is relevant to the present context and of independent interest.

2. Prerequisites

We start by briefly recalling the relevant parts of the Hardy space over the unit polydisc. Let $n \geq 1$, and let $\mathbb{D}^n$ be the open unit polydisc in $\mathbb{C}^n$. The *Hardy space* $H^2(\mathbb{D}^n)$ over $\mathbb{D}^n$ is the Hilbert space of all holomorphic functions f on $\mathbb{D}^n$ such that

$$\|f\|_{H^2(\mathbb{D}^n)} = \left(\sup_{0 \leq r < 1} \int_{\mathbb{T}^n} |f(re^{i\theta_1}, \ldots, re^{i\theta_n})|^2 \, d\theta \right)^{\frac{1}{2}} < \infty,$$

where $d\theta$ is the normalized Lebesgue measure on the torus $\mathbb{T}^n$, the distinguished boundary of $\mathbb{D}^n$. It is well known that $H^2(\mathbb{D}^n)$ is a reproducing kernel Hilbert space corresponding to the Szegö kernel $\mathbb{S}_n$ on $\mathbb{D}^n$, where

$$\mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^{n} (1 - z_i \bar{w}_i)^{-1} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

Clearly

$$\mathbb{S}_n^{-1}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{0 \leq |\boldsymbol{k}| \leq n} (-1)^{|\boldsymbol{k}|} \boldsymbol{z}^{\boldsymbol{k}} \bar{\boldsymbol{w}}^{\boldsymbol{k}},$$

where $|\boldsymbol{k}| = \sum_{i=1}^{n} k_i$ and $0 \leq k_i \leq 1$ for all $i = 1, \ldots, n$. Here we use the notation $\boldsymbol{z}$ for the n-tuple $(z_1, \ldots, z_n)$ in $\mathbb{C}^n$. Also for any multi-index $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ and $\boldsymbol{z} \in \mathbb{C}^n$, we write $\boldsymbol{z}^{\boldsymbol{k}} = z_1^{k_1} \cdots z_n^{k_n}$.

Let $\mathcal{E}$ be a Hilbert space, and let $H^2_{\mathcal{E}}(\mathbb{D}^n)$ denote the $\mathcal{E}$-valued Hardy space over $\mathbb{D}^n$. Then $H^2_{\mathcal{E}}(\mathbb{D}^n)$ is the $\mathcal{E}$-valued reproducing kernel Hilbert space with the $\mathcal{B}(\mathcal{E})$-valued kernel function

$$(\boldsymbol{z}, \boldsymbol{w}) \mapsto \mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w}) I_{\mathcal{E}} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

In the sequel, by virtue of the canonical unitary $U : H^2_{\mathcal{E}}(\mathbb{D}^n) \to H^2(\mathbb{D}^n) \otimes \mathcal{E}$ defined by

$$U(\boldsymbol{z}^{\boldsymbol{k}} \eta) = \boldsymbol{z}^{\boldsymbol{k}} \otimes \eta \qquad (\boldsymbol{k} \in \mathbb{Z}_+^n, \eta \in \mathcal{E}),$$

we shall often identify the vector valued Hardy space $H^2(\mathbb{D}^n) \otimes \mathcal{E}$ with $H^2_{\mathcal{E}}(\mathbb{D}^n)$. Let $(M_{z_1}, \ldots, M_{z_n})$ denote the n-tuple of multiplication operators on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ by the coordinate functions $\{z_i\}_{i=1}^n$, that is,

$$(M_{z_i} f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}),$$

for all $f \in H^2_{\mathcal{E}}(\mathbb{D}^n)$, $\boldsymbol{w} \in \mathbb{D}^n$ and $i = 1, \ldots, n$. It is well known and easy to check that

$$\|M_{z_i} f\| = \|f\|,$$

and

$$\|M_{z_i}^{*m} f\| \to 0,$$

as $m \to \infty$ and for all $f \in H^2_{\mathcal{E}}(\mathbb{D}^n)$, that is, M_{z_i} defines a shift (see the definition of shift below) on $H^2_{\mathcal{E}}(\mathbb{D}^n)$, $i = 1, \ldots, n$. If $n > 1$, then it also follows easily that

$$M_{z_i} M_{z_j} = M_{z_j} M_{z_i},$$

and

$$M_{z_i}^* M_{z_j} = M_{z_j} M_{z_i}^*,$$

for all $1 \leq i < j \leq n$. Therefore, $(M_{z_1}, \ldots, M_{z_n})$ is an n-tuple of *doubly commuting* shifts on $H^2_{\mathcal{E}}(\mathbb{D}^n)$. Evidently the shift M_{z_i} on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ can be identified with $M_{z_i} \otimes I_{\mathcal{E}}$ on $H^2(\mathbb{D}^n) \otimes \mathcal{E}$. This canonical identification will be used throughout the rest of the paper.

We recall that a closed subspace $\mathcal{S} \subseteq H^2_{\mathcal{E}}(\mathbb{D}^n)$ is called an *invariant subspace* for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ if

$$z_i \mathcal{S} \subseteq \mathcal{S},$$

for all $i = 1, \ldots, n$.

Now we review and adapt some standard techniques for shift operators which are useful for our purposes (see [16] for more details). Let $\mathcal{H}$ be a Hilbert space. Let V be an isometry on $\mathcal{H}$, that is, $\|Vf\| = \|f\|$ for all $f \in \mathcal{H}$. Then V is said to be a *shift* [13] if there is no non trivial reducing subspace of $\mathcal{H}$ on which V is unitary. Equivalently, an isometry V on $\mathcal{H}$ is a shift if V is pure, that is, $\|V^{*m}f\| \to 0$ for all $f \in \mathcal{H}$. Now if V is a shift on $\mathcal{H}$, then

$$\mathcal{H} = \bigoplus_{m=0}^{\infty} V^m \mathcal{W},$$

where $\mathcal{W}$ is the *wandering subspace* [13] for V, that is, $\mathcal{W} = \ker V^* = \mathcal{H} \ominus V\mathcal{H}$. Hence the natural map $\Pi_V : \mathcal{H} \to H^2_{\mathcal{W}}(\mathbb{D})$ defined by

$$\Pi_V(V^m \eta) = z^m \eta,$$

for all $m \geq 0$ and $\eta \in \mathcal{W}$, is a unitary operator and

$$\Pi_V V = M_z \Pi_V.$$

Following Wold [23] and von Neumann [22], we call Π_V the *Wold-von Neumann decomposition* of the shift V.

We will need the following representation theorem for commutators of shifts proved in [16]. Here we only sketch this proof and refer the reader to [16] for more details.

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert space. Let V be a shift on $\mathcal{H}$ and C be a bounded operator on $\mathcal{H}$. Let Π_V be the Wold-von Neumann decomposition of V, $M = \Pi_V C \Pi_V^*$, and let*

$$\Theta(z) = P_{\mathcal{W}}(I_{\mathcal{H}} - zV^*)^{-1} C|_{\mathcal{W}} \qquad (z \in \mathbb{D}).$$

Then $CV = VC$ if and only if $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ and

$$M = M_\Theta.$$

Sketch of proof: For the necessary part, let $CV = VC$. Then $MM_z = M_z M$, and so

$$M = M_\Theta,$$

for some (unique) bounded analytic function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ [18]. Let $z \in \mathbb{D}$ and $\eta \in \mathcal{W}$. Since $\Theta(z)\eta = (M_\Theta \eta)(z)$, it follows that

$$\begin{aligned}\Theta(z)\eta &= (\Pi_V C \Pi_V^* \eta)(z)\\ &= (\Pi_V C\eta)(z),\end{aligned}$$

as $\Pi_V^* \eta = \eta$. Now a simple computation shows that (cf. [16])

$$I_{\mathcal{H}} = \sum_{m=0}^{\infty} V^m P_{\mathcal{W}} V^{*m},$$

in the strong operator topology, from which it follows that

$$C\eta = \sum_{m=0}^{\infty} V^m P_{\mathcal{W}} V^{*m} C\eta,$$

and so

$$\begin{aligned}\Theta(z)\eta &= (\Pi_V(\sum_{m=0}^{\infty} V^m P_{\mathcal{W}} V^{*m} C\eta))(z)\\ &= (\sum_{m=0}^{\infty} M_z^m (P_{\mathcal{W}} V^{*m} C\eta))(z).\end{aligned}$$

Using the fact that $P_{\mathcal{W}} V^{*m} C\eta \in \mathcal{W}$ for all $m \geq 0$, from here we get

$$\begin{aligned}\Theta(z)\eta &= \sum_{m=0}^{\infty} z^m (P_{\mathcal{W}} V^{*m} C\eta)\\ &= P_{\mathcal{W}}(I_{\mathcal{H}} - zV^*)^{-1} C\eta.\end{aligned}$$

The sufficient part easily follows from the fact that $\Pi_V^* M \Pi_V = C$. This proves the theorem. ∎

As usual, here $H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ denotes the Banach algebra of all $\mathcal{B}(\mathcal{E})$-valued bounded analytic functions on the open unit disc $\mathbb{D}$ (cf. [18]).

3. Main results

With the above preparation, we now turn to the representations of joint invariant subspaces of vector-valued Hardy spaces. Let n be a positive integer. Let $\mathcal{E}$ be a Hilbert space, and consider the vector-valued Hardy space $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$. Our strategy here is to identify M_{z_1} on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ with the multiplication operator M_z on the $H^2_{\mathcal{E}}(\mathbb{D}^n)$-valued Hardy space on the disc $\mathbb{D}$. Then we show that under this identification, the remaining operators $\{M_{z_2}, \ldots, M_{z_{n+1}}\}$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ can be represented as the multiplication operators by n simple and constant $\mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}^n))$-valued functions on $\mathbb{D}$. For this we need a few more notations.

For each Hilbert space $\mathcal{L}$, for the sake of notational ease, define

$$\mathcal{L}_n = H^2(\mathbb{D}^n) \otimes \mathcal{L}.$$

When $\mathcal{L} = \mathbb{C}$, we simply write $\mathcal{L}_n = H_n$, that is,

$$H_n = H^2(\mathbb{D}^n).$$

Also, for each $i = 1, \ldots, n$, we define

$$\kappa_{\mathcal{L},i}(w) = M_{z_i} \otimes I_{\mathcal{L}},$$

for all $w \in \mathbb{D}$, and write

$$\kappa_{\mathcal{L},i} = \kappa_i,$$

when $\mathcal{L}$ is clear from the context. It is evident that $\kappa_i \in H^\infty_{\mathcal{B}(\mathcal{L}_n)}(\mathbb{D})$ is a constant function and M_{κ_i} on $H^2_{\mathcal{L}_n}(\mathbb{D})$, defined by

$$M_{\kappa_i} f = \kappa_i f \qquad (f \in H^2_{\mathcal{L}_n}(\mathbb{D})),$$

is a shift on $H^2_{\mathcal{L}_n}(\mathbb{D})$ for all $i = 1, \ldots, n$.

Now we return to the invariant subspaces of $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$. First we identify $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ with $H^2(\mathbb{D}) \otimes \mathcal{E}_n$ by the natural unitary map $\hat{U} : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2(\mathbb{D}) \otimes \mathcal{E}_n$ defined by

$$\hat{U}(z_1^{k_1} z_2^{k_2} \cdots z_{n+1}^{k_{n+1}} \eta) = z^{k_1} \otimes (z_1^{k_2} \cdots z_n^{k_{n+1}} \eta),$$

for all $k_1, \ldots, k_{n+1} \geq 0$ and $\eta \in \mathcal{E}$. Then it is clear that

$$\hat{U} M_{z_1} = (M_z \otimes I_{\mathcal{E}_n}) \hat{U}.$$

Moreover, a simple computation shows that

$$\hat{U} M_{z_{1+i}} = (I_{H^2(\mathbb{D})} \otimes K_i) \hat{U},$$

where K_i is the multiplicational operator M_{z_i} on $\mathcal{E}_n$, that is

$$K_i = M_{z_i},$$

for all $i = 1, \ldots, n$. Therefore, the tuples $(M_{z_1}, M_{z_2}, \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ and $(M_z \otimes I_{\mathcal{E}_n}, I_{H^2(\mathbb{D})} \otimes K_1, \ldots, I_{H^2(\mathbb{D})} \otimes K_n)$ on $H^2(\mathbb{D}) \otimes \mathcal{E}_n$ are unitarily equivalent. We further identify $H^2(\mathbb{D}) \otimes \mathcal{E}_n$ with the $\mathcal{E}_n$-valued Hardy space $H^2_{\mathcal{E}_n}(\mathbb{D})$ by the canonical unitary map $\tilde{U} : H^2(\mathbb{D}) \otimes \mathcal{E}_n \to H^2_{\mathcal{E}_n}(\mathbb{D})$ defined by

$$\tilde{U}(z^k \otimes \eta) = z^k \eta,$$

for all $k \geq 0$ and $\eta \in \mathcal{E}_n$. Clearly

$$\tilde{U}(M_z \otimes I_{\mathcal{E}_n}) = M_z \tilde{U}.$$

Now for each $i = 1, \ldots, n$, define the constant $\mathcal{B}(\mathcal{E}_n)$-valued (analytic) function on $\mathbb{D}$ by

$$\kappa_i(z) = K_i,$$

for all $z \in \mathbb{D}$. Then $\kappa_i \in H^\infty_{\mathcal{B}(\mathcal{E}_n)}(\mathbb{D})$, and the multiplication operator M_{κ_i} on $H^2_{\mathcal{E}_n}(\mathbb{D})$, defined by

$$(M_{\kappa_i}(z^m \eta))(w) = w^m (K_i \eta),$$

for all $m \geq 0$, $\eta \in \mathcal{E}_n$ and $w \in \mathbb{D}$, is a shift on $H^2_{\mathcal{E}_n}(\mathbb{D})$. It is now easy to see that

$$\tilde{U}(I_{H^2(\mathbb{D})} \otimes K_i) = M_{\kappa_i} \tilde{U}.$$

for all $i = 1, \ldots, n$. Finally, by setting

$$U = \tilde{U} \hat{U},$$

it follows that $U : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{E}_n}(\mathbb{D})$ is a unitary operator and

$$U M_{z_1} = M_z U,$$

and

$$U M_{z_{1+i}} = M_{\kappa_i} U,$$

for all $i = 1, \ldots, n$. This proves the vector-valued version of the first half of the statement of Theorem 1.1:

Theorem 3.1. *Let $\mathcal{E}$ be a Hilbert space. Then $(M_{z_1}, M_{z_2} \dots, M_{z_{n+1}})$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ and $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent, where $\kappa_i \in H^\infty_{\mathcal{B}(\mathcal{E}_n)}(\mathbb{D})$ is the constant function*

$$\kappa_i(w) = M_{z_i} \in \mathcal{B}(\mathcal{E}_n),$$

for all $w \in \mathbb{D}$ and $i = 1, \dots, n$.

Now we proceed to prove the remaining half of Theorem 1.1 in the vector-valued Hardy space setting. Let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Set

$$V = M_z|_{\mathcal{S}},$$

and

$$V_i = M_{\kappa_i}|_{\mathcal{S}},$$

for all $i = 1, \dots, n$. Clearly, $(V, V_1, \dots, V_n)$ is a commuting tuple of isometries on $\mathcal{S}$. Note that if $f \in \mathcal{S}$, then

$$\begin{aligned}\|V_i^{*m} f\|_{\mathcal{S}} &= \|P_{\mathcal{S}} M_{\kappa_i}^{*m} f\||_{\mathcal{S}} \\ &\leq \|M_{\kappa_i}^{*m} f\||_{H^2_{\mathcal{E}_n}(\mathbb{D})},\end{aligned}$$

that is, V_i, $i = 1, \dots, n$, is a shift on $\mathcal{S}$, and similarly V is also a shift on $\mathcal{S}$. Let $\mathcal{W} = \mathcal{S} \ominus V\mathcal{S}$ denote the wandering subspace for V, that is

$$\begin{aligned}\mathcal{W} &= \ker V^* \\ &= \ker P_{\mathcal{S}} M_z^*,\end{aligned}$$

and let $\Pi_V : \mathcal{S} \to H^2_{\mathcal{W}}(\mathbb{D})$ be the Wold-von Neumann decomposition of V on $\mathcal{S}$ (see Section 2). Then Π_V is a unitary operator and

$$\Pi_V V = M_z \Pi_V.$$

Since

$$V V_i = V_i V,$$

applying Theorem 2.1 to V_i, we obtain

$$\Pi_V V_i = M_{\Phi_i} \Pi_V,$$

where

$$\Phi_i(w) = P_{\mathcal{W}} (I_{\mathcal{S}} - w V^*)^{-1} V_i|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$, $\Phi_i \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$, M_{Φ_i} is a shift on $H^2_{\mathcal{W}}(\mathbb{D})$ since V_i is a shift on $\mathcal{S}$ and $i = 1, \dots, n$. Now since Π_V is unitary, we obtain that

$$\Pi_V^* M_z = V \Pi_V^*,$$

and

$$\Pi_V^* V_i = M_{\Phi_i} \Pi_V^*,$$

for all $i = 1, \dots, n$. Finally, if we let $i_{\mathcal{S}}$ denote the inclusion map $i_{\mathcal{S}} : \mathcal{S} \hookrightarrow H^2_{\mathcal{E}_n}(\mathbb{D})$, then $\Pi_{\mathcal{S}} : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{\mathcal{E}_n}(\mathbb{D})$ is an isometry, where

$$\Pi_{\mathcal{S}} = i_{\mathcal{S}} \circ \Pi_V^*.$$

Clearly $\Pi_{\mathcal{S}} \Pi_{\mathcal{S}}^* = i_{\mathcal{S}} i_{\mathcal{S}}^*$. This implies that

$$\operatorname{ran} \Pi_{\mathcal{S}} = \operatorname{ran} i_{\mathcal{S}},$$

and so

$$\operatorname{ran} \Pi_{\mathcal{S}} = \mathcal{S}.$$

Now, using $i_{\mathcal{S}} V = M_z i_{\mathcal{S}}$ and $i_{\mathcal{S}} V_j = M_{\kappa_j} i_{\mathcal{S}}$, we have

$$\Pi_{\mathcal{S}} M_z = M_z \Pi_{\mathcal{S}},$$

and

$$\Pi_{\mathcal{S}} M_{\Phi_i} = M_{\kappa_i} \Pi_{\mathcal{S}},$$

for all $i = 1, \dots, n$. From the first equality it follows that there exists an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}, \mathcal{E}_n)}(\mathbb{D})$ such that

$$\Pi_{\mathcal{S}} = M_\Theta.$$

This and the second equality implies that

$$\kappa_i \Theta = \Theta \Phi_i,$$

for all $i = 1, \dots, n$. Moreover, $\operatorname{ran} \Pi_{\mathcal{S}} = \mathcal{S}$ yields

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}).$$

To prove that $(M_{\Phi_1}, \dots, M_{\Phi_n})$ is a commuting tuple, observe that

$$\begin{aligned} M_{\Phi_i} M_{\Phi_j} \Pi_V &= M_{\Phi_i} \Pi_V V_j \\ &= \Pi_V V_i V_j \\ &= \Pi_V V_j V_i \\ &= M_{\Phi_j} M_{\Phi_i} \Pi_V, \end{aligned}$$

and so

$$M_{\Phi_i} M_{\Phi_j} = M_{\Phi_j} M_{\Phi_i},$$

for all $i, j = 1, \dots, n$. For the converse, let us begin by observing that if $\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D})$ for some inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}, \mathcal{E}_n)}(\mathbb{D})$, then $\mathcal{S}$ is invariant for M_z and

$$P_{\mathcal{S}} M_z^* P_{\mathcal{S}} = P_{\mathcal{S}} M_z^*.$$

In particular

$$P_{\mathcal{S}} M_z^*|_{\mathcal{S}} = P_{\mathcal{S}} M_z^* \in \mathcal{B}(\mathcal{S}),$$

and so $\{\Phi_1, \dots, \Phi_n\}$ is a well-defined set of $\mathcal{B}(\mathcal{W})$-valued analytic functions on $\mathbb{D}$. Furthermore, if $(M_{\Phi_1}, \dots, M_{\Phi_n})$ is an n-tuple of commuting shifts on $H^2_{\mathcal{W}}(\mathbb{D})$ (so, in particular, $\Phi_i \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ for all $i = 1, \dots, n$. See Remark 3.3) and $\kappa_i \Theta = \Theta \Phi_i$, then it follows obviously that $\kappa_i \mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \dots, n$, that is, $\mathcal{S}$ is invariant for $(M_{\kappa_1}, \dots, M_{\kappa_n})$. This proves the last part of Theorem 1.1 in the vector-valued Hardy space setting:

Theorem 3.2. *Let $\mathcal{E}$ be a Hilbert space, $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be a closed subspace, and let $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$. Then $\mathcal{S}$ is invariant for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ if and only if $(M_{\Phi_1}, \dots, M_{\Phi_n})$ is an n-tuple of commuting shifts on $H^2_{\mathcal{W}}(\mathbb{D})$ and there exists an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D})$ such that*

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$

and

$$\kappa_i \Theta = \Theta \Phi_i,$$

where

$$\Phi_i(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - wP_{\mathcal{S}}M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$ and $i = 1, \dots, n$.

A few remarks are in order.

Remark 3.3. Note that since $\|wP_{\mathcal{S}}M_z^*\| < 1$ for all $w \in \mathbb{D}$, the $\mathcal{B}(\mathcal{W})$-valued function Φ_i, as defined in the above theorem, is analytic on $\mathbb{D}$. Here the boundedness condition (or the shift condition) on M_{Φ_i} on $H^2_{\mathcal{W}}(\mathbb{D})$ assures that $\Phi_i \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ for all $i = 1, \dots, n$.

Remark 3.4. Clearly, one obvious necessary condition for a closed subspace $\mathcal{S}$ of $H^2_{\mathcal{E}_n}(\mathbb{D})$ to be invariant for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ is that $\mathcal{S}$ is invariant for M_z, and, consequently

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$

is the classical Beurling, Lax and Halmos representation of $\mathcal{S}$, where $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$ is the wandering subspace for $M_z|_{\mathcal{S}}$ and $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D})$ is the (unique up to a unitary constant right factor; see Section 5) Beurling, Lax and Halmos inner function. Moreover, since $\kappa_i \mathcal{S} \subseteq \mathcal{S}$, another condition which is evidently necessary (by Douglas's range inclusion theorem) is that

$$\kappa_i \Theta = \Theta \Gamma_i,$$

for some $\Gamma_i \in \mathcal{B}(H^2_{\mathcal{W}}(\mathbb{D}))$, $i = 1, \dots, n$. In the above theorem, we prove that Γ_i is explicit, that is

$$\Gamma_i = \Phi_i \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D}),$$

for all $i = 1, \dots, n$, and $(\Gamma_1, \dots, \Gamma_n)$ is an n-tuple of commuting shifts on $H^2_{\mathcal{W}}(\mathbb{D})$. This is probably the most non-trivial part of our treatment to the invariant subspace problem in the present setting.

Remark 3.5. Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let $\mathcal{W}$, Θ and

$$\{\Phi_i\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D}),$$

be as in Theorem 3.2. Now it follows from $P_{\mathcal{S}} = M_\Theta M_\Theta^*$ that

$$P_{\mathcal{S}} M_z^{*m} = M_\Theta M_z^{*m} M_\Theta^*,$$

for all $m \geq 0$. Hence the equality

$$(I_{\mathcal{S}} - wP_{\mathcal{S}} M_z^*)^{-1} = \sum_{m=0}^{\infty} w^m P_{\mathcal{S}} M_z^{*m},$$

yields

$$(I_{\mathcal{S}} - wP_{\mathcal{S}} M_z^*)^{-1} = M_\Theta (I_{H^2_{\mathcal{W}}(\mathbb{D})} - wM_z^*)^{-1} M_\Theta^*,$$

so that

$$\Phi_i(w) = P_{\mathcal{W}} M_\Theta (I_{H^2_{\mathcal{W}}(\mathbb{D})} - wM_z^*)^{-1} M_\Theta^* M_{\kappa_i}|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$.

A well known consequence of the Beurling, Lax and Halmos theorem (cf. page 239, Foias and Frazho [8]) implies that a closed subspace $\mathcal{S} \subseteq H^2_{\mathcal{E}}(\mathbb{D})$ is invariant for M_z if and only if $\mathcal{S} \cong H^2_{\mathcal{F}}(\mathbb{D})$ for some Hilbert space $\mathcal{F}$ with

$$\dim \mathcal{F} \leq \dim \mathcal{E}.$$

More specifically, if $\mathcal{S}$ is a closed invariant subspace of $H^2_{\mathcal{E}}(\mathbb{D})$ and if $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$, then the pure isometry $M_z|_{\mathcal{S}}$ on $\mathcal{S}$ and M_z on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent, and $\dim \mathcal{W} \leq \dim \mathcal{E}$. The above theorem sets the stage for a similar result.

Corollary 3.6. *Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$, and*

$$\Phi_i(w) = P_{\mathcal{W}} (I_{\mathcal{S}} - wP_{\mathcal{S}} M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{W}} \qquad (w \in \mathbb{D}),$$

for all $i = 1, \ldots, n$. Then $(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \ldots, M_{\kappa_n}|_{\mathcal{S}})$ on $\mathcal{S}$ and $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent.

Proof. Let $\mathcal{W}$, Θ and $\{\Phi_i\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ be as in Theorem 3.2. Then it follows that

$$X : H^2_{\mathcal{W}}(\mathbb{D}) \to \Theta H^2_{\mathcal{W}}(\mathbb{D}) = \mathcal{S},$$

is a unitary operator, where

$$X = M_\Theta.$$

It is now clear that X intertwines $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ and

$$(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \ldots, M_{\kappa_n}|_{\mathcal{S}}),$$

on $\mathcal{S}$. This completes the proof of the corollary. ■

Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be an invariant subspace for M_z. Then $\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D})$, where $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$ and $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D})$ is the Beurling, Lax and Halmos inner function. A natural question arises in connection with Remark 3.4: Under what additional condition(s) on Θ is $\mathcal{S}$ also invariant for $(M_{\kappa_1}, \ldots, M_{\kappa_n})$? An answer to this question directly follows, with appropriate reformulation, from Theorem 3.2 and Remark 3.5:

Theorem 3.7. *Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be an invariant subspace for M_z on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let $\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D})$, where $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$ and $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D})$ is the Beurling Lax and Halmos inner function. Set*

$$\Phi_i(w) = P_{\mathcal{W}} M_\Theta (I_{H^2_{\mathcal{W}}(\mathbb{D})} - wM_z^*)^{-1} M_\Theta^* M_{\kappa_i}|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$ and $i = 1, \dots, n$. Then $\mathcal{S}$ is invariant for $(M_{\kappa_1}, \dots, M_{\kappa_n})$ if and only if $(M_{\Phi_1}, \dots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ is an n-tuple of commuting shifts, and

$$\kappa_i \Theta = \Theta \Phi_i,$$

for all $i = 1, \dots, n$. Moreover, in this case, $(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \dots, M_{\kappa_n}|_{\mathcal{S}})$ on $\mathcal{S}$ and $(M_z, M_{\Phi_1}, \dots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent.

Thus the n-tuples of commuting shifts

$$(M_{\Phi_1}, \dots, M_{\Phi_n}) \text{ on } H^2_{\mathcal{L}}(\mathbb{D}),$$

for Hilbert spaces $\mathcal{L}$ and inner multipliers $\{\Phi_i\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{L})}(\mathbb{D})$, yielding invariant subspaces of vector-valued Hardy spaces over $\mathbb{D}^{n+1}$ are distinguished among the general n-tuples of commuting shifts by the fact that

$$\Phi_i(w) = P_{\mathcal{L}}(I_{\mathcal{S}} - wP_{\mathcal{S}} M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{L}} \qquad (w \in \mathbb{D}),$$

where $\mathcal{S} = \Theta H^2_{\mathcal{L}}(\mathbb{D})$ for some inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{L},\mathcal{E}_n)}(\mathbb{D})$, and

$$\kappa_i \Theta = \Theta \Phi_i,$$

for all $i = 1, \dots, n$. Moreover, in view of Remark 3.5, the above condition is equivalent to the condition that

$$\Phi_i(w) = P_{\mathcal{W}} M_\Theta (I_{H^2_{\mathcal{L}}(\mathbb{D})} - wM_z^*)^{-1} M_\Theta^* M_{\kappa_i}|_{\mathcal{W}},$$

for some inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{L},\mathcal{E}_n)}(\mathbb{D})$ such that

$$\kappa_i \Theta = \Theta \Phi_i,$$

for all $i = 1, \dots, n$.

4. Representations of model isometries

In connection with Theorem 3.1 (or part (i) of Theorem 1.1), a natural question arises: Given a Hilbert space $\mathcal{E}$, how to identify Hilbert spaces $\mathcal{F}$ and $\mathcal{B}(\mathcal{F})$-valued multipliers $\{\Psi\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{F})}(\mathbb{D})$ such that $(M_z, M_{\Psi_1}, \dots, M_{\Psi_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$ and $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent. More generally, given a Hilbert space $\mathcal{E}$, characterize $(n+1)$-tuples of commuting shifts on Hilbert spaces that are unitarily equivalent to $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$.

This question has a simple answer, although a rigorous proof of it involves some technicalities. More specifically, the answer to this question is related to a numerical invariant, the rank of an operator associated with the Szegö kernel on $\mathbb{D}^{n+1}$. First, however, we need a few more definitions.

Let $(T_1, \ldots, T_m)$ be an m-tuple of commuting contractions on a Hilbert space $\mathcal{H}$. Define the *defect operator* [12] corresponding to $(T_1, \ldots, T_m)$ as

$$\mathbb{S}_m^{-1}(T_1, \ldots, T_m) = \sum_{0 \leq |\boldsymbol{k}| \leq m} (-1)^{|\boldsymbol{k}|} T_1^{k_1} \cdots T_m^{k_m} T_1^{*k_1} \cdots T_m^{*k_m},$$

where $0 \leq k_i \leq 1$, $i = 1, \ldots, m$. This definition is motivated by the representation of the Szegö kernel on the polydisc $\mathbb{D}^m$ (see Section 2). We say that $(T_1, \ldots, T_m)$ is of *rank* p ($p \in \mathbb{N} \cup \{\infty\}$) if

$$\text{rank } [\mathbb{S}_m^{-1}(T_1, \ldots, T_m)] = p,$$

and we write

$$\text{rank } (T_1, \ldots, T_m) = p.$$

The defect operators plays an important role in multivariable operator theory (cf. [10, 12]). For instance, if $\mathcal{E}$ is a Hilbert space, then the defect operator of the multiplication operator tuple $(M_{z_1}, \ldots, M_{z_n})$ on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ is given by

$$\mathbb{S}_n^{-1}(M_{z_1}, \ldots, M_{z_n}) = P_{H^2_c(\mathbb{D}^n)} \otimes I_{\mathcal{E}},$$

where $P_{H^2_c(\mathbb{D}^n)}$ denotes the orthogonal projection of $H^2(\mathbb{D}^n)$ onto the one dimensional space of constant functions. Furthermore, as is evident from the definition (and also see the proof of Theorem 3.1), the defect operator for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ is given by

$$\mathbb{S}_{n+1}^{-1}(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n}) = P_{H^2_c(\mathbb{D})} \otimes P_{H^2_c(\mathbb{D}^n)} \otimes I_{\mathcal{E}}.$$

In particular,

$$\dim \mathcal{E} = \text{rank } (M_z, M_{\kappa_1}, \ldots, M_{\kappa_n}) = \text{rank } (M_{z_1}, \ldots, M_{z_n}).$$

Now let $\mathcal{E}$ and $\mathcal{K}$ be Hilbert spaces, and let $(V, V_1 \ldots, V_n)$ be an $(n+1)$-tuple of commuting shifts on $\mathcal{K}$. Suppose that $(V, V_1 \ldots, V_n)$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $\mathcal{K}$ and $H^2_{\mathcal{E}_n}(\mathbb{D})$, respectively, are unitarily equivalent. In this case, it is necessary that M_z on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and V on $\mathcal{K}$ are unitarily equivalent. As $VV_i = V_i V$ and $V_i V_j = V_j V_i$ for all $i, j = 1, \ldots, n$, Theorem 2.1 implies that $(V, V_1, \ldots, V_n)$ and $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent, where $\mathcal{W} = \mathcal{K} \ominus V\mathcal{K}$, and

$$\Phi_i(z) = P_{\mathcal{W}}(I_{\mathcal{K}} - zV^*)^{-1} V_i|_{\mathcal{W}},$$

for all $z \in \mathbb{D}$ and $i = 1, \ldots, n$. Since $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ is doubly commuting, another necessary condition is that $(V, V_1, \ldots, V_{n+1})$ is doubly commuting. In particular, $V^* V_i = V_i V^*$, and so

$$V^{*m} V_i = V_i V^{*m},$$

for all $m \geq 0$ and $i = 1, \ldots, n$. Using $V^{*m}|_{\mathcal{W}} = 0$ for all $m \geq 1$, this implies that $\Phi_i(z) = P_{\mathcal{W}} V_i|_{\mathcal{W}}$ for all $z \in \mathbb{D}$. Again using $VV_i^* = V_i^* V$, we have

$$V_i(I - VV^*) = (I - VV^*)V_i,$$

for all $i = 1, \ldots, n$. This implies that $\mathcal{W}$ is a reducing subspace for V_i, and hence we obtain

$$\Phi_i(z) = V_i|_{\mathcal{W}},$$

that is, Φ_i is a constant shift-valued function on $\mathbb{D}$ for all $i = 1, \ldots, n$. This observation leads to the following proposition:

Proposition 4.1. *Let $(V, V_1, \ldots, V_n)$ be an $(n+1)$-tuple of doubly commuting shifts on some Hilbert space $\mathcal{H}$. Let $\mathcal{W} = \mathcal{H} \ominus V\mathcal{H}$, and let*

$$\Phi_i(z) = V_i|_{\mathcal{W}} \qquad (i = 1, \ldots, n),$$

for all $z \in \mathbb{D}$. Then $\mathcal{W}$ is reducing for V_i, $i = 1, \ldots, n$, and $(V, V_1, \ldots, V_n)$ and $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ are unitarily equivalent.

In particular, if $\mathcal{L}$ is a Hilbert space and $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{L}}(\mathbb{D})$, for some $\{\Phi_i\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{L})}(\mathbb{D})$, is a tuple of doubly commuting shifts, then

$$\Phi_i(z) = \Phi_i(0) \qquad (z \in \mathbb{D}),$$

that is, Φ is a constant function for all $i = 1, \ldots, n$.

Now we return to $(V, V_1 \ldots, V_n)$, which in turn is an $(n+1)$-tuple of doubly commuting shifts on $\mathcal{H}$. For simplicity of notation, set $U_1 = V$, $U_{i+1} = V_i$ for all $i = 1, \ldots, n$, and let

$$\mathcal{D} = \operatorname{ran} \mathbb{S}^{-1}_{n+1}(V, V_1, \ldots, V_n) = \bigcap_{i=1}^{n+1} \ker U_i^*,$$

is the wandering subspace for $(V, V_1, \ldots, V_n)$ (cf. [19]). From here, one can use the fact that (cf. Theorem 3.3 in [19])

$$\mathcal{H} = \bigoplus_{\boldsymbol{k} \in \mathbb{Z}_+^{n+1}} U^{\boldsymbol{k}} \mathcal{D},$$

to prove that the map $\Gamma : \mathcal{H} \to H^2_{\mathcal{D}}(\mathbb{D}^{n+1})$ defined by

$$\Gamma(U^{\boldsymbol{k}} \eta) = \boldsymbol{z}^{\boldsymbol{k}} \eta \qquad (\boldsymbol{k} \in \mathbb{Z}_+^{n+1}, \eta \in \mathcal{D}),$$

is a unitary and

$$\Gamma U_i = M_{z_i} \Gamma,$$

for all $i = 1, \ldots, n+1$. Therefore, $(V, V_1, \ldots, V_n)$ on $\mathcal{H}$ and $(M_{z_1}, \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{D}}(\mathbb{D}^{n+1})$ are unitarily equivalent. In addition, if $\mathcal{E}$ is a Hilbert space, and

$$\dim \mathcal{E} = \operatorname{rank}\,(V, V_1, \ldots, V_n) \;\; (= \dim \mathcal{D}),$$

then it follows that (see the equivalence of (ii) and (v) of Theorem 3.3 in [19]) $(M_{z_1}, \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{D}}(\mathbb{D}^{n+1})$ and $(M_{z_1}, \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ are unitarily equivalent. But then Theorem 3.1 yields immediately that $(M_{z_1}, \ldots, M_{z_{n+1}})$ on

$H^2_{\mathcal{D}}(\mathbb{D}^{n+1})$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent. This gives the following:

Theorem 4.2. *In the setting of Proposition 4.1 the following hold:* $(V, V_1, \ldots, V_n)$ *on* $\mathcal{H}$, $(M_z, M_{\Psi_1}, \ldots, M_{\Psi_n})$ *on* $H^2_{\mathcal{W}}(\mathbb{D})$, *and* $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ *on* $H^2_{\mathcal{E}_n}(\mathbb{D})$ *are unitarily equivalent, where* $\mathcal{E}$ *is a Hilbert space and*

$$\dim \mathcal{E} = \operatorname{rank}(V, V_1, \ldots, V_n).$$

Therefore, an $(n+1)$-tuple of doubly commuting shift operators

$$(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n}),$$

is completely determined by the numerical invariant rank $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$:

Corollary 4.3. *Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be Hilbert spaces. Let* $(M_z, M_{\Psi_1}, \ldots, M_{\Psi_n})$ *be an* $(n+1)$*-tuple of commuting shifts on* $H^2_{\mathcal{F}}(\mathbb{D})$. *Then* $(M_z, M_{\Psi_1}, \ldots, M_{\Psi_n})$ *on* $H^2_{\mathcal{F}}(\mathbb{D})$ *and* $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ *on* $H^2_{\mathcal{E}_n}(\mathbb{D})$ *are unitarily equivalent if and only if*

$$(M_z, M_{\Psi_1}, \ldots, M_{\Psi_n})$$

is doubly commuting and

$$\dim \mathcal{E} = \operatorname{rank}(M_z, M_{\Psi_1}, \ldots, M_{\Psi_n}).$$

The above corollary should be compared with the uniqueness of the multiplicity of shift operators on Hilbert spaces [13].

5. Nested invariant subspaces and uniqueness

Now we proceed to the description of nested invariant subspaces of $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two closed invariant subspaces for

$$(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n}) \text{ on } H^2_{\mathcal{E}_n}(\mathbb{D}).$$

Let $\mathcal{W}_j = \mathcal{S} \ominus z\mathcal{S}_j$, and let

$$\Phi_{j,i}(w) = P_{\mathcal{W}_j}(I_{\mathcal{S}_j} - wP_{\mathcal{S}_j}M_z^*)^{-1}M_{\kappa_i}|_{\mathcal{W}_j},$$

for all $w \in \mathbb{D}$, $j = 1, 2$, and $i = 1, \ldots, n$. Hence by Theorem 3.2 there exists an inner function $\Theta_j \in H^\infty_{\mathcal{B}(\mathcal{W}_j, \mathcal{E}_n)}(\mathbb{D})$ such that

$$\mathcal{S}_j = \Theta_j H^2_{\mathcal{W}_j}(\mathbb{D}),$$

and

$$\kappa_i \Theta_j = \Theta_j \Phi_{j,i}, \tag{5.1}$$

for all $j = 1, 2$, and $i = 1, \ldots, n$. Now, let

$$\mathcal{S}_1 \subseteq \mathcal{S}_2,$$

that is

$$\Theta_1 H^2_{\mathcal{W}_1}(\mathbb{D}) \subseteq \Theta_2 H^2_{\mathcal{W}_2}(\mathbb{D}).$$

Then there exists an inner multiplier $\Psi \in H^\infty_{\mathcal{B}(\mathcal{W}_1,\mathcal{W}_2)}(\mathbb{D})$ [8] such that

$$\Theta_1 = \Theta_2 \Psi.$$

Using this in (5.1), we get

$$\begin{aligned}\Theta_2 \Psi \Phi_{1,i} &= \Theta_1 \Phi_{1,i}\\ &= \kappa_i \Theta_1\\ &= \kappa_i \Theta_2 \Psi\\ &= \Theta_2 \Phi_{2,i} \Psi,\end{aligned}$$

and so

$$\Psi \Phi_{1,i} = \Phi_{2,i} \Psi,$$

for all $i = 1, \ldots, n$. On the other hand, given two invariant subspaces $\mathcal{S}_j = \Theta_j H^2_{\mathcal{W}_j}(\mathbb{D})$, $j = 1, 2$, for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ described as above, if there exists an inner multiplier $\Psi \in H^\infty_{\mathcal{B}(\mathcal{W}_1,\mathcal{W}_2)}(\mathbb{D})$ such that $\Theta_1 = \Theta_2 \Psi$, then it readily follows that $\mathcal{S}_1 \subseteq \mathcal{S}_2$. We state this in the following theorem:

Theorem 5.1. *Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S}_1 = \Theta_1 H^2_{\mathcal{W}_1}(\mathbb{D})$ and $\mathcal{S}_2 = \Theta_2 H^2_{\mathcal{W}_2}(\mathbb{D})$ be two invariant subspaces for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let*

$$\Phi_{j,i}(w) = P_{\mathcal{W}_j}(I_{\mathcal{S}_j} - w P_{\mathcal{S}_j} M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{W}_j},$$

for all $w \in \mathbb{D}$, $j = 1, 2$, and $i = 1, \ldots, n$. Then $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if there exists an inner multiplier $\Psi \in H^\infty_{\mathcal{B}(\mathcal{W}_1,\mathcal{W}_2)}(\mathbb{D})$ such that $\Theta_1 = \Theta_2 \Psi$ and $\Psi \Phi_{1,i} = \Phi_{2,i} \Psi$ for all $i = 1, \ldots, n$.

We now proceed to prove the uniqueness of the representations of invariant subspaces as described in Theorem 3.2. Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S}$ be an invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Let $\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D})$ and

$$\kappa_i \Theta = \Theta \Phi_i \qquad (i = 1, \ldots, n),$$

in the notation of Theorem 3.2. Now assume that $\tilde{\Theta} \in H^\infty_{\mathcal{B}(\tilde{\mathcal{W}})}(\mathbb{D})$ is an inner function, for some Hilbert space $\tilde{\mathcal{W}}$, and

$$\mathcal{S} = \tilde{\Theta} H^2_{\tilde{\mathcal{W}}}(\mathbb{D}).$$

Also assume that

$$\kappa_i \tilde{\Theta} = \tilde{\Theta} \tilde{\Phi}_i,$$

for some shift $M_{\tilde{\Phi}_i}$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ and $i = 1, \ldots, n$. Then as an application of the uniqueness of the Beurling, Lax and Halmos inner functions (cf. Theorem 2.1 in page 239 [8]) to

$$\Theta H^2_{\mathcal{W}}(\mathbb{D}) = \tilde{\Theta} H^2_{\tilde{\mathcal{W}}}(\mathbb{D}),$$

we get

$$\Theta = \tilde{\Theta} \tau,$$

for some unitary operator (constant in z) $\tau : \mathcal{W} \to \tilde{\mathcal{W}}$. Then, the previous line of argument shows that

$$\tau \Phi_i = \tilde{\Phi}_i \tau,$$

for all $i = 1, \dots, n$. This proves the uniqueness of the representations of invariant subspaces in Theorem 3.2.

Theorem 5.2. *In the setting of Theorem 3.2, if* $\mathcal{S} = \tilde{\Theta} H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ *and* $\kappa_i \tilde{\Theta} = \tilde{\Theta}\tilde{\Phi}_i$ *for some Hilbert space* $\tilde{\mathcal{W}}$*, inner function* $\tilde{\Theta} \in H^\infty_{\mathcal{B}(\tilde{\mathcal{W}})}(\mathbb{D})$ *and shift* $M_{\tilde{\Phi}_i}$ *on* $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$*,* $i = 1, \dots, n$*, then there exists a unitary operator (constant in* z*)* $\tau : \mathcal{W} \to \tilde{\mathcal{W}}$ *such that*

$$\Theta = \tilde{\Theta} \tau,$$

and

$$\tau \Phi_i = \tilde{\Phi}_i \tau,$$

for all $i = 1, \dots, n$*.*

6. Applications

In this section, first, we explore a natural connection between the intertwining maps on vector-valued Hardy space over $\mathbb{D}$ and the commutators of the multiplication operators on the Hardy space over $\mathbb{D}^{n+1}$. Then, as a noteworthy added benefit to our approach, we compute a complete set of unitary invariants for invariant subspaces of vector-valued Hardy space over $\mathbb{D}^{n+1}$. We also test our main results on invariant subspaces unitarily equivalent to $H^2_{\mathcal{E}_n}(\mathbb{D})$. As a by-product, we obtain some useful results about the structure of invariant subspaces for the Hardy space. We begin with the following definition.

Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be two Hilbert spaces. Let $\mathcal{S}$ and $\tilde{\mathcal{S}}$ be invariant subspaces for the $(n+1)$-tuples of multiplication operators on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$, respectively. We say that $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are *unitarily equivalent*, and write $\mathcal{S} \cong \tilde{\mathcal{S}}$, if there is a unitary map $U : \mathcal{S} \to \tilde{\mathcal{S}}$ such that

$$U M_z|_{\mathcal{S}} = M_z|_{\tilde{\mathcal{S}}} U \quad \text{and} \quad U M_{\kappa_i}|_{\mathcal{S}} = M_{\kappa_i}|_{\tilde{\mathcal{S}}} U,$$

for all $i = 1, \dots, n$.

6.1. INTERTWINING MAPS. Recall that, given a Hilbert space $\mathcal{E}$, there exists a unitary operator $U_{\mathcal{E}} : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{E}_n}(\mathbb{D})$ (see Section 3) such that

$$U_{\mathcal{E}} M_{z_1} = M_z U_{\mathcal{E}},$$

and

$$U_{\mathcal{E}} M_{z_{i+1}} = M_{\kappa_i} U_{\mathcal{E}},$$

for all $i = 1, \dots, n$. Let $\mathcal{F}$ be another Hilbert space, and let $X : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{F}}(\mathbb{D}^{n+1})$ be a bounded linear operator such that

$$X M_{z_i} = M_{z_i} X, \tag{6.1}$$

for all $i = 1, \ldots, n+1$. Set

$$X_n = U_{\mathcal{F}} X U_{\mathcal{E}}^*.$$

Then $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ is bounded and

$$X_n M_z = M_z X_n \quad \text{and} \quad X_n M_{\kappa_i} = M_{\kappa_i} X_n, \tag{6.2}$$

for all $i = 1, \ldots, n$. Conversely, a bounded linear operator $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ satisfying (6.2) yields a canonical bounded linear map $X : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{F}}(\mathbb{D}^{n+1})$, namely

$$X = U_{\mathcal{F}}^* X_n U_{\mathcal{E}}$$

such that (6.1) holds. Moreover, this construction shows that

$$X \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}^{n+1}), H^2_{\mathcal{F}}(\mathbb{D}^{n+1}))$$

is a contraction (respectively, isometry, unitary, etc.) if and only if

$$X_n \in \mathcal{B}(H^2_{\mathcal{E}_n}(\mathbb{D}), H^2_{\mathcal{F}_n}(\mathbb{D}))$$

is a contraction (respectively, isometry, unitary, etc.).

For brevity, any map satisfying (6.2) will be referred to *module maps*.

6.2. A COMPLETE SET OF UNITARY INVARIANTS. Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be Hilbert spaces, and let $\{\Psi_1, \ldots, \Psi_n\} \subseteq H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ and $\{\tilde{\Psi}_1, \ldots, \tilde{\Psi}_n\} \subseteq H^\infty_{\mathcal{B}(\tilde{\mathcal{E}})}(\mathbb{D})$. We say that $\{\Psi_1, \ldots, \Psi_n\}$ and $\{\tilde{\Psi}_1, \ldots, \tilde{\Psi}_n\}$ coincide if there exists a unitary operator $\tau : \mathcal{E} \to \tilde{\mathcal{E}}$ such that

$$\tau \Psi_i(z) = \tilde{\Psi}_i(z) \tau,$$

for all $z \in \mathbb{D}$ and $i = 1, \ldots, n$.

Now let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ and $\tilde{\mathcal{S}} \subseteq H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$ be invariant subspaces for

$$(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$$

on $H^2_{\mathcal{E}_n}(\mathbb{D})$, and $H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$, respectively. Let $\mathcal{S} \cong \tilde{\mathcal{S}}$. By Theorem 3.7, this implies that

$$(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n}) \text{ on } H^2_{\mathcal{W}}(\mathbb{D}),$$

and $(M_z, M_{\tilde{\Phi}_1}, \ldots, M_{\tilde{\Phi}_n})$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ are unitarily equivalent, where $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$, $\tilde{\mathcal{W}} = \tilde{\mathcal{S}} \ominus z\tilde{\mathcal{S}}$ and

$$\Phi_i(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - wP_{\mathcal{S}}M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{W}},$$

and

$$\tilde{\Phi}_i(w) = P_{\tilde{\mathcal{W}}}(I_{\tilde{\mathcal{S}}} - wP_{\tilde{\mathcal{S}}}M_z^*)^{-1} M_{\kappa_i}|_{\tilde{\mathcal{W}}},$$

for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$. Let $U : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ be a unitary map such that

$$UM_z = M_z U,$$

and

$$UM_{\Phi_i} = M_{\tilde{\Phi}_i} U,$$

for all $i = 1, \ldots, n$. The former condition implies that

$$U = I_{H^2(\mathbb{D})} \otimes \tau,$$

for some unitary operator $\tau : \mathcal{W} \to \tilde{\mathcal{W}}$, and so the latter condition implies that

$$\tau \Phi_i(z) = \tilde{\Phi}_i(z) \tau,$$

for all $z \in \mathbb{D}$ and $i = 1, \ldots, n$. Therefore $\{\Phi_1, \ldots, \Phi_n\}$ and $\{\tilde{\Phi}_1, \ldots, \tilde{\Phi}_n\}$ coincide. To prove the converse, assume now that the above equality holds for a given unitary operator $\tau : \mathcal{W} \to \tilde{\mathcal{W}}$. Obviously $U = I_{H^2(\mathbb{D})} \otimes \tau$ is a unitary from $H^2_{\mathcal{W}}(\mathbb{D})$ to $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$. Clearly $UM_z = M_z U$ and $UM_{\Phi_i} = M_{\tilde{\Phi}_i} U$ for all $i = 1, \ldots, n$. So we have the following theorem on a complete set of unitary invariants for invariant subspaces:

Theorem 6.1. *Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be Hilbert spaces. Let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ and $\tilde{\mathcal{S}} \subseteq H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$ be invariant subspaces for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$, respectively. Then $\mathcal{S} \cong \tilde{\mathcal{S}}$ if and only if $\{\Phi_1, \ldots, \Phi_n\}$ and $\{\tilde{\Phi}_1, \ldots, \tilde{\Phi}_n\}$ coincide.*

Now, if we consider the Beurling, Lax and Halmos representations of the given invariant subspaces $\mathcal{S}$ and $\tilde{\mathcal{S}}$ as

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$

and

$$\tilde{\mathcal{S}} = \tilde{\Theta} H^2_{\tilde{\mathcal{W}}}(\mathbb{D}),$$

where $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}, \mathcal{E}_n)}(\mathbb{D})$ and $\tilde{\Theta} \in H^\infty_{\mathcal{B}(\tilde{\mathcal{W}}, \tilde{\mathcal{E}}_n)}(\mathbb{D})$, then, in view of Remark 3.5, the multipliers in Theorem 6.1 can be represented as

$$\Phi_i(w) = P_{\mathcal{W}} M_\Theta (I_{H^2_{\mathcal{W}}(\mathbb{D})} - w M_z^*)^{-1} M_\Theta^* M_{\kappa_i}|_{\mathcal{W}},$$

and

$$\tilde{\Phi}_i(w) = P_{\tilde{\mathcal{W}}} M_{\tilde{\Theta}} (I_{H^2_{\tilde{\mathcal{W}}}(\mathbb{D})} - w M_z^*)^{-1} M_{\tilde{\Theta}}^* M_{\kappa_i}|_{\tilde{\mathcal{W}}},$$

for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$.

6.3. UNITARILY EQUIVALENT INVARIANT SUBSPACES. Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert spaces, and let $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ be a module map. If X_n is an isometry, then the closed subspace $\mathcal{S} \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ defined by

$$\mathcal{S} = X_n(H^2_{\mathcal{E}_n}(\mathbb{D})),$$

is invariant for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$ and $\mathcal{S} \cong H^2_{\mathcal{E}_n}(\mathbb{D})$. In other words, the tuples $(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \ldots, M_{\kappa_n}|_{\mathcal{S}})$ on $\mathcal{S}$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent. Conversely, let $\mathcal{S} \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$, and let $\mathcal{S} \cong H^2_{\mathcal{E}_n}(\mathbb{D})$ for some Hilbert space $\mathcal{E}$. Let $\tilde{X}_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to \mathcal{S}$ be the unitary map which intertwines $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$

on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $(M_z|_{\mathcal{S}}, M_{\kappa_1}|_{\mathcal{S}}, \ldots, M_{\kappa_n}|_{\mathcal{S}})$ on $\mathcal{S}$. Suppose that $i_{\mathcal{S}} : \mathcal{S} \hookrightarrow H^2_{\mathcal{F}_n}(\mathbb{D})$ is the inclusion map. Then

$$X_n = i_{\mathcal{S}} \circ \tilde{X}_n,$$

is an isometry from $H^2_{\mathcal{E}_n}(\mathbb{D})$ to $H^2_{\mathcal{F}_n}(\mathbb{D})$, $X_n M_z = M_z X_n$, $X_n M_{\kappa_i} = M_{\kappa_i} X_n$ for all $i = 1, \ldots, n$, and

$$\text{ran } X_n = \mathcal{S}.$$

Therefore, if $\mathcal{S} \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ is a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$, then $\mathcal{S} \cong H^2_{\mathcal{E}_n}(\mathbb{D})$, for some Hilbert space $\mathcal{E}$, if and only if there exists an isometric module map $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ such that $\mathcal{S} = X_n(H^2_{\mathcal{E}_n}(\mathbb{D}))$. Now, it also follows from the discussion at the beginning of this section that $X : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{F}}(\mathbb{D}^{n+1})$ (corresponding to the module map X_n) is an isometry and $XM_{z_i} = M_{z_i}X$ for all $i = 1, \ldots, n$. Then Theorem 7.1 tells us that

$$\dim \mathcal{E} \leq \dim \mathcal{F}.$$

Therefore, we have the following theorem:

Theorem 6.2. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert spaces, and let $\mathcal{S} \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$. Then $\mathcal{S} \cong H^2_{\mathcal{E}_n}(\mathbb{D})$ if and only if there exists an isometric module map $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ such that*

$$\mathcal{S} = X_n H^2_{\mathcal{E}_n}(\mathbb{D}).$$

Moreover, in this case

$$\dim \mathcal{E} \leq \dim \mathcal{F}.$$

Of particular interest is the case when $\mathcal{F} = \mathbb{C}$. In this case (see Section 3) the tensor product Hilbert space $\mathcal{F}_n = H^2(\mathbb{D}^n) \otimes \mathbb{C}$ is denoted by H_n, that is, $H_n = H^2(\mathbb{D}^n)$.

Corollary 6.3. *Let $\mathcal{S} \subseteq H^2_{H_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{H_n}(\mathbb{D})$. Then $\mathcal{S} \cong H^2_{H_n}(\mathbb{D})$ if and only if there exists an isometric module map $X_n : H^2_{H_n}(\mathbb{D}) \to H^2_{H_n}(\mathbb{D})$ such that*

$$\mathcal{S} = X_n(H^2_{H_n}(\mathbb{D})).$$

The above result, in the polydisc setting, was first observed by Agrawal, Clark and Douglas (see Corollary 1 in [1]). Also see Mandrekar [17].

We now proceed to analyze doubly commuting invariant subspaces. Let $\mathcal{F}$ be a Hilbert space, and let $\mathcal{S} \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$. Set

$$V = M_z|_{\mathcal{S}},$$

and

$$V_i = M_{\kappa_i}|_{\mathcal{S}},$$

for all $i = 1, \ldots, n$. We say that $\mathcal{S}$ is *doubly commuting* if $V_i^* V_j = V_j V_i^*$ for all $1 \leq i < j \leq n$, and $V V_l^* = V_l^* V$ for all $l = 1, \ldots, n$.

Now let $\mathcal{E}$ be a Hilbert space, and suppose that $H^2_{\mathcal{E}_n}(\mathbb{D}) \cong \mathcal{S}$. In view of Theorem 6.2 this implies that $(V, V_1, \ldots, V_n)$ on $\mathcal{S}$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent. Because $H^2_{\mathcal{E}_n}(\mathbb{D})$ is doubly commuting this immediately implies that $\mathcal{S}$ is doubly commuting.

Conversely, let $\mathcal{S}$ be doubly commuting. From Theorem 3.7 we readily conclude that $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ and $(V, V_1, \ldots, V_n)$ on $\mathcal{S}$ are unitarily equivalent.

Applying Theorem 4.2 with $(M_z, M_{\Phi_1}, \ldots, M_{\Phi_n})$ in place of

$$(M_z, M_{\Psi_1}, \ldots, M_{\Psi_n}),$$

we see that $(V, V_1, \ldots, V_n)$ on $\mathcal{S}$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent, where $\mathcal{E}$ is a Hilbert space. Now, proceeding as in the proof of the necessary part of Theorem 6.2 one checks that there exists a module isometry $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ such that

$$\text{ran } X_n = \mathcal{S}.$$

This proves the following variant of Theorem 6.2:

Theorem 6.4. *Let $\mathcal{F}$ be a Hilbert space. An invariant subspace $\mathcal{S} \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ is doubly commuting if and only if there exists a Hilbert space $\mathcal{E}$ and an isometric module map $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ such that*

$$\mathcal{S} = X_n H^2_{\mathcal{E}_n}(\mathbb{D}).$$

Moreover, in this case

$$\dim \mathcal{E} \leq \dim \mathcal{F}.$$

The above result, in the polydisc setting, was first observed by Mandrekar [17]. Also this should be compared with the discussion prior to Corollary 3.6 on the application of the classical Beurling, Lax and Halmos theorem to invariant subspaces of the Hardy space over the unit disc.

7. Appendix: An inequality on fibre dimensions

Given a Hilbert space $\mathcal{E}$, the n-tuple of multiplication operators by the coordinate functions z_i, $i = 1, \ldots, n$, on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ is denoted by $(M^{\mathcal{E}}_{z_1}, \ldots, M^{\mathcal{E}}_{z_n})$. Whenever $\mathcal{E}$ is clear from the context, we will omit the superscript $\mathcal{E}$. Clearly, one can regard $\mathcal{E}$ as a closed subspace of $H^2_{\mathcal{E}}(\mathbb{D}^n)$ by identifying $\mathcal{E}$ with the constant $\mathcal{E}$-valued functions on $\mathbb{D}^n$.

In this appendix, we aim to prove the following result:

Theorem 7.1. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces and let $X : H^2_{\mathcal{E}_1}(\mathbb{D}^n) \to H^2_{\mathcal{E}_2}(\mathbb{D}^n)$ be an isometry. If*
$$XM_{z_i}^{\mathcal{E}_1} = M_{z_i}^{\mathcal{E}_2} X,$$
for all $i = 1, \dots, n$, then
$$\dim \mathcal{E}_1 \leq \dim \mathcal{E}_2.$$

We believe that the above result (possibly) follows from the boundary behavior of bounded analytic functions following the classical case $n = 1$ (see the remark at the end of this appendix). Here, however, we take a shorter approach than generalizing the classical theory of bounded analytic functions on the unit polydisc. We first prove the L^2-version of the above statement.

Theorem 7.2. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces and let $\tilde{X} : L^2_{\mathcal{E}_1}(\mathbb{T}^n) \to L^2_{\mathcal{E}_2}(\mathbb{T}^n)$ be an isometry. If*
$$\tilde{X} M_{e^{i\theta_j}} = M_{e^{i\theta_j}} \tilde{X},$$
for all $j = 1, \dots, n$, then
$$\dim \mathcal{E}_1 \leq \dim \mathcal{E}_2.$$

Proof. By the triviality, we can assume that
$$m := \dim \mathcal{E}_2 < \infty.$$
Let $\{\eta_j\}_{j=1}^m$ be an orthonormal basis for $\mathcal{E}_2$. Since $\{e_{\boldsymbol{k}} : \boldsymbol{k} \in \mathbb{Z}^n\}$, where
$$e_{\boldsymbol{k}} = \prod_{j=1}^n e^{ik_j\theta_j} \qquad (\boldsymbol{k} \in \mathbb{Z}^n),$$
is an orthonormal basis for $L^2(\mathbb{T}^n)$, this implies that $\{e_{\boldsymbol{k}}\eta_j : \boldsymbol{k} \in \mathbb{Z}^n, j = 1, \dots, n\}$ is an orthonormal basis for $L^2_{\mathcal{E}_2}(\mathbb{T}^n)$. Let $\{f_j : j \in J\}$ be an orthonormal basis for $\tilde{X}(\mathcal{E}_1)$, where J is a subset of $\mathbb{Z}_+$. In view of the intertwining property of $\tilde{X}$, this implies that $\{e_{\boldsymbol{k}} f_j : \boldsymbol{k} \in \mathbb{Z}^n, j \in J\}$ is an orthonormal basis for
$$\widetilde{X}(L^2_{\mathcal{E}_1}(\mathbb{T}^n)) \subseteq L^2_{\mathcal{E}_2}(\mathbb{T}^n),$$
and so, an orthonormal set in $L^2_{\mathcal{E}_2}(\mathbb{T}^n)$. It follows from the Parseval's identity that
$$\begin{aligned}
\dim \mathcal{E}_1 &= \dim(\tilde{X}\mathcal{E}_1) \\
&= \sum_{j\in J} \|f_j\|^2 \\
&= \sum_{j\in J} \sum_{l=1}^m \sum_{\boldsymbol{k}\in\mathbb{Z}^n} |\langle M^{\boldsymbol{k}}_{e^{i\theta}} \eta_l, f_j\rangle|^2 \\
&= \sum_{j\in J} \sum_{l=1}^m \sum_{\boldsymbol{k}\in\mathbb{Z}^n} |\langle \eta_l, M^{\boldsymbol{k}}_{e^{i\theta}} f_j\rangle|^2 \\
&= \sum_{j\in J} \sum_{l=1}^m \sum_{\boldsymbol{k}\in\mathbb{Z}^n} |\langle \eta_l, e_{\boldsymbol{k}} f_j\rangle|^2,
\end{aligned}$$

on the one hand, and on the other, by Bessel's Inequality,

$$\begin{aligned} m &= \sum_{l=1}^{m} \|\eta_l\|^2 \\ &\geq \sum_{l=1}^{m} \sum_{j \in J} \sum_{\boldsymbol{k} \in \mathbb{Z}^n} |\langle \eta_l, e_{\boldsymbol{k}} f_j \rangle|^2. \end{aligned}$$

This proves $\dim \mathcal{E}_1 \leq m$ and completes the proof of the theorem. ∎

Proof of Theorem 7.1: Define $\tilde{X}$ on $\{e_{\boldsymbol{k}}\eta : \boldsymbol{k} \in \mathbb{Z}^n, \eta \in \mathcal{E}_1\}$ by

$$\tilde{X}(e_{\boldsymbol{k}}\eta) = e_{\boldsymbol{k}} X\eta,$$

for all $\boldsymbol{k} \in \mathbb{Z}^n$ and $\eta \in \mathcal{E}_1$. The intertwining property of the isometry X then gives

$$\langle \tilde{X}(e_{\boldsymbol{k}}\eta), \tilde{X}(e_{\boldsymbol{l}}\zeta) \rangle_{L^2_{\mathcal{E}_2}(\mathbb{T}^n)} = \langle e_{\boldsymbol{k}}\eta, e_{\boldsymbol{l}}\zeta \rangle_{L^2_{\mathcal{E}_1}(\mathbb{T}^n)},$$

for all $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{Z}^n$ and $\eta, \zeta \in \mathcal{E}_1$. Therefore this map extends uniquely to an isometry, denoted again by $\tilde{X}$ from $L^2_{\mathcal{E}_1}(\mathbb{T}^n)$ to $L^2_{\mathcal{E}_2}(\mathbb{T}^n)$, such that

$$\tilde{X} M_{e^{i\theta_j}} = M_{e^{i\theta_j}} \tilde{X},$$

for all $j = 1, \ldots, n$. The result then easily follows from Theorem 7.2. ∎

If $X : H^2_{\mathcal{E}_1}(\mathbb{D}^n) \to H^2_{\mathcal{E}_2}(\mathbb{D}^n)$ is an isometry, and if $XM_{z_i} = M_{z_i}X$ for all $i = 1, \ldots, n$, then it is easy to see that

$$X = M_\Theta,$$

for some isometric multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_1,\mathcal{E}_2)}(\mathbb{D}^n)$ (that is, $M_\Theta : H^2_{\mathcal{E}_1}(\mathbb{D}^n) \to H^2_{\mathcal{E}_2}(\mathbb{D}^n)$ is an isometry). In the case $n = 1$, the conclusion of Theorem 7.1 follows from the boundary behavior of bounded analytic functions on the open unit disc: M_Θ is an isometry if and only if $\Theta(e^{i\theta})$ is isometry a.e. on $\mathbb{T}$ (cf. [18]). Unlike the proof of the classical case $n = 1$, our proof does not use the boundary behavior of Θ.

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