IDEMPOTENT, MODEL, AND TOEPLITZ OPERATORS ATTAINING THEIR NORMS

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Abstract. We study idempotent, model, and Toeplitz operators that attain the norm. Notably, we prove that if $\mathcal{Q}$ is a backward shift invariant subspace of the Hardy space $H^2(\mathbb{D})$, then the model operator $S_{\mathcal{Q}}$ attains its norm. Here $S_{\mathcal{Q}} = P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$, the compression of the shift M_z on the Hardy space $H^2(\mathbb{D})$ to $\mathcal{Q}$.

1. Introduction

Let $\mathcal{E}$ be a Hilbert space (here all Hilbert spaces are separable and over $\mathbb{C}$) and T be a bounded linear operator on $\mathcal{E}$ ($T \in \mathscr{B}(\mathcal{E})$ in short). Then T is said to be *norm-attaining* (in short $T \in \mathcal{NA}$) if there exists a non-zero vector $f \in \mathcal{E}$ such that

$$\|Tf\|_{\mathcal{E}} = \|T\|_{\mathscr{B}(\mathcal{E})} \|f\|_{\mathcal{E}}.$$

As far as the theory of bounded linear operators is concerned, it is perhaps very natural to study operators that attain the norm. It is also worth to point out that compact operators are always norm attaining. While the norm attaining property at the Banach space level has been studied extensively (for instance, see [1, 12, 13]), the same for operators on Hilbert spaces has so far received far less attention (however, see [8, 14, 15, 16]). On the other hand, in 1965 Brown and Douglas [5, Lemma 2], in answering a question of H. Helson [11, page 12], established a close connection between arithmetic of inner functions, Toeplitz operators, and norm attaining operators on Hilbert spaces. Curiously, this is also intimately connected with the Sarason's commutant lifting theorem [17].

In this paper, we study norm attainment of three classical (and fairly non-compact in nature) Hilbert space operators, namely, Toeplitz operators, model operators, and idempotent operators. Toeplitz operators are one of the most important concrete operators, where concept of a model operator is one of the most useful in operator theory and function theory having important applications in various fields. Model operators also play the role of building blocks in the basic theory of linear operators [3, 18]. Idempotent operators (also known as oblique projections) are yet another

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2010 *Mathematics Subject Classification.* 47B07, 47A46, 47A10, 47B35, 30J05.

Key words and phrases. Toeplitz operators, model operators, idempotents, Hardy space, inner functions.

concrete (but complex) class of operators that plays a significant role in many definite problems in operator theory and operator algebras.

In Section 2 we study idempotent operators. Let T be an idempotent operator (that is, $T^2 = T$) on some Hilbert space. In Theorem 2.2 we prove that $T \in \mathcal{NA}$ if and only if the *Buckholtz operator* $T + T^* - I$ is in $\mathcal{NA}$. Observe that the Buckholtz operator [6, 7] is a self-adjoint operator.

Section 3 deals with model operators. Let M_z denote the forward shift (or the multiplication operator by the coordinate function z) on the Hardy space $H^2(\mathbb{D})$. Let $\mathcal{Q}$ be a closed M_z^*-invariant subspace of $H^2(\mathbb{D})$ (see Section 3 for more details). The *model operator* $S_{\mathcal{Q}}$ is the compression of M_z to $\mathcal{Q}$, that is, $S_{\mathcal{Q}} = P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$. Theorem 3.3 says that

$$S_{\mathcal{Q}} \in \mathcal{NA}.$$

Of course, the above $\mathcal{Q}$ is associated with an (essentially unique) inner function $\theta \in H^\infty(\mathbb{D})$ (because of Beurling), that is, $\mathcal{Q} = \mathcal{Q}_\theta$, where $\mathcal{Q}_\theta = H^2(\mathbb{D})/\theta H^2(\mathbb{D})$. The representing Beurling inner function θ plays an essential role in the proof of the above result. We also obtain norm attaintment result for the general Sz.-Nagy and Foiaş model operators (vector-valued counterpart of $S_{\mathcal{Q}}$).

The final section, Section 4, of this paper deals with Toeplitz and analytic Toeplitz operators. In Theorem 4.1 we prove that a Toeplitz operator T_Φ with operator-valued symbol Φ is in $\mathcal{NA}$ if and only if that Φ satisfies certain inner function divisibility criterion. This result recovers Brown and Douglas classification of norm attaining Toeplitz operators with scalar-valued symbols. We also discuss the case of analytic Toeplitz operators (see Theorem 4.4) and Laurent operators (see Proposition 4.7). Examples 4.5 and 4.6 bring out more insight between scalar and vector-valued Toeplitz operators.

Before we proceed to the main content of this paper, in the following we collect some useful results (see [8, Corollary 2.4, Proposition 2.5] and [15, Theorem 2.4]):

Theorem 1.1. *Let $T \in \mathscr{B}(\mathcal{E})$. The following are equivalent:*
(i) $T \in \mathcal{NA}$.
(ii) $T^* \in \mathcal{NA}$.
(iii) $TT^* \in \mathcal{NA}$.
(iv) $\|T\|^2$ is in the point spectrum of TT^*.

This will be used frequently in what follows.

2. Idempotent Operators

It is evident that any orthogonal projection on a Hilbert space is norm attaining. Here we deal with the issue of norm attainment of idempotent operators. We begin with an example of an idempotent which is not a norm attaining operator.

Example 2.1. Define a linear operator $T : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ by

$$T(\{\alpha_n\}_{n=1}^{\infty}) = \Big\{\alpha_1, 0, \alpha_3, (1-\frac{1}{3})\alpha_3, \alpha_5, (1-\frac{1}{5})\alpha_5, \ldots, \alpha_{2n+1}, (1-\frac{1}{2n+1})\alpha_{2n+1}, \ldots\Big\}.$$

Clearly, T is an idempotent operator. Note that for any $\alpha = \{\alpha_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$, we have

$$\|T\alpha\|^2 = \sum_{n=0}^{\infty} \left\{1 + \left(1 - \frac{1}{2n+1}\right)^2\right\} |\alpha_{2n+1}|^2 < 2\|\alpha\|^2, \tag{2.1}$$

and hence, $\|T\| \leq \sqrt{2}$. Furthermore, for all $n \geq 1$, we also have

$$\|T(e_{2n+1})\|^2 = 1 + \left(1 - \frac{1}{2n+1}\right)^2,$$

which implies that

$$\|T\| \geq \sqrt{1 + \left(1 - \frac{1}{2n+1}\right)^2},$$

and hence $\|T\| \geq \sqrt{2}$. We conclude that $\|T\| = \sqrt{2}$. Finally, by (2.1), it follows that $T \notin \mathcal{NA}$.

We turn now to classifying idempotent operators that admit the norm. We begin by recalling a geometric construction of idempotent operators. Let $T \in \mathscr{B}(\mathcal{E})$ be an idempotent. Let P denote the orthogonal projection onto $\operatorname{ran} T$. Then, by Feldman, Krupnik and Markus [9, Equation (1.8)] (also see [4, 19]), there exists $X \in \mathscr{B}(\ker P, \operatorname{ran} P)$ such that

$$T = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix},$$

on $\operatorname{ran} P \oplus \ker P$. Set $A = I + XX^*$. Then

$$TT^* = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that $\|A\| = 1 + \|X\|^2$ and $\|T\|^2 = \|TT^*\| = \|A\|$.

We are now ready to prove our first classification result. Observe that the *Buckholtz operator* $T + T^* - I$ is a self adjoint operator.

Theorem 2.2. *Let $T \in \mathscr{B}(\mathcal{E})$ be an idempotent. Then $T \in \mathcal{NA}$ if and only if*

$$T + T^* - I \in \mathcal{NA}.$$

Proof. We continue to use the above notation. Set $B = I + X^*X$. It is easy to see that $T + T^* - I = \begin{bmatrix} I & X \\ X^* & -I \end{bmatrix}$, and hence

$$(T + T^* - I)(T + T^* - I)^* = (T + T^* - I)^2 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Since $\|B\| = \|A\| = 1 + \|X\|^2$, it readily follows that

$$\|T + T^* - I\|^2 = 1 + \|X\|^2.$$

Now let $T \in \mathcal{NA}$. By Theorem 1.1, we infer that $\|T\|^2$ is an eigenvalue of TT^*. Then there exists a non-zero vector $f \in \text{ran } P$ such that $Af = \|A\|f$. It follows that

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} f \\ 0 \end{bmatrix} = (1 + \|X\|^2) \begin{bmatrix} f \\ 0 \end{bmatrix},$$

which, along with Theorem 1.1 implies that $T + T^* - I \in \mathcal{NA}$.
Conversely, assume that $T + T^* - I \in \mathcal{NA}$. By Theorem 1.1, there exists a non-zero vector

$$\begin{bmatrix} f \\ g \end{bmatrix} \in \text{ran } P \oplus \ker P.$$

such that

$$Af = (I + XX^*)f = \left(1 + \|X\|^2\right) f, \tag{2.2}$$

and

$$Bg = (I + X^*X)g = \left(1 + \|X\|^2\right) g. \tag{2.3}$$

If $f \neq 0$, then the matrix representation of TT^* and (2.2) imply that $T \in \mathcal{NA}$. Now assume that $f = 0$ and $g \neq 0$. Multiplying (2.3) from left by X we get

$$(I + XX^*)Xg = (1 + \|X\|^2)Xg. \tag{2.4}$$

If $Xg = 0$, then (2.3) implies that $g = g + \|X\|^2 g$. Since $g \neq 0$, we get $X = 0$. Hence T is an orthogonal projection and, consequently, $T \in \mathcal{NA}$. On the other hand, if $Xg \neq 0$, then by (2.4), we obtain

$$AXg = \|A\|Xg.$$

Then, the matrix representation of TT^* together with Theorem 1.1 asserts that $T \in \mathcal{NA}$. This completes the proof of the theorem. □

Remark 2.3. The present proof of Theorem 2.2 is due to the referee which is more elegant than our original proof. Our original proof here had been based on the argument of Ando [2, Theorem 2.6, Theorem 3.9].

The proof of the corollary below now follows easily from Theorem 2.2 and Theorem 1.1:

Corollary 2.4. *Let $T \in \mathscr{B}(H)$ be an idempotent operator. Then $T \in \mathcal{NA}$ if and only if either $\|T\|$ or $-\|T\|$ is an eigenvalue of $T + T^* - I$.*

In particular, an idempotent operator is norm attaining if and only if the corresponding Buckholtz operator is norm attaining. Now, we return to the idempotent T in Example 2.1 and validate the above corollary. First note that

$$T^*(\{\alpha_n\}_{n=1}^\infty) = \left\{\alpha_1, 0, \alpha_3 + \left(1 - \frac{1}{3}\right)\alpha_4, 0, \alpha_5 + \left(1 - \frac{1}{5}\right)\alpha_6, 0, \dots\right\}.$$

Then

$$(T + T^* - I)(\{\alpha_n\}_{n=1}^\infty) = \left\{\alpha_1, -\alpha_2, \alpha_3 + (1 - \frac{1}{3})\alpha_4, (1 - \frac{1}{3})\alpha_3 - \alpha_4, \dots\right\}.$$

If $(T + T^* - I)(\{\alpha_n\}_{n=1}^\infty) = \pm\sqrt{2}(\{\alpha_n\}_{n=1}^\infty)$ for some $\{\alpha_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$, then the above implies that $\alpha_n = 0$ for all n. By Corollary 2.4, we conclude that $T \notin \mathcal{NA}$.

We also have the following general result:

Theorem 2.5. *Let $T \in \mathscr{B}(\mathcal{E})$ be an idempotent operator. Then $T \in \mathcal{NA}$ if and only if there exists $f \in \mathcal{E}$ such that $Tf \neq 0$, $P_{ran\ T^*}Tf = f$, and*

$$T^* P_{(ran\ T^*)^\perp} Tf = (\|T\|^2 - 1)f.$$

Proof. Suppose $T \in \mathcal{NA}$. By Theorem 1.1, there exists a non-zero vector $h \in \mathcal{E}$ such that $TT^*h = \|T\|^2 h$. Hence

$$\|T\|^2 h = TT^*h = T^2T^*h = T(\|T\|^2 h) = \|T\|^2 Th,$$

which implies that $Th = h$. Set $h = f \oplus g \in \text{ran } T^* \oplus (\text{ran } T^*)^\perp$. Since $T(f+g) = f+g$, $g \in (\text{ran } T^*)^\perp = \ker\ T$, we get $Tf = f + g$, and hence we obtain

$$P_{\text{ran } T^*} Tf = f \text{ and } P_{(\text{ran } T^*)^\perp} Tf = g.$$

Now $TT^*(f + g) = \|T\|^2(f + g)$ and $f \in \text{ran } T^*$ implies that

$$\|T\|^2(f + g) = Tf + TT^*g = f + g + TT^*g,$$

and hence $TT^*g = (\|T\|^2 - 1)(f + g)$. Then $P_{\text{ran } T^*}(TT^*g) = (\|T\|^2 - 1)f$, which, along with $P_{\text{ran } T^*}Tf = f$ implies that $P_{\text{ran } T^*}T(T^*g - (\|T\|^2 - 1)f) = 0$. Thus

$$T(T^*g - (\|T\|^2 - 1)f) \in (\text{ran } T^*)^\perp = \ker\ T,$$

and hence $T(T^*g - (\|T\|^2 - 1)f) = 0$ as $T^2 = T$. Then $T^*g - (\|T\|^2 - 1)f \in (\text{ran } T^*)^\perp$, where, on the other hand, $f \in \text{ran } T^*$ implies that $T^*g - (\|T\|^2 - 1)f \in \text{ran } T^*$. This is possible only when

$$T^*g - (\|T\|^2 - 1)f = 0,$$

which, along with $P_{(\text{ran } T^*)^\perp}Tf = g$, implies $T^*P_{(\text{ran } T^*)^\perp}Tf = (\|T\|^2 - 1)f$.
Conversely, assume $Tf \neq 0$ for some $f \in \mathcal{E}$, and assume that $P_{\text{ran } T^*}Tf = f$ and $T^*P_{(\text{ran } T^*)^\perp}Tf = (\|T\|^2 - 1)f$. Set $g = P_{(\text{ran } T^*)^\perp}Tf$. Then $Tf = f + g$ and $T^*g = (\|T\|^2 - 1)f$. Since

$$TT^*Tf = TT^*f + TT^*g = Tf + TT^*g,$$

we have $TT^*Tf = \|T\|^2 Tf$. Then Theorem 1.1 implies that $T \in \mathcal{NA}$, and completes the proof of the theorem. $\square$

In particular, an idempotent operator T is in $\mathcal{NA}$ if and only if 1 and $\|T\|^2 - 1$ are eigenvalues of $P_{\text{ran } T^*}T$ and $T^* P_{(\text{ran } T^*)^\perp} T$, respectively, corresponding to a common eigenvector.

3. Model Operators

In this section we will treat model operators that attain the norm. We begin with some standard terminology. Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$, and let $\mathcal{E}$ be a Hilbert space. The $\mathcal{E}$-valued *Hardy space* $H^2_{\mathcal{E}}(\mathbb{D})$ (or $H^2(\mathbb{D})$ if $\mathcal{E} = \mathbb{C}$) over $\mathbb{D}$ is the Hilbert space of all $\mathcal{E}$-valued analytic functions $f = \sum\limits_{n=0}^{\infty} a_n z^n$, $a_n \in \mathcal{E}$, on $\mathbb{D}$ such that

$$\|f\| = (\sum_{n=0}^{\infty} \|a_n\|^2)^{\frac{1}{2}} < \infty.$$

Given another Hilbert space $\mathcal{E}_*$, we denote by $H^\infty_{\mathscr{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D})$ (or $H^\infty_{\mathscr{B}(\mathcal{E})}(\mathbb{D})$ if $\mathcal{E}_* = \mathcal{E}$) the set of $\mathscr{B}(\mathcal{E}_*,\mathcal{E})$-valued bounded analytic functions on $\mathbb{D}$. Also recall that a function $\Theta \in H^\infty_{\mathscr{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D})$ is called *inner* if $\Theta(z)$ (via radial limits) is an isometry for all z a.e. in $\mathbb{T}$. If $\mathcal{E} = \mathcal{E}_* = \mathbb{C}$, then $H^\infty_{\mathscr{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D})$ will be denoted by $H^\infty(\mathbb{D})$. In particular, a function $\varphi \in H^\infty(\mathbb{D})$ is inner if and only if $|\varphi(z)| = 1$ for all z a.e. in $\mathbb{T}$.

Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{Q} \subseteq H^2_{\mathcal{E}}(\mathbb{D})$ be a closed subspace. We say that $\mathcal{Q}$ is a *Sz.-Nagy and Foiaş model space* if $\mathcal{Q}$ is M_z^*-invariant. In this case, the *Sz.-Nagy and Foiaş model operator* $S_{\mathcal{Q}}$ is the compression of M_z on $\mathcal{Q}$, that is

$$S_{\mathcal{Q}} = P_{\mathcal{Q}} M_z|_{\mathcal{Q}}.$$

If $\mathcal{E} = \mathbb{C}$, we simply say that $\mathcal{Q}$ is a *model space* and $S_{\mathcal{Q}}$ is a *model operator*. In this paper we always assume that $\{0\} \subsetneq \mathcal{Q} \subsetneq H^2_{\mathcal{E}}(\mathbb{D})$. Observe that Sz.-Nagy and Foiaş model operators essentially represents the set of all contractions T such that $T^{*n} \to 0$ in the strong operator topology [18].

Proposition 3.1. *Suppose $\mathcal{Q} \subseteq H^2_{\mathcal{E}}(\mathbb{D})$ be a Sz.-Nagy and Foiaş model space. Then there exists a non-zero vector $f \in \mathcal{Q}$ such that $f(0) = 0$ if and only if $S_{\mathcal{Q}} \in \mathcal{NA}$ and $\|S_{\mathcal{Q}}\| = 1$.*

Proof. If we set $\mathcal{C} := \{h \in \mathcal{Q} : \|S_{\mathcal{Q}} h\| = \|h\|\}$, then

$$\mathcal{C} = \{h \in \mathcal{Q} : zh \in \mathcal{Q}\}.$$

Indeed, if $h \in \mathcal{C}$, then

$$\|h\| = \|S_{\mathcal{Q}} h\| = \|P_{\mathcal{Q}} M_z h\| \leq \|M_z h\| = \|h\|,$$

and hence $\|P_{\mathcal{Q}}(zh)\| = \|zh\|$. This implies that $zh \in \mathcal{Q}$, as $P_{\mathcal{Q}}$ is an orthogonal projection of $H^2_{\mathcal{E}}(\mathbb{D})$ onto $\mathcal{Q}$. On the other hand, if $h \in \mathcal{Q}$ and $zh \in \mathcal{Q}$, then clearly

$$\|S_{\mathcal{Q}}h\| = \|P_{\mathcal{Q}}zh\| = \|M_z h\| = \|h\|.$$

Now, let $\|S_{\mathcal{Q}}\| = 1$ and $S_{\mathcal{Q}} \in \mathcal{NA}$. This implies that $\mathcal{C} \neq \{0\}$. Then there exists a non-zero vector $h \in \mathcal{Q}$ such that $f := zh \in \mathcal{Q}_\theta$. Clearly, $f(0) = 0$. Conversely, let $f \in \mathcal{Q}$ be a non-zero vector such that $f(0) = 0$. Then $f = zg \in \mathcal{Q}$ for some $g \in H^2_{\mathcal{E}}(\mathbb{D})$. Since $\mathcal{Q}$ is M_z^*-invariant, it follows that $g = M_z^* f \in \mathcal{Q}$. Then

$$\|S_{\mathcal{Q}}g\| = \|P_{\mathcal{Q}}(zg)\| = \|zg\| = \|g\|,$$

which implies that $\|S_{\mathcal{Q}}\| = 1$ and $S_{\mathcal{Q}} \in \mathcal{NA}$. ☐

We now concentrate on the special case when $\mathcal{Q}$ is a model space, that is, $\mathcal{Q} \subseteq H^2(\mathbb{D})$. First, we recall that the Hardy space $H^2(\mathbb{D})$ is also a reproducing kernel Hilbert space corresponding to the Szegő kernel c on $\mathbb{D}$, where

$$c(z, w) = (1 - z\bar{w})^{-1} \qquad (z, w \in \mathbb{D}).$$

Then the set of *kernel functions* $\{c(\cdot, w) : w \in \mathbb{D}\}$ forms a total set in $H^2(\mathbb{D})$, and satisfies the *reproducing property* $f(w) = \langle f, c(\cdot, w)\rangle_{H^2(\mathbb{D})}$, for all $w \in \mathbb{D}$ and $f \in H^2(\mathbb{D})$. Now suppose that $\mathcal{Q}$ is a model space. Then the classical Beurling theorem yields

$$\mathcal{Q} = \mathcal{Q}_\theta := H^2(\mathbb{D})/\theta H^2(\mathbb{D}),$$

where $\theta \in H^\infty(\mathbb{D})$ is an inner function (which is unique up to multiplication by a scalar of modulus one). Then the corresponding model operator $S_{\mathcal{Q}}$, denoted by S_θ, is given by $S_\theta = P_{\mathcal{Q}_\theta} M_z|_{\mathcal{Q}_\theta}$, where $P_{\mathcal{Q}_\theta}$ is the orthogonal projection of $H^2(\mathbb{D})$ onto $\mathcal{Q}_\theta$. One can easily prove that

$$c_\theta(z, w) = \frac{1 - \theta(z)\overline{\theta(w)}}{1 - z\bar{w}} \qquad (z, w \in \mathbb{D}),$$

defines the reproducing kernel function of $\mathcal{Q}_\theta$. In particular

$$c_\theta(z, 0) = 1 - \theta(z)\overline{\theta(0)},$$

and hence

$$\|c_\theta(\cdot, 0)\| = (1 - |\theta(0)|^2)^{\frac{1}{2}}.$$

The following result complements Proposition 3.1:

Proposition 3.2. *Let $\theta \in H^\infty(\mathbb{D})$ be inner. Then $\|S_\theta\| = |\theta(0)|$ if and only if $\|S_\theta\| < 1$ and $S_\theta \in \mathcal{NA}$.*

Proof. Suppose $\|S_\theta\| < 1$ and $S_\theta \in \mathcal{NA}$. Then, by Theorem 1.1, there exists a non-zero vector $f \in \mathcal{Q}_\theta$ such that

$$(\|S_\theta\|^2 I_{\mathcal{Q}_\theta} - S_\theta S_\theta^*)f = 0.$$

It is easy to see that $S_\theta S_\theta^* = I_{\mathcal{Q}_\theta} - c_\theta(\cdot,0)\otimes c_\theta(\cdot,0)$. Then

$$(\|S_\theta\|^2-1)f + \langle f, c_\theta(\cdot,0)\rangle c_\theta(\cdot,0) = 0. \tag{3.1}$$

Taking inner product with $c_\theta(\cdot,0)$, we have

$$(\|S_\theta\|^2-1)\langle f, c_\theta(\cdot,0)\rangle + \left(1-|\theta(0)|^2\right)\langle f, c_\theta(\cdot,0)\rangle = 0,$$

as $\|c_\theta(\cdot,0)\|^2 = 1-|\theta(0)|^2$. Note that since $\|S_\theta\| < 1$ and $f \neq 0$, (3.1) implies that $\langle f, c_\theta(\cdot,0)\rangle \neq 0$. Then we have $\|S_\theta\|^2 - 1 + (1-|\theta(0)|^2) = 0$, and hence $\|S_\theta\| = |\theta(0)|$. Conversely, if $\|S_\theta\| = |\theta(0)|$, then

$$\begin{aligned}(\|S_\theta\|^2 I_{\mathcal{Q}_\theta} - S_\theta S_\theta^*)c_\theta(\cdot,0) &= \Big(|\theta(0)|^2 I_{\mathcal{Q}_\theta} - (I_{\mathcal{Q}_\theta} - c_\theta(\cdot,0)\otimes c_\theta(\cdot,0))\Big)c_\theta(\cdot,0)\\ &= |\theta(0)|^2 c_\theta(\cdot,0) - c_\theta(\cdot,0) + \|c_\theta(\cdot,0)\|^2 c_\theta(\cdot,0)\\ &= (|\theta(0)|^2 - 1 + (1-|\theta(0)|^2))c_\theta(\cdot,0)\\ &= 0,\end{aligned}$$

that is, $\|S_\theta\|^2$ is in the point spectrum of $S_\theta S_\theta^*$. This along with Theorem 1.1 shows that $S_\theta \in \mathcal{NA}$. Finally, since θ is inner, by the maximum modulus principle we conclude that $1 > |\theta(0)| = \|S_\theta\|$. This completes the proof. $\square$

We now proceed to the most definite result of this section. For any $\lambda \in \mathbb{D}$, we denote by b_λ the *Blaschke factor* corresponding to λ, that is

$$b_\lambda(z) = \frac{z-\lambda}{1-\bar{\lambda}z} \qquad (z \in \mathbb{D}).$$

Theorem 3.3. *Let $\theta \in H^\infty(\mathbb{D})$ be an inner function. Then $S_\theta \in \mathcal{NA}$. Moreover, $\|S_\theta\| = 1$ if and only if dim$\mathcal{Q}_\theta > 1$.*

Proof. First we consider $\dim\mathcal{Q}_\theta < \infty$. Suppose $\dim\mathcal{Q}_\theta = 1$. Then there exists $\lambda \in \mathbb{D}$ such that $\theta = b_\lambda$ and $\mathcal{Q}_\theta = \mathbb{C}c(\cdot,\lambda)$. Then $S_\theta^* c(\cdot,\lambda) = \bar{\lambda}c(\cdot,\lambda)$ implies that

$$\|S_\theta\| = |\lambda| = |\theta(0)| < 1.$$

Now let $\dim\mathcal{Q}_\theta = n (>1)$. Then $\theta = \prod\limits_{i=1}^n b_{\lambda_i}$ for some $\{\lambda_i\}_{i=1}^n \subseteq \mathbb{D}$. Suppose $\lambda_p \neq \lambda_q$ for some $p \neq q$ and $1 \leq p, q \leq n$, and suppose

$$f = c(\cdot,\lambda_p) - c(\cdot,\lambda_q).$$

If $\lambda = \lambda_i$ for all $i = 1,\ldots,n$, then we consider f as

$$f = \overline{\lambda}c(\cdot,\lambda) - \partial c(\cdot,\lambda),$$

where the partial derivative is with repect to the first variable. In either case, $f \in \mathcal{Q}_\theta$ and $f(0) = 0$, and thus, by Proposition 3.1, $\|S_\theta\| = 1$ and $S_\theta \in \mathcal{NA}$ (of course, the latter conclusion is trivial as S_θ is a finite rank operator).

We now consider the infinite dimensional case. Assume first that θ is an infinite Blaschke product, that is

$$\theta(z) = \prod_{n=1}^{\infty} \alpha_n b_{\lambda_n}(z) \qquad (z \in \mathbb{D}),$$

where $\sum_{k=1}^{\infty}(1-|\lambda_k|) < \infty$ and $|\alpha_n| = 1$, for all $n \geq 1$. Then $\mathcal{Q}_\theta = \overline{\text{span}}\{c(\cdot, \lambda_n) : n \geq 1\}$. Since $S_\theta^* c(\cdot, \lambda_m) = \bar{\lambda}_m c(\cdot, \lambda_m)$ for all $m \geq 1$, and $|\lambda_n| \to 1$ as $n \to \infty$, it follows that

$$1 = \sup_{n \in \mathbb{N}} |\lambda_n| \leq \|S_\theta\| \leq 1,$$

which implies that $\|S_\theta\| = 1$. Again we set $f = c(\cdot, \lambda_p) - c(\cdot, \lambda_q)$ or $f = \overline{\lambda} c(\cdot, \lambda) - \partial c(\cdot, \lambda)$, as above, and conclude that $f(0) = 0$ for some $f \in \mathcal{Q}_\theta$. Then Proposition 3.1 again implies that $S_\theta \in \mathcal{NA}$.

Now assume that θ is a singular inner function. By Frostman's theorem, there exist distinct λ_1 and λ_2 in $\mathbb{D}$ such that $c_\theta(\cdot, \lambda_1), c_\theta(\cdot, \lambda_2) \in \mathcal{Q}_\theta$. Set

$$f = \Big(1 - \overline{\theta(\lambda_2)}\theta(0)\Big) c_\theta(\cdot, \lambda_1) - \Big(1 - \overline{\theta(\lambda_1)}\theta(0)\Big) c_\theta(\cdot, \lambda_2).$$

Then $f \in \mathcal{Q}_\theta$ and $f(0) = 0$, as $c_\theta(0, w) = 1 - \theta(0)\overline{\theta(w)}$, $w \in \mathbb{D}$. By Theorem 3.2 it follows that $S_\theta \in \mathcal{NA}$ and $||S_\theta|| = 1$.

Finally, let $\theta = \theta_b \theta_s$, where θ_b is the Blaschke product formed by the zeros of θ and θ_s is the corresponding singular factor of θ. Since $\theta_s | \theta$, it easily follows that (by the Douglas range inclusion theorem)

$$\mathcal{Q}_{\theta_s} \subseteq \mathcal{Q}_\theta.$$

We can then use the singular inner function part above to find a $f \in \mathcal{Q}_{\theta_s} \subseteq \mathcal{Q}_\theta$ such that $f(0) = 0$. Consequently, $S_\theta \in \mathcal{NA}$ and $\|S_\theta\| = 1$. This completes the proof of the theorem. □

Remark 3.4. The latter conclusion of Theorem 3.3 is not new, and it essentially follows from [10, Section 7, Corollary 3]. In fact, [10, Section 7] deals with the problem of norm attaining symbols of truncated Toeplitz operators: given $\varphi \in L^\infty(\mathbb{T})$, when does $\|A_\varphi^\theta\|_{\mathscr{B}(\mathcal{Q}_\theta)} = \|\varphi\|_\infty$, where $A_\varphi^\theta = P_{\mathcal{Q}_\theta} L_\varphi|_{\mathcal{Q}_\theta}$ (see Section 4) is the truncated Toeplitz operator. This problem, a priori, is different from our norm attaining operators. On the other hand, our approach, like divisibility of functions in model spaces as in the proof of Proposition 3.1, certainly relies on classical technique as in [5] and [10].

4. Toeplitz Operators

In this section we consider Toeplitz and Laurent operators that admit the norm. We first introduce the classical vector-valued Hilbert measure spaces. Suppose $\mathcal{E}$ is a

Hilbert space. Let $L^2_{\mathcal{E}}(\mathbb{T})$ (here $\mathbb{T} = \partial\mathbb{D}$) denote the Hilbert space of all square $\mathcal{E}$-valued (Lebesgue) integrable functions on $\mathbb{T}$, that is

$$L^2_{\mathcal{E}}(\mathbb{T}) = \left\{ f : \mathbb{T} \to \mathcal{E} \text{ measurable} : \|f\| = \Big[\int_{\mathbb{T}} \|f(z)\|^2_{\mathcal{E}} \, dm(z) \Big]^{\frac{1}{2}} < \infty \right\},$$

where m is the normalized Lebesgue measure on $\mathbb{T}$. The Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$ also can be identified (via radial limits) to the subspace (which we will denote again by $H^2_{\mathcal{E}}(\mathbb{D})$) of $\mathcal{E}$-valued functions f in $L^2_{\mathcal{E}}(\mathbb{T})$ such that $\hat{f}(n) = 0$, $n < 0$, where $\hat{f}(n)$ is the n-th Fourier coefficient of f. Given another Hilbert space $\mathcal{E}_*$, we denote by $L^\infty_{\mathscr{B}(\mathcal{E}_*, \mathcal{E})}(\mathbb{T})$ (or $L^\infty_{\mathscr{B}(\mathcal{E})}(\mathbb{T})$ if $\mathcal{E}_* = \mathcal{E}$) the set of $\mathscr{B}(\mathcal{E}_*, \mathcal{E})$-valued bounded functions on $\mathbb{T}$.

Now we turn to the main content of this section. We begin with Laurent operators: Let $\mathcal{E}_*$ and $\mathcal{E}$ be Hilbert spaces. For $\Phi \in L^\infty_{\mathscr{B}(\mathcal{E}_*, \mathcal{E})}(\mathbb{T})$, the *Laurent operator* $L_\Phi : L^2_{\mathcal{E}_*}(\mathbb{T}) \to L^2_{\mathcal{E}}(\mathbb{T})$ is defined by $(L_\Phi f)(z) = \Phi(z) f(z)$, $z \in \mathbb{T}$. In this case, L_Φ is bounded and $\|L_\Phi\| = \|\Phi\|_\infty$. The *Toeplitz operator* $T_\Phi : H^2_{\mathcal{E}_*}(\mathbb{D}) \to H^2_{\mathcal{E}}(\mathbb{D})$ with (operator-valued) symbol Φ is defined by

$$T_\Phi = P_{H^2_{\mathcal{E}}(\mathbb{D})} L_\Phi|_{H^2_{\mathcal{E}_*}(\mathbb{D})},$$

where $P_{H^2_{\mathcal{E}}(\mathbb{D})}$ is the orthogonal projection of $L^2_{\mathcal{E}}(\mathbb{T})$ onto $H^2_{\mathcal{E}}(\mathbb{D})$. In particular, when $\mathcal{E}_* = \mathcal{E}$, L_z is the bilateral shift, and $T_z = P_{H^2_{\mathcal{E}}(\mathbb{D})} L_z|_{H^2_{\mathcal{E}}(\mathbb{D})} = M_z$, as the symbol z is analytic. It is well known that $\|T_\Phi\| = \|\Phi\|_\infty$ (cf. [3, Theorem 1.7, page 112]).

The following theorem provides a complete characterization of norm attaining operator valued Toeplitz operators.

Theorem 4.1. *Let $\Phi \in L^\infty_{\mathscr{B}(\mathcal{E})}(\mathbb{T})$, and suppose $\|T_\Phi\| = 1$. Then $T_\Phi \in \mathcal{NA}$ if and only if there exist a Hilbert space $\mathcal{E}_*$ and inner functions $\Theta, \Psi \in H^\infty_{\mathscr{B}(\mathcal{E}_*, \mathcal{E})}(\mathbb{D})$ such that $\Theta = \Phi\Psi$. Moreover, in this case $T_\Theta = T_\Phi T_\Psi$.*

Proof. As in the proof of Proposition 3.1, we set $\mathcal{C} = \{h \in H^2_{\mathcal{E}}(\mathbb{D}) : \|T_\Phi h\| = \|h\|\}$. If $h \in \mathcal{C}$, then

$$\|h\| = \|P_{H^2_{\mathcal{E}}(\mathbb{D})}(\Phi h)\| \leq \|\Phi h\| \leq \|h\|,$$

and hence $\|P_{H^2_{\mathcal{E}}(\mathbb{D})}(\Phi h)\| = \|\Phi h\|$, or, equivalently, $\Phi h \in H^2_{\mathcal{E}}(\mathbb{D})$. In particular, $\|L_\Phi h\| = \|\Phi h\| = \|h\|$ implies $\langle (I - L_\Phi^* L_\Phi) h, h \rangle = 0$, and hence $L_\Phi^* L_\Phi h = h$. As the reverse direction is obvious, we have

$$\mathcal{C} = \{h \in H^2_{\mathcal{E}}(\mathbb{D}) : \Phi h \in H^2_{\mathcal{E}}(\mathbb{D}) \text{ and } L_\Phi^* L_\Phi h = h\}.$$

In particular, $\mathcal{C}$ is a closed subspace of $H^2_{\mathcal{E}}(\mathbb{D})$. Moreover, if $h \in \mathcal{C}$ and $f \in L^2_{\mathcal{E}}(\mathbb{T}) \ominus H^2_{\mathcal{E}}(\mathbb{D})$, then

$$\|L_\Phi(zh)\| = \|L_\Phi L_z h\| = \|L_\Phi h\| = \|h\| = \|zh\|,$$

and

$$\langle \Phi z h, f \rangle_{L^2_{\mathcal{E}}(\mathbb{T})} = \langle \Phi h, L_z^* f \rangle_{L^2_{\mathcal{E}}(\mathbb{T})} = 0,$$

implies that $\mathcal{C}$ is an M_z-invariant subspace of $H^2_{\mathcal{E}}(\mathbb{D})$.

Now suppose $T_\phi \in \mathcal{NA}$. Then $\mathcal{C} \neq \{0\}$, and hence by Beurling, Lax and Halmos theorem [18, Chapter V, Theorem 3.3], there exist a Hilbert space $\mathcal{E}_*$ and an inner function $\Psi \in H^\infty_{\mathscr{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D})$ such that $\mathcal{C} = \Psi H^2_{\mathcal{E}_*}(\mathbb{D})$. Moreover, $\Phi\mathcal{C} \subseteq H^2_{\mathcal{E}}(\mathbb{D})$ implies that

$$\Phi\Psi H^2_{\mathcal{E}_*}(\mathbb{D}) \subseteq H^2_{\mathcal{E}}(\mathbb{D}).$$

Evidently, there exists $\Theta \in H^\infty_{\mathscr{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D})$ such that $\Theta = \Phi\Psi$. Moreover, if $f \in H^2_{\mathcal{E}_*}(\mathbb{D})$, then

$$\|\Theta f\| = \|\Phi\Psi f\| = \|\Psi f\| = \|f\|,$$

which implies that $\Theta \in H^\infty_{\mathscr{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D})$ is an inner function.
For the converse, observe that since Θ and Ψ are inner, for each $f \in H^2_{\mathcal{E}_*}(\mathbb{D})$ we have

$$\|T_\Phi(\Psi f)\| = \|P_{H^2_{\mathcal{E}}(\mathbb{D})}\Phi\Psi f\| = \|P_{H^2_{\mathcal{E}}(\mathbb{D})}\Theta f\| = \|\Theta f\| = \|f\| = \|\Psi f\|,$$

which implies that $T_\Phi \in \mathcal{NA}$.
The final part is standard: $T_\Theta = T_\Phi T_\Psi$ follows from the fact that Θ and Ψ are inner and $\Theta = \Phi\Psi$. □

If $\mathcal{E} = \mathbb{C}$, then the above theorem reduces to the Brown and Douglas [5, Lemma 2] classification of norm attaining Toeplitz operators with scalar-valued symbols:

Corollary 4.2. *Let $\varphi \in L^\infty(\mathbb{T})$ and suppose $\|\varphi\|_\infty = 1$. Then $T_\varphi \in \mathcal{NA}$ if and only if there exist inner functions $\psi, \theta \in H^\infty(\mathbb{D})$ such that $T_\varphi = T^*_\psi T_\theta$.*

Proof. In this case, $\mathcal{E}_* = \mathbb{C}$. The result now follows from the observation that $T_\varphi T_\psi = T_\psi T_\varphi$. □

Now we turn to norm attaining analytic Toeplitz operators. Let $\mathcal{E}$ be a Hilbert space, and let $\Phi \in H^\infty_{\mathscr{B}(\mathcal{E})}(\mathbb{D})$. The *analytic Toeplitz operator* (or *multiplication operator*) $M_\Phi : H^2_{\mathcal{E}}(\mathbb{D}) \to H^2_{\mathcal{E}}(\mathbb{D})$ with symbol Φ is defined by

$$(M_\Phi f)(z) = \Phi(z)f(z) \qquad (f \in H^2_{\mathcal{E}}(\mathbb{D}), z \in \mathbb{D}).$$

It is known that $\|M_\Phi\| = \|\Phi\|_\infty$, and M_Φ is an isometry if and only if Φ is inner [18, Proposition 2.2].
The following vector-valued analogue of F. and M. Riesz theorem is certainly well known, but we have not been able to trace an explicit reference in the literature.

Lemma 4.3. *If f is a non-zero function in $H^2_{\mathcal{E}}(\mathbb{D})$, then the measure of the set $\{z \in \mathbb{T} : f(z) = 0\}$ is zero.*

Proof. Let $f \in H^2_{\mathcal{E}}(\mathbb{D})$ and suppose $E = \{z : f(z) = 0\}$, and $f(z) = \sum_{k=0}^\infty a_k z^k$, $z \in \mathbb{D}$. For each $\eta \in \mathcal{E}$ we define $f_\eta : \mathbb{D} \to \mathbb{C}$ by $f_\eta(z) = \langle f(z), \eta\rangle_{\mathcal{E}}$, $z \in \mathbb{D}$. Clearly $f_\eta(z) = \sum\limits_{k=0}^{\infty} \langle a_k, \eta\rangle_{\mathcal{E}} z^k$. Hence $f_\eta \in H^2(\mathbb{D})$, as

$$\sum |\langle a_k, \eta\rangle|^2 \leq \|\eta\|^2 \sum \|a_k\|^2 = \|\eta\|^2\|f\|^2 < \infty,$$

by the Cauchy-Schwarz's inequality. Moreover, if $z \in \mathbb{T}$, then

$$f_\eta(z) = \lim_{r\to 1^-} f_\eta(rz) = \lim_{r\to 1^-} \langle f(rz), \eta\rangle = \langle f(z), \eta\rangle,$$

which implies $f_\eta(z) = 0$ on E for all $\eta \in \mathcal{E}$. If $m(E) > 0$, then by the classical F. and M. Riesz theorem, $f_\eta = 0$ for each $\eta \in \mathcal{E}$, and hence $f = 0$. □

We are now ready to prove the following theorem:

Theorem 4.4. *Let $\Phi \in H^\infty_{\mathscr{B}(\mathcal{E})}(\mathbb{D})$.*
(i) If $M_\Phi \in \mathcal{NA}$, then $\|\Phi(z)\| = \|\Phi\|_\infty$, $z \in \mathbb{T}$ a.e.
(ii) If $\mathcal{E} = \mathbb{C}$, then $M_\Phi \in \mathcal{NA}$ if and only if $\frac{1}{\|\Phi\|_\infty}\Phi$ is inner.

Proof. (i) Suppose M_Φ attains its norm at $f \in H^2_{\mathcal{E}}(\mathbb{D})$. Then $\|M_\Phi f\| = \|M_\Phi\|\|f\| = \|\Phi\|_\infty\|f\|$ implies that

$$\int_{\mathbb{T}} \|\Phi(z)f(z)\|^2\, dm(z) = \int_{\mathbb{T}} \|\Phi\|^2_\infty \|f(z)\|^2\, dm(z),$$

and hence

$$\int_{\mathbb{T}} \|\Phi\|^2_\infty \|f(z)\|^2\, dm(z) \geq \int_{\mathbb{T}} \|\Phi(z)\|^2 \|f(z)\|^2\, dm(z) \geq \int_{\mathbb{T}} \|\Phi\|^2_\infty \|f(z)\|^2\, dm(z),$$

whence $\int_{\mathbb{T}} \|\Phi\|^2_\infty \|f(z)\|^2\, dm(z) = \int_{\mathbb{T}} \|\Phi(z)\|^2 \|f(z)\|^2\, dm(z)$. Lemma 4.3 then implies that $\|\Phi(z)\| = \|\Phi\|$, $z \in \mathbb{T}$ a.e. as desired.
(ii) In view of part (i), it is enough to observe that $M_{\frac{1}{\|\Phi\|_\infty}\Phi}$ is an isometry whenever $\frac{1}{\|\Phi\|_\infty}\Phi$ is inner. □

The converse of Theorem 4.4(i) does not hold:

Example 4.5. In the setting of Proposition 4.4, consider $\mathcal{E} = \ell^2(\mathbb{N})$ and the compact operator $K : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by

$$K\Big(\{\alpha_n\}_{n=1}^\infty\Big) = \Big\{\alpha_1, \frac{\alpha_2}{2}, \frac{\alpha_3}{3}, \dots\Big\}.$$

Note that $I - K \notin \mathcal{NA}$. Indeed, for any non-zero sequence $\{\alpha_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$, we have

$$\|(I-K)\{\alpha_1, \alpha_2, \alpha_3, \dots\}\|^2 = \sum_{n=1}^\infty \left(1 - \frac{1}{n}\right)^2 |\alpha_n|^2 < \sum_{n=1}^\infty |\alpha_n|^2.$$

Define the constant function $\Phi : \mathbb{D} \to \mathscr{B}(\mathcal{E})$ by $\Phi(z) = I - K$, $z \in \mathbb{D}$. Clearly $\|M_\Phi\| = \|\Phi\| = \|I - K\| = 1$. We shall show that $M_\Phi \notin \mathcal{NA}$. Suppose towards a contradiction that $M_\Phi \in \mathcal{NA}$. Then there exists a non-zero f in $H^2_{\mathcal{E}}(\mathbb{D})$ such that $\|M_\Phi f\| = \|f\|$, and so

$$\int_{\mathbb{T}} \Big(\|f(z)\|^2 - \|(I-K)f(z)\|^2\Big) dm(z) = 0.$$

Since $\|I-K\|=1$, we have

$$\|(I-K)f(z)\| = \|f(z)\| \qquad (z \in \mathbb{T}\, a.e.).$$

However, $I-K \notin \mathcal{NA}$, which is a contradiction.

Example 4.6. Now we comment on the inner function property of Φ in the statement of Theorem 4.4. There, unlike the scalar case, if $\Phi \in H^\infty_{\mathscr{B}(\mathcal{E})}(\mathbb{D})$ with $\|\Phi\| = 1$ and $M_\Phi \in \mathcal{NA}$, then Φ need not be inner. For instance, consider the backward shift S^* on $\mathcal{E} = \ell^2(\mathbb{N})$, that is

$$S^* e_n = \begin{cases} e_{n-1} & \text{if } n \geq 2 \\ 0 & \text{if } n = 1, \end{cases}$$

and define the constant function $\Phi(z) = S^*$, $z \in \mathbb{D}$. Of course, $\Phi(z) = S^*$ for all $z \in \mathbb{T}$, and hence, it follows that Φ is not inner. On the other hand, define $f \in H^2_{\mathcal{E}}(\mathbb{D})$ by $f(z) = e_2$, $z \in \mathbb{D}$, where $e_2(i) = \delta_{2,i}$. Then $\|f\| = 1$ and

$$\|M_\Phi f\|^2 = \int_{\mathbb{T}} \|\Phi(z) f(z)\|^2 \, dm(z) = \int_{\mathbb{T}} \|S^* e_2\|^2 \, dm(z) = 1 = \|\Phi\|^2,$$

and hence $M_\Phi \in \mathcal{NA}$.

We conclude the paper with norm attaining Laurent operators. In this setting, the results and the ideas are similar to that of Theorem 4.4. We first illustrate the scalar case: Let $\varphi \in L^\infty(\mathbb{T})$. Suppose $f \neq 0$ in $L^2(\mathbb{T})$ satisfy $\|L_\varphi f\| = \|\varphi\|_\infty \|f\|_{L^2(\mathbb{T})}$. Then

$$\left(\|\varphi\|_\infty^2 - |\varphi(z)|^2\right) |f(z)|^2 = 0 \qquad (z \in \mathbb{T}\ a.e.).$$

If $E = \{z \in \mathbb{T} : f(z) = 0\}$, then $m(E) = 0$, and hence $|\varphi(z)| = \|\varphi\|_\infty$ for all $z \in E$, where $E \subseteq \mathbb{T}$ and $m(E) > 0$. Conversely, if $m(E) > 0$ and $|\varphi(z)| = \|\varphi\|_\infty$ for all $z \in E$, then M_φ attain its norm at χ_E. This proves the following:

Proposition 4.7. *Let $\varphi \in L^\infty(\mathbb{T})$. Then $L_\varphi \in \mathcal{NA}$ if and only if there exists a measurable set $A \subseteq \mathbb{T}$ such that $m(A) > 0$ and $|\varphi(z)| = \|\varphi\|_\infty$ for all $z \in A$.*

A similar (but one directional as in Theorem 4.4) statement is valid for operator-valued Laruant operators: Let $\Phi \in L^\infty_{\mathscr{B}(\mathcal{E})}(\mathbb{T})$. If $L_\Phi \in \mathcal{NA}$, then there exists a measurable set $A \subseteq \mathbb{T}$ such that $m(A) > 0$ and $\|\Phi(z)\| = \|\Phi\|$, $z \in A$. Again, the converse fails to hold: Example 4.5 serves the purpose.

Acknowledgement: We are grateful to the referee for a careful reading of the paper and useful suggestions. We are especially thankful to the referee for suggesting the elegant proof of Theorem 2.2. The research of first and second author is supported by the Theoretical Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore Centre, India. The third author is supported in part by the Mathematical Research Impact Centric Support, MATRICS (MTR/2017/000522), and Core Research

Grant (CRG/2019/000908), by SERB, Department of Science & Technology (DST), and NBHM (NBHM/R.P.64/2014), Government of India. The research of the fourth author is supported by the NBHM postdoctoral fellowship, Department of Atomic Energy (DAE), Government of India (File No: 0204/3/2020/R&D-II/2445).

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