MAXIMAL CONTRACTIVE TUPLES

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ABSTRACT. Maximality of a contractive tuple of operators is considered. A characterization for a contractive tuple to be maximal is obtained. The notion of maximality for a submodule of the Drury-Arveson module on the d-dimensional unit ball $\mathbb{B}_d$ is defined. For $d = 1$, it is shown that every submodule of the Hardy module over the unit disc is maximal. But for $d \geq 2$ we prove that any homogeneous submodule or submodule generated by polynomials is not maximal. A characterization of maximal submodules is obtained.

1. INTRODUCTION

Let $T = (T_1, \dots, T_d)$ be a d-tuple of bounded linear operators on some Hilbert space $\mathcal{H}$. We say that T is a *row contraction*, or, *contractive tuple* if the row operator $(T_1, \dots, T_d) : \mathcal{H}^d \to \mathcal{H}$ is a contraction or equivalently $\sum_{i=1}^d T_i T_i^* \leq I_{\mathcal{H}}$. The *defect operator* $D_T := (I - \sum_{i=1}^d T_i T_i^*)^{1/2}$ and the *defect dimension* $\Delta_T := \dim[\overline{\text{ran}} D_T]$ associated with the contractive tuple T are important invariants in operator theory. For instance, a pair of shift operators are unitary equivalent if and only if the defect dimensions are the same. The same result holds true for d-tuple of pure isometries with orthogonal ranges ([Po$_2$]). In order to extract more information about contractive tuples one can proceed further to form a sequence of defect indices (defined below).

The defect sequence and the notion of maximality for a single contraction were introduced in [GaW]. In a recent paper these were extended for d-tuple of operators ([BDS]). The main aim of this paper is to characterize maximal contractive tuples in the commuting as well as non-commuting case. In the non-commuting setup it turns out that the tuple consists of restrictions of creation operators on the full Fock space to an invariant subspace is always maximal but in the commuting setup the same conclusion does not hold. Examples of submodules of the Drury-Arveson module are given to illustrate the above fact and a characterization of maximal submodules is also obtained.

The plan of the paper is as follows. After introducing the completely positive map associated to a contractive tuple we define the defect sequence and obtain its properties in Section 2. In Section 3, we provide a characterization for maximal contractive tuples and consequently establish some relations between the minimal function of a particular type of single pure contraction and the dimension of the Hilbert space on which the contraction acts. In the last section we investigate the maximality for the tuple $(M_{z_1}|_{\mathcal{S}}, \cdots, M_{z_d}|_{\mathcal{S}})$, where $\mathcal{S}$ is

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a proper submodule of the Drury-Arveson module and the tuple $(M_{z_1}, \cdots, M_{z_d})$ is the d-shift of the Drury-Arveson module.

2. The defect sequence

In this section we define the defect sequence of a contractive tuple and study its properties. Some of the results can be found in ([BDS]) and we provide proofs of them using simple and different method. We fix for this section a *contractive d-tuple* $T = (T_1, \dots, T_d)$ of operators acting on a Hilbert space $\mathcal{H}$ in which the tuple T is not necessarily commuting and the Hilbert space $\mathcal{H}$ is infinite dimensional in general unless otherwise we specify it.

We begin by defining the completely positive map associated to the contractive tuple T (cf. [Arv], [Pot]) as follows:

$$\Psi_T : B(\mathcal{H}) \to B(\mathcal{H}), \quad X \mapsto \sum_{i=1}^{d} T_i X T_i^*. \tag{1}$$

This map is essential for simplifying the study of defect sequences. The following decreasing chain of operator inequality

$$I \geq \Psi_T(I) \geq \Psi_T^2(I) \geq \dots$$

is immediate from the contractivity of T and the positivity of Ψ_T. The contractive tuple T is said to be *pure* if $\Psi_T^n(I) \to 0$ in the strong operator topology (S.O.T.) as $n \to \infty$.

The following rule of multiplication for operator tuples is in use. Let $\Lambda = \{1, \dots, d\}$. For $n \in \mathbb{N}$, we denote T^n by the following d^n-tuple of operators

$$T^n = (T_{i_1} T_{i_2} \dots T_{i_n} : i_j \in \Lambda, j = 1, \dots, n),$$

and for $n = 1$ we set $T^1 := T$. In particular, for $n = 2$, T^2 is the following d^2-tuple

$$(T_1^2, T_1 T_2, \dots, T_1 T_d, T_2 T_1, T_2^2, \dots, T_d^2) : \mathcal{H}^{d^2} \to \mathcal{H}.$$

Under this rule of multiplication note that $\Psi_{T^2}(X) = \Psi_T(\Psi_T(X)) = \Psi_T^2(X)$ for all $X \in B(\mathcal{H})$, where Ψ_T is as in (1).

Definition. The defect operator of T, denoted by D_T, is the bounded linear operator on $\mathcal{H}$ defined by

$$D_T := (I - \sum_{i=1}^{d} T_i T_i^*)^{1/2} = (I - \Psi_T(I))^{1/2}.$$

The *first defect index* Δ_T is the dimension of the *first defect space* $\mathcal{D}_T$, where

$$\mathcal{D}_T := \overline{\text{ran}} D_T = \overline{\text{ran}} D_T^2 = \overline{\text{ran}}(I - \Psi_T(I)).$$

The *nth defect index* of the tuple T is the dimension of the *nth defect space* $\overline{\text{ran}} D_{T^n}$, where

$$D_{T^n}^2 = I - \Psi_{T^n}(I) = I - \Psi_T^n(I).$$

We denote by $\mathcal{D}_n$ the nth defect space of a contractive tuple T, where context dictates the tuple T. Here we note the following identity

$$I - \Psi_T^n(I) = [I - \Psi_T(I)] + \Psi_T[I - \Psi_T(I)] + \cdots + \Psi_T^{n-1}[I - \Psi_T(I)]$$

$$= \sum_{i=0}^{n-1} \Psi_T^i(I - \Psi_T(I)). \tag{2}$$

The properties of the defect sequence are as follows.

Proposition 2.1. (i) *The defect spaces of T are increasing subspaces of $\mathcal{H}$, that is, for $n \leq k$, $\mathcal{D}_n \subset \mathcal{D}_k$.*
(ii)$\Delta_T^n \leq \Delta_T^k$*, for all $n \leq k$.*
(iii) *For $n \in \mathbb{N}$, $\Delta_T^n \leq (1 + d + d^2 + \cdots + d^{n-1})\Delta_T$.*

Proof. (i) For $k \geq n$, by row contractivity of T we have

$$I \geq \Psi_T^n(I) \geq \Psi_T^k(I).$$

Therefore,

$$0 \leq I - \Psi_T^n(I) \leq I - \Psi_T^k(I),$$

and consequently

$$\overline{\text{ran}}(I - \Psi_T^n(I)) \subseteq \overline{\text{ran}}(I - \Psi_T^k(I)).$$

Thus (i) follows.
(ii) Follows immediately from (i).
(iii) From (2) and the fact that Ψ_T^l is the sum of d^l terms, we have

$$\dim[\overline{\text{ran}}\ \Psi_T^l(I - \Psi_T(I))] \leq d^l \Delta_T,$$

for all $l \in \mathbb{N}$, and the result follows. ☐

Remarks. (i) If T is a single contraction, that is, if $d = 1$ then $\Delta_T^n \leq n\Delta_T$ for all $n \in \mathbb{N}$ ([GaW]).
(ii) If T is a commuting d-tuple then $\Delta_T^n \leq \left(\sum_{k=0}^{n-1} \binom{k+d-1}{d-1}\right)\Delta_T$.

Before we provide the explicit expression of the defect spaces we need the following lemma.

Lemma 2.2. *For each $n \in \mathbb{N}$, $T|_{\mathcal{D}_n^d} : \mathcal{D}_n^d \to \mathcal{D}_{n+1}$, where $\mathcal{D}_n^d$ is the direct sum of d copies of $\mathcal{D}_n$.*

Proof. First note that,

$$\sum_{i=1}^d T_i(I - \Psi_T^n(I))^{1/2}(I - \Psi_T^n(I))^{1/2}T_i^* = \sum_{i=1}^d T_iT_i^* - \sum_{i=1}^d T_i\Psi_T^n(I)T_i^*$$

$$\leq I - \Psi_T^{n+1}(I).$$

Letting

$$R := (T_1, \dots, T_d) \begin{pmatrix} D_{T^n} & 0 & 0 & \cdots & 0 \\ 0 & D_{T^n} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{T^n} \end{pmatrix}_{d\times d},$$

we have from the above inequality that $RR^* \leq I - \Psi_T^{n+1}(I)$. Thus

$$\overline{\operatorname{ran}} R \subseteq \overline{\operatorname{ran}}(I - \Psi_T^{n+1}(I)),$$

and this completes the proof. ☐

Remark. A simple induction argument shows that $T^n|_{\mathcal{D}_l^{d^n}} : \mathcal{D}_l^{d^n} \to \mathcal{D}_{l+n}$ for all $l, n \in \mathbb{N}$.

The following expression in terms of the first defect space is quite useful to compute the higher order defect spaces and is used throughout the paper.

Proposition 2.3. *The defect spaces of a contractive tuple T has the following form:*

$$\mathcal{D}_n = \mathcal{D}_1 \vee T(\mathcal{D}_1^d) \vee T^2(\mathcal{D}_1^{d^2}) \vee \cdots \vee T^{n-1}(\mathcal{D}_1^{d^{n-1}})$$

for all $n \in \mathbb{N}$.

Proof. $\mathcal{D}_n \subseteq \mathcal{D}_1 \vee T(\mathcal{D}_1^d) \vee T^2(\mathcal{D}_1^{d^2}) \vee \cdots \vee T^{n-1}(\mathcal{D}_1^{d^{n-1}})$ follows from (2) and the other inclusion follows from the previous lemma. ☐

Remark. Let $n < m$. Then as $I - \Psi_T^m = (I - \Psi_T^n) + \Psi_T^n(I - \Psi_T^{m-n}(I))$, we have $\mathcal{D}_m \subseteq \mathcal{D}_n \vee T^n(\mathcal{D}_{m-n}^{d^n})$. The other inclusion follows from the remark after Lemma 2.2. Thus $\mathcal{D}_m = \mathcal{D}_n \vee T^n(\mathcal{D}_{m-n}^{d^n})$.

Corollary 2.4. *If $\Delta_T^n = \Delta_T^{n+1}$ for some $n \in \mathbb{N}$ then $\Delta_T^n = \Delta_T^m$ for all $m > n$.*

Proof. If $\Delta_T^n = \Delta_T^{n+1}$ for some $n \in \mathbb{N}$ then clearly $\mathcal{D}_n = \mathcal{D}_{n+1}$. Note that $\mathcal{D}_{n+2} = \mathcal{D}_{n+1} \vee T^{n+1}(\mathcal{D}_1^{d^{n+1}})$ and $T^{n+1}(\mathcal{D}_1^{d^{n+1}}) = T\Big(\big(T^n(\mathcal{D}_1^{d^n})\big)^d\Big)$. Since $T^n(\mathcal{D}_1^{d^n}) \subset \mathcal{D}_{n+1} = \mathcal{D}_n$ we have $\mathcal{D}_{n+2} = \mathcal{D}_n$. Thus an induction argument gives the result. ☐

The above two propositions are from [BDS], Theorem 2.2 and Theorem 2.4, but the method used here will illuminate further studies in this direction.

3. Maximal tuples of operators

In this section we study the notion of maximality for contractive tuples. A necessary and sufficient condition for a contractive tuple to be maximal is obtained. In the rest of this note we only consider contractive tuples with finite first defect index. For this purpose we use the terminology 'finite rank' following Arveson and others. We say a contractive tuple T has finite rank if $\Delta_T < \infty$. The reader should not confuse with the dimension of the range of the tuple T.

The following notations are used throughout this section. Let $\Lambda = \{1, 2, \dots, d\}$ be a fixed index set. For every $k \in \mathbb{N}$, let $F(k, \Lambda)$ be the set of all functions from $\{1, 2, \dots, k\}$ to Λ, and set

$$F := \cup_{k=0}^{\infty} F(k, \Lambda), \quad F_{n]} := \cup_{k=0}^{n} F(k, \Lambda), \tag{3}$$

where $F(0, \Lambda)$ stands for $\emptyset$. For $T = (T_1, T_2, \dots, T_d)$, a d-tuple of operators, and $f \in F(k, \Lambda)$, we set

$$T_f := T_{f(1)} T_{f(2)} \dots T_{f(k)} \quad \text{and } T_\emptyset := I. \tag{4}$$

Definition. Let T be a contractive d-tuple of operators on an infinite dimensional Hilbert space $\mathcal{H}$ with finite rank. The tuple T is called *maximal* if

$$\Delta_T^n = (1 + d + \cdots + d^{n-1}) \Delta_T$$

for all $n \in \mathbb{N}$. Moreover, for finite dimensional $\mathcal{H}$, T is maximal if

$$\Delta_T^n = \begin{cases} (1 + d + \cdots + d^{n-1}) \Delta_T & \text{if } (1 + d + \cdots + d^{n-1}) \Delta_T \le \dim \mathcal{H}, \\ \dim \mathcal{H} & \text{otherwise} \end{cases},$$

for all $n \in \mathbb{N}$.

The maximality of a commuting contractive d-tuple is defined by interchanging the number $(1 + d + \cdots + d^{n-1})$ with $\sum_{k=0}^{n-1} \binom{k+d-1}{d-1}$ in the above definition.

It is clear from the properties of defect sequence that if $\Delta_T^n = (1 + d + \cdots + d^{n-1}) \Delta_T$ (or $\Delta_T^n = (\sum_{k=0}^{n-1} \binom{k+d-1}{d-1}) \Delta_T$ in the commuting case) for some $n \ge 2$, then $\Delta_T^l = (1 + d + \cdots + d^{l-1}) \Delta_T$ (respectively, $\Delta_T^l = (\sum_{k=0}^{l-1} \binom{k+d-1}{d-1}) \Delta_T$) for all $l \le n$. Thus for a non-commuting or commuting tuple, once the sequence of numbers Δ_T^n departs from the sequence of maximal possible values, it never returns.

Remark. For a non-commuting or commuting contractive tuple T on an infinite dimensional Hilbert space $\mathcal{H}$, let $\Delta_T = n$ and $\{\xi_i : i = 1, \dots, n\}$ be a basis of $\mathcal{D}_1$. Then T is maximal if and only if the set

$$\{T_f \xi_i : f \in F, i = 1, \dots, n\},$$

is linearly independent, where T_f is as in (4).

The tuple of multiplication operators on the Drury-Arveson module by the co-ordinate functions and the tuple of creation operators on the full Fock space are the standard examples of commuting and non-commuting maximal tuples respectively. An example of non-commuting, pure and maximal tuple which is not unitarily equivalent to the tuple of creation operators on the full Fock space $\mathcal{F}_d^2$, can be found in [BDS]. There it is shown that for a certain type of co-invariant subspace $\mathcal{M}$ of $\mathcal{F}_d^2$, the compression of creation tuple $(P_{\mathcal{M}} S_1|_{\mathcal{M}}, \cdots, P_{\mathcal{M}} S_d|_{\mathcal{M}})$ is maximal. This tuple is pure, maximal but need not be unitarily equivalent to the tuple of creation operators on the full Fock space.

By the above definition, a single contraction T acting on a Hilbert space $\mathcal{H}$ with $\Delta_T = 1$ is maximal if $\Delta_T^n = n$ for all $n \le \dim \mathcal{H}$. The next theorem provides a large class of maximal contractions.

Theorem 3.1. *Let T be a single pure contraction on $\mathcal{H}$ with $\Delta_T = 1$. Then $\Delta_{T^n} = n$, for $0 \leq n \leq \dim \mathcal{H}$.*

Proof. Since T is a pure contraction with $\Delta_T = 1$, so T is unitarily equivalent to the operator $R = P_{H_\theta} M_z|_{H_\theta}$ where $H_\theta = H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$ is a co-invariant subspace of the Hardy space $H^2(\mathbb{D})$ over the unit disc and $\theta \in H^\infty(\mathbb{D})$ is an inner function. Then it is enough to prove the theorem for the contraction R. A simple calculation reveals that $D_R = P_{H_\theta} P_{\mathbb{C}} P_{H_\theta}$ and as $\Delta_T = \Delta_R = 1$, we have $P_{\mathbb{C}} P_{H_\theta} \neq 0$ and $\operatorname{ran}(P_{\mathbb{C}} P_{H_\theta}) = \mathbb{C}$. Note that (cf. [Ber])

$$P_{H_\theta}(1) = (I - P_{H_\theta^\perp})1 = 1 - \overline{\theta(0)}\theta.$$

Then the first defect space of R is

$$\mathcal{D}_1 = \operatorname{span}\{1 - \overline{\theta(0)}\theta\}.$$

By the following elementary calculation we have

$$R(1 - \overline{\theta(0)}\theta) = P_{H_\theta}(z - \overline{\theta(0)} z\theta) = (I - P_{H_\theta^\perp})z = z - (\overline{\theta(0)}z + \overline{\theta'(0)})\theta.$$

Then by Proposition 2.3,

$$\mathcal{D}_2 = \operatorname{span}\{1 - \overline{\theta(0)}\theta, z - (\overline{\theta(0)}z + \overline{\theta'(0)})\theta\}.$$

An easy induction argument yields

$$R^n(1 - \overline{\theta(0)}\theta) = (I - P_{H_\theta^\perp})z^n = z^n - \big(\overline{\theta(0)}z^n + \overline{\theta'(0)}z^{n-1} + \cdots + \overline{\theta^{(n)}(0)}\big)\theta,$$

for all $n \geq 0$. Therefore, one has the explicit expression of $\mathcal{D}_n$ as follows. Let us denote

$$v_i = z^i - \big(\overline{\theta(0)}z^i + \overline{\theta'(0)}z^{i-1} + \cdots + \overline{\theta^{(i)}(0)}\big)\theta \quad (i \in \mathbb{N}).$$

Then $\mathcal{D}_n = \operatorname{span}\{v_i : i = 1, \ldots, n\}$. Now suppose that $\dim(\mathcal{D}_l) = \dim(\mathcal{D}_{l+1})$ for some $l \in \mathbb{N}$. Then as $\Delta_R = 1$ and the defect sequence is an increasing sequence it suffices to prove that $H_\theta = \mathcal{D}_l$. For a contradiction let $f \in H_\theta \ominus \mathcal{D}_l$. Then for all $i \in \mathbb{N}$, $\langle f, \theta z^i \rangle = 0$, and $\langle f, v_i \rangle = 0$ together implies that $\langle f, z^i \rangle = 0$ for all i. Thus $f = 0$ and the proof follows. ☐

The above result is due to [GaW], Theorem 1.4. However, our proof is different, being more analytic and explicit. For a tuple of operators T with $\Delta_T = 1$, purity of T is not enough to ensure that T is maximal. An example of a pure tuple T with $\Delta_T = 1$ which is not maximal can be found in [BDS].

Set $\mathcal{D}_\infty := \vee_{n=1}^\infty \mathcal{D}_n$, where $\mathcal{D}_n$s are the defect spaces of T. The multi-variable analogue of the previous theorem is as follows.

Proposition 3.2. *Let T be a pure contractive d-tuple of operators on a Hilbert space $\mathcal{H}$. Then $\mathcal{H} = \mathcal{D}_\infty$.*

Proof. First note that

$$\mathcal{D}_\infty^\perp = \bigcap_{n\geq 1} \mathcal{D}_n^\perp = \bigcap_{n \geq 1} \ker(I - T^n T^{n*}).$$

Therefore, if $x \in \mathcal{D}_\infty^\perp$ then $\|x\| = \|T^{n*}x\|$ for all $n \in \mathbb{N}$. Since T is pure we have $\Psi_T^n(I) \to 0$ in S.O.T. as $n \to \infty$. In particular, $\langle x, \Psi_T^n(I)x \rangle = \|(T^n)^* x\|^2 \to 0$ as $n \to \infty$. We have thus obtained that $x = 0$ concluding the proof. ☐

Corollary 3.3. *Let T be a pure contractive tuple of finite rank acting on an infinite dimensional Hilbert space $\mathcal{H}$. Then $\Delta_T^m \neq \Delta_T^n$ for $m \neq n$.*

Proof. Suppose $\Delta_T^m = \Delta_T^n$ for $m < n$. We know $\mathcal{D}_m \subset \mathcal{D}_n$ and the assumption that they have the same finite dimension implies $\mathcal{D}_m = \mathcal{D}_n$. Then by Corollary 2.4 $\mathcal{D}_k = \mathcal{D}_m$ for all $k \geq m$, and so $\mathcal{D}_\infty = \mathcal{D}_m$ is finite dimensioned, a contradiction. □

Now we provide a characterization of maximal contractive tuples of rank 1.

Theorem 3.4. *Let T be a contractive d-tuple acting on an infinite dimensional Hilbert space $\mathcal{H}$ with $\Delta_T = 1$. Then the following are equivalent:*
(i) *T is maximal.*
(ii) *There is no polynomial P in d non-commuting variables such that*

$$P(T_1, \ldots, T_d)|_{\mathcal{D}_1} = 0.$$

Proof. Since $\Delta_T = 1$, let $\mathcal{D}_1 = \mathbb{C}\xi$ for some $\xi \in \mathcal{H}$. Then by Proposition 2.3

$$\mathcal{D}_n = \text{span}\{T_f \xi : f \in F_{n-1]}\},$$

where T_f is as in (4). Note that $|F(n, \Lambda)| = d^n$. Thus T is maximal if and only if for all $n \in \mathbb{N}$, the set $\{T_f \xi : f \in F_{n-1]}\}$ is linearly independent. Now it is clear that sets of the above type are linearly independent if and only if (ii) holds. This concludes the proof. □

Remarks. (i) In the above theorem if we assume $1 < \Delta_T < \infty$ then the second condition is necessary for T to be maximal. But the following example shows that it is not sufficient. Let $T = (r_1 \oplus S_1, \ldots, r_d \oplus S_d)$ be the d-tuple of operators on $\mathbb{C} \oplus \mathcal{F}_d^2$, where $r_i \in \mathbb{C}$ with $\sum_{i=1}^d |r_i|^2 < 1$ and S_i is the creation operator on $\mathcal{F}_d^2$, the full Fock space over $\mathbb{C}^d$ (defined below), $i = 1, \ldots, d$. It is not difficult to see that T satisfies the second condition of the above theorem and $\Delta_T = 2$ but it is not maximal.

(ii) A commuting contractive d-tuple T on an infinite dimensional Hilbert space $\mathcal{H}$ with $\Delta_T = 1$ is maximal if and only if

$$P(T_1, \ldots, T_d)|_{\mathcal{D}_1} \neq 0,$$

for any polynomial P in d commuting variables.

(iii) Let M_{z_i} be the co-ordinate multiplication operator on the Drury-Arveson module (see [Arv], [Dru] or Section 4) H_d^2 for all $i = 1, \ldots, d$. Then consider the tuple

$$M = (P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} M_{z_d}|_{\mathcal{Q}}),$$

where $\mathcal{Q}$ is a quotient module of H_d^2 given by $\mathcal{Q} = H_d^2 \ominus \theta H_d^2$ and θ is a multiplier. Let $\Delta_M = 1$ and $\mathcal{Q}$ be infinite dimensional. Then note that $\theta f \in \mathcal{Q}^\perp$ for any $f \in H_d^2$ and this implies $\theta(M) = 0$. So by the second remark if θ is a polynomial then M is not maximal. Even if $\theta = p/q$ is a rational function, where p and q are polynomials in d commuting variables and q has poles off $\overline{\mathbb{B}}_d$, the closed unit ball of $\mathbb{C}^d$, then also M is not maximal. This can be seen by establishing the fact that $p(M)|_{\mathcal{D}_1} = 0$.

Corollary 3.5. *Let T be as in the above theorem. Then the following are equivalent:*
(i) $\Delta_T^n = 1 + d + d^2 + \cdots + d^{n-1}$ *for all* $n \leq m$ *and* $\Delta_T^n < 1 + d + \cdots + d^{n-1}$ *for all* $n > m$.
(ii) *There is no non-commuting polynomial P with d variable of degree less than m such that $P(T)|_{\mathcal{D}_1} = 0$ and there is a non-commuting polynomial Q with d variable of degree m such that $Q(T)|_{\mathcal{D}_1} = 0$.*

Proof. By the same argument as in the proof of the above theorem it follows that the dimension of $\mathcal{D}_n$ is maximal for some n if and only if there is no polynomial P of degree smaller than n such that $P(T)|_{\mathcal{D}_1} = 0$. This completes the proof. ☐

For a single contraction acting on a finite dimensional Hilbert space $\mathcal{H}$, we have the following immediate corollary.

Corollary 3.6. *Let T be a single contraction acting on a finite dimensional Hilbert space $\mathcal{H}$ with $\Delta_T = 1$. Then the following are equivalent:*
(i)

$$\Delta_T^n = \begin{cases} n, & n \leq m \leq \dim \mathcal{H} \\ m, & n > m \end{cases}.$$

(ii) *The degree of the minimal polynomial of T is at least m and there is a polynomial P of degree m such that $P(T)|_{\mathcal{D}_1} = 0$.*

Now we recall some of the work of Popescu ([Po_2]) in order to characterize pure maximal tuples of operators. We denote by $\mathcal{F}_d^2$ the full Fock space over the d-dimensional Hilbert space $\mathbb{C}^d$ with orthonormal basis $(e_1, e_2, \ldots, e_d)$. It is often represented by

$$\mathcal{F}_d^2 = \mathbb{C} \oplus_{m \geq 1} (\mathbb{C}^d)^{\otimes m}$$

but we use the notation (3) to give an alternative description as follows. If $f \in F(k, \Lambda)$, let

$$e_f = e_{f(1)} \otimes e_{f(2)} \otimes \cdots \otimes e_{f(k)}, \quad \text{and for } k = 0, \ e_\emptyset = \omega.$$

We call ω the vacuum vector. Then $\mathcal{F}_d^2$ is the Hilbert space with basis $\{e_f : f \in F\}$. For each $n \in \mathbb{N}$, we set

$$\Gamma_{n]} := \text{span}\{e_f : f \in F_{n]}\}.$$

The *Creation operators* on $\mathcal{F}_d^2$ are denoted by $S_i, i = 1, \ldots, d$, and defined by

$$S_i : \mathcal{F}_d^2 \to \mathcal{F}_d^2, \quad \psi \mapsto e_i \otimes \psi, \quad (i = 1, \ldots, d).$$

The complete characterization of invariant subspaces (consequently co-invariant subspaces) for these creation operators on $\mathcal{F}_d^2$ by Popescu (see [Po_1], [Po_2]) is given in the next theorem.

Theorem 3.7 (Popescu)**.** *If $\mathcal{S} \subset \mathcal{F}_d^2$ is invariant for each $S_1, \ldots, S_d$, then there exists a sequence $\{\phi_j\}_{j \in J}$ of orthogonal inner functions such that*

$$\mathcal{S} = \oplus_{j \in J} \mathcal{F}_d^2 \otimes \phi_j.$$

Moreover, this representation is essentially unique.

The model for a pure non-commuting d-tuple of contractions is the compression of the creation operators to a co-invariant subspace (see [Po_1]).

Theorem 3.8 (Popescu). *Let T be a pure non-commuting contractive d-tuple. Then $T \cong \big(P_{\mathcal{Q}}(S_1 \otimes I_{\mathcal{D}_1})|_{\mathcal{Q}}, \dots, P_{\mathcal{Q}}(S_d \otimes I_{\mathcal{D}_1})|_{\mathcal{Q}}\big)$, where $\mathcal{Q}$ is a co-invariant subspace for the creation tuples $(S_1 \otimes I_{\mathcal{D}_1}, \dots, S_d \otimes I_{\mathcal{D}_1})$, $\mathcal{D}_1$ is the first defect space of T and $P_{\mathcal{Q}}$ denotes the projection onto $\mathcal{Q}$.*

The co-invariant subspace appearing in the above theorem is the image of the Poisson kernel $K(T)$ corresponding to the tuple T defined by $K(T) : \mathcal{H} \to \mathcal{F}^2_d \otimes \mathcal{D}_1,\ h \mapsto (\xi_0, \xi_1, \dots)$, where $\xi_0 = \omega \otimes D_T h$ and for $k \geq 1$,

$$\xi_k = \sum_{f \in F(k,\Lambda)} e_f \otimes D_T (T_f)^* h.$$

In this case, $K(T)$ is an isometry. Moreover,

$$K(T)T_i^* = (S_i^* \otimes I_{\mathcal{D}_1})K(T),$$

for all $1 \leq i \leq n$, and

$$K(T)^* : \mathcal{F}^2_d \otimes \mathcal{D}_1 \to \mathcal{H}, \quad e_f \otimes \xi \mapsto T_f D_T \xi. \tag{5}$$

A characterization of maximal non-commuting pure tuples is obtained in the next theorem.

Theorem 3.9. *Let T be a non-commuting pure contractive d-tuple of operators on an infinite dimensional Hilbert space $\mathcal{H}$ with $\Delta_T = 1$. Then the following are equivalent:*
(i) *T is maximal.*
(ii) *There is no polynomial P in d non-commuting variables such that $P(T)|_{\mathcal{D}_1} = 0$.*
(iii) *$T \cong (P_{\mathcal{Q}} S_1|_{\mathcal{Q}}, \dots, P_{\mathcal{Q}} S_d|_{\mathcal{Q}})$, where $\mathcal{Q}$ is the co-invariant subspace of the creation tuple such that $\dim[\operatorname{ran} P_{\mathcal{Q}}|_{\Gamma_{n]}}] = 1 + d + \dots + d^n$ for all $n \in \mathbb{N}$.*
(iv) *For any $n \in \mathbb{N}$, $(\Gamma_{n]} \otimes \mathcal{D}_1) \cap \ker K(T)^* = \{0\}$ where $K(T)^*$ is the adjoint of the Poisson kernel as in (5).*

Proof. (i) $\Leftrightarrow$ (ii) follows from Theorem 3.4.
(i) $\Leftrightarrow$ (iii)
It is follows from Theorem 3.8 that $T \cong (P_{\mathcal{Q}} S_1|_{\mathcal{Q}}, \dots, P_{\mathcal{Q}} S_d|_{\mathcal{Q}})$ where $\mathcal{Q}$ is a co-invariant subspace for the creation tuple. Thus it is enough to show that the tuple $(P_{\mathcal{Q}} S_1|_{\mathcal{Q}}, \dots, P_{\mathcal{Q}} S_d|_{\mathcal{Q}})$ is maximal if and only if $\dim[\operatorname{ran}(P_{\mathcal{Q}}|_{\Gamma_{n]}})] = 1 + d + \dots + d^n$ for all n. Now we calculate the defect spaces of the tuple as follows:

$$\mathcal{D}_1 = \operatorname{ran}(P_{\mathcal{Q}} - \sum_{i=1}^{d} P_{\mathcal{Q}} S_i S_i^*|_{\mathcal{Q}}) = P_{\mathcal{Q}} P_{\mathbb{C}\omega} P_{\mathcal{Q}}.$$

Since the the first defect dimension is one it follows that $\mathcal{D}_1 = \operatorname{span}\{\xi := P_{\mathcal{Q}}(\omega)\}$. Note that $P_{\mathcal{Q}} S_i P_{\mathcal{Q}}(\xi) = P_{\mathcal{Q}} S_i (I - P_{\mathcal{Q}^\perp})(\omega) = P_{\mathcal{Q}}(e_i)$ as $\mathcal{Q}^\perp$ is an invariant subspace for each S_i, $i = 1, \dots, d$. By an induction argument one can show that for any $k \in \mathbb{N}$ and $f \in F(k, \Lambda)$, $P_q S_{f(1)} P_{\mathcal{Q}} \dots P_{\mathcal{Q}} S_{f(k)} \xi = P_{\mathcal{Q}}(e_f)$. Then by Proposition 2.3,

$$\mathcal{D}_n = \operatorname{span}\{P_{\mathcal{Q}}(e_f) : f \in F_{n-1]}\}$$

for all n. Therefore $\mathcal{D}_n = \operatorname{ran}(P_{\mathcal{Q}}|_{\Gamma_{n-1]}})$ and the claim follows.
(ii) $\Leftrightarrow$ (iv)

Since the first defect space is one dimensional then $\mathcal{D}_1 = \mathbb{C}\xi$ for some $\xi \in \mathcal{H}$. Now as $\operatorname{ran} D_T = \operatorname{ran} D_T^2$ we have $D_T\xi = \lambda\xi$ for some non-zero scalar λ. By definition of $K(T)^*$ it follows that $\sum_{f\in F_{k]}} a_f e_f \otimes \xi \in \ker K(T)^*$ if and only if $P(T)(\xi) = 0$, where $P = \sum_{f\in F_{k]}} \lambda a_f Z_f$, $Z_f = z_{f(1)} \dots z_{f(k)}$ and $k \in \mathbb{N}$, proving the theorem.

□

Remarks. (i) The last equivalent condition in the above theorem is independent of the assumption $\Delta_T = 1$. More precisely, a pure contractive d-tuple T with finite rank is maximal if and only if (iv) holds. To see this, let $\Delta_T = n$ and $\mathcal{D}_1 = \operatorname{span}\{\phi_1, \dots, \phi_n\}$. Set $\psi_i := D_T(\phi_i)$, $i = 1, \dots n$. Now as $\operatorname{ran} D_T^2 = \mathcal{D}_1$ and $\operatorname{ran} D_T = \operatorname{span}\{\phi_1, \dots, \phi_n\}$ we also have $\mathcal{D}_1 = \operatorname{span}\{\psi_1, \dots \psi_n\}$. Thus T is maximal if and only if the set $\{T_f\psi_i : f \in F_{k]}, i = 1, \dots, n\}$ is linearly independent for all $k \in \mathbb{N}$. Then the claim readily follows from the following equivalent conditions:

$$\sum_{i=1,\dots,n, f\in F_{k]}} a_{f,i} e_f \otimes \phi_i \in \ker K(T)^* \Leftrightarrow \sum_{i=1,\dots,n, f\in F_{k]}} a_{f,i} T_f \psi_i = 0,$$

for all $k \in \mathbb{N}$.
(ii) The condition (iii) in the above theorem can be made independent of the assumption $\Delta_T = 1$ as follows. If $\Delta_T = k$ then $T \cong (P_{\mathcal{Q}}(S_1 \otimes I_{\mathbb{C}^k})|_{\mathcal{Q}}, \dots, P_{\mathcal{Q}}(S_d \otimes I_{\mathbb{C}^k})|_{\mathcal{Q}})$ where $\mathcal{Q}$ is a joint co-invariant subspace for the ampliated creation tuple. Then the equivalence condition of maximality in this case is the following:

$$\dim[\operatorname{ran} P_{\mathcal{Q}}|_{\Gamma_{n]}\otimes\mathbb{C}^k}] = (1 + d + \dots + d^n)k$$

for all $n \in \mathbb{N}$.

By Proposition 3.2, we know that if T is a single pure contraction T on a Hilbert space $\mathcal{H}$ then $\mathcal{D}_\infty = \mathcal{H}$. So for every polynomial p such that $p(T)|_{\mathcal{D}_1} = 0$, $0 = T^n p(T)\xi = p(T)T^n\xi$ for $\xi \in \mathcal{D}_1$ and $n \in \mathbb{N}$. Now as $\mathcal{D}_\infty = \overline{\operatorname{span}}\{T^n\xi : n \in \mathbb{N}, \xi \in \mathcal{D}_1\}$ we have $p(T) = 0$. Thus for a single pure contraction T,

$$p(T)|_{\mathcal{D}_1} = 0 \Leftrightarrow p(T) = 0$$

for any polynomial p. This observation helps us to find a connection with minimal function as follows. We denote by $H^\infty(\mathbb{D})$ the multiplier algebra of the Hardy space $H^2(\mathbb{D})$ over the unit disc.

Theorem 3.10. *Let T be a single pure contraction on a Hilbert space $\mathcal{H}$ with $\Delta_T = 1$.*
(a) *If $\mathcal{H}$ is infinite dimensional and there is a non-zero function $m \in H^\infty(\mathbb{D})$ such that $m(T) = 0$ then m cannot be a polynomial.*
(b) *If $\mathcal{H}$ is finite dimensional then the degree of the minimal polynomial for T is $\dim \mathcal{H}$.*

Proof. Part (a) follows from the above discussion and Theorem 3.4 and the fact that T is maximal. For part (b) note that the maximality of the operator T implies the defect spaces in this case are as follows:

$$\Delta_T^n = \begin{cases} n, & n \le \dim \mathcal{H} \\ \dim \mathcal{H}, & n > \dim \mathcal{H} \end{cases}.$$

Then by the Corollary 3.6 we have the degree of the minimal polynomial is at least $\dim \mathcal{H}$ and this completes the proof. ☐

Remark. It is well known that any single pure contraction T on a Hilbert space $\mathcal{H}$ with $\Delta_T = 1$ is unitarily equivalent to $P_{H_\theta} M_z|_{H_\theta}$, where $H_\theta = H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$ is a co-invariant subspace for the co-ordinate multiplication operator M_z and θ is an inner function. In this case the minimal function of T is θ (see [SzF], Chapter 3, Proposition 4.3). Then the above theorem tells us that if $\mathcal{H}$ is infinite dimensional then θ cannot be polynomial and if θ is a polynomial then the dimension of $\mathcal{H}$ is indeed the same as the degree of θ.

4. Maximal submodules of H^2_d

This section concerns the maximality of submodules of the Drury-Arveson module ([Dru], [Arv]). We denote by H^2_d the Drury-Arveson module over the unit ball $\mathbb{B}_d$ and defined by the reproducing kernel $K_\lambda(z) = \frac{1}{1-<\lambda,z>}$, where $< \lambda, z >= \sum_{j=1}^d z_j \overline{\lambda_j}$ and $\lambda, z \in \mathbb{B}_d$. For $d = 1$, $H^2_1 = H^2(\mathbb{D})$ the Hardy space over the unit disc. The multiplication operators M_{z_i} by the co-ordinate functions z_i, $i = 1, \ldots, d$, turns H^2_d to a Hilbert module over $\mathbb{C}[\boldsymbol{z}] := \mathbb{C}[z_1, \ldots, z_d]$ as follows:

$$\mathbb{C}[\boldsymbol{z}] \times H^2_d \to H^2_d,\ (p, h) \mapsto p(M_{z_1}, \ldots, M_{z_d})h.$$

A closed subspace $\mathcal{S}$ of H^2_d is said to be *submodule* of H^2_d if $M_{z_i}\mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \ldots, d$. Let $\mathcal{S}$ be a submodule of H^2_d and let $R_{\mathcal{S}} := (M_{z_1}|_{\mathcal{S}}, \cdots, M_{z_d}|_{\mathcal{S}})$ be the restriction of the d-shift to $\mathcal{S}$. It readily follows that the d-tuple $R_{\mathcal{S}}$ is contractive. A submodule $\mathcal{S}$ of H^2_d is *maximal* if $R_{\mathcal{S}}$ is of finite rank and is maximal. For $d = 1$, let $\mathcal{S} \subset H^2_d$ be a submodule of the Hardy space over the unit disc. Then $R_{\mathcal{S}} := M_z|_{\mathcal{S}}$ is a pure isometry ($R_{\mathcal{S}}^{*n} \to 0$ in S.O.T. as $n \to \infty$) with multiplicity one. In other words, that $R_{\mathcal{S}} \cong M_z$. Consequently we have the following result.

Theorem 4.1. *Any submodule $\mathcal{S}$ of $H^2(\mathbb{D})$ is maximal.*

For $d \geq 2$ the above theorem does not hold in general as we show next. For the *rest of the section* we assume $d \geq 2$.

Before proceeding, we recall a result concerning the defect space and the multipliers of submodules of the Drury-Arveson module (for details see [Arv], [GRS], [McT]). First note that the defect operator $D_{R_{\mathcal{S}}}$ and the defect index of the tuple $R_{\mathcal{S}}$ are given by

$$D_{R_{\mathcal{S}}} = (P_{\mathcal{S}} - \sum_{i=1}^d M_{z_i} P_{\mathcal{S}} M_{z_i}^*)^{1/2},$$

and

$$\Delta_{R_{\mathcal{S}}} = \dim[\overline{\operatorname{ran}} D_{R_{\mathcal{S}}}],$$

where $P_{\mathcal{S}}$ is the orthogonal projection in $B(H^2_d)$ with range $\mathcal{S}$.

Theorem 4.2. *Let $\mathcal{S}$ be a submodule of H^2_d with $\Delta_{R_{\mathcal{S}}} = n$. Then there exists $\phi_i \in \operatorname{ran} D_{R_{\mathcal{S}}}, i = 1, \ldots, n$, such that each ϕ_i is a multiplier,*

$$P_{\mathcal{S}} = \sum_{i=1}^n M_{\phi_i} M_{\phi_i}^*,$$

and the submodule $\mathcal{S}$ is generated by $\{\phi_i\}_{i=1}^n$.

By the above theorem one can describe all the defect spaces of $R_{\mathcal{S}}$ in terms of the generators of $\mathcal{S}$ as follows. Let $\Delta_{R_{\mathcal{S}}} = n$ and $\phi_i, i = 1, \dots, n$, as above such that $P_{\mathcal{S}} = \sum_i M_{\phi_i} M_{\phi_i}^*$. Then

$$\begin{aligned} D_{R_{\mathcal{S}}}^2 &= P_{\mathcal{S}} - \sum_{i=1}^d M_{z_i} P_{\mathcal{S}} M_{z_i}^* \\ &= \sum_{k=1}^n M_{\phi_k} (I_{H_d^2} - \sum_{i=1}^d M_{z_i} M_{z_i}^*) M_{\phi_k}^* \\ &= \sum_{k=1}^n M_{\phi_k} |1\rangle\langle 1| M_{\phi_k}^* \\ &= \sum_{k=1}^n |\phi_k\rangle\langle\phi_k|, \end{aligned}$$

where $|f\rangle\langle g|$ denote the rank one operator that takes h to $\langle g, h\rangle f$ for all $f, g, h \in H_d^2$. Thus

$$\mathcal{D}_1 = \text{span}\{\phi_i : i = 1, \dots, n\},$$

and by Proposition 2.3,

$$\mathcal{D}_m = \text{span}\{z_1^{j_1} \cdots z_d^{j_d} \phi_i : i = 1, \dots n \ \text{ and } \sum_{t=1}^d j_t = m - 1\},$$

for all $m \in \mathbb{N}$.

Now we investigate the question of maximality of a homogeneous submodule $\mathcal{S}$ when $\Delta_{R_{\mathcal{S}}}$ is finite.

Theorem 4.3. *Suppose $\mathcal{S}$ is a proper homogeneous submodule of H_d^2 with $\Delta_{R_{\mathcal{S}}} < \infty$. Then $\mathcal{S}$ is not maximal.*

Proof. Let $\Delta_{R_{\mathcal{S}}} = n$. By assumption $\mathcal{S}$ is proper so $n \geq 2$. Note that a submodule is homogeneous if and only if it is generated by homogeneous polynomials. Consequently, there exists an orthonormal basis of $\mathcal{S}$ consisting of homogeneous polynomials, and hence there exists polynomials p_i, $i = 1, \dots, n$, such that $\mathcal{D}_1 = \text{span}\{p_i : i = 1, \dots, n\}$. For the contradiction suppose $\mathcal{S}$ is maximal then by maximality of the tuple $R_{\mathcal{S}}$, the set $\{z_1^{j_1} \cdots z_d^{j_d} p_i : i = 1, \dots, n \ \text{ and } j_1, \dots, j_d \in \mathbb{N}\}$ is linearly independent. However since p_ks are polynomials and $n \geq 2$, the set cannot be linearly independent. This concludes the proof. □

Corollary 4.4. *Let $\mathcal{S}$ be a submodule of H_d^2 with $\Delta_{R_{\mathcal{S}}} = n$ and $P_{\mathcal{S}} = \sum_{i=1}^n M_{p_i} M_{p_i}^*$ for non-constant polynomials p_is. Then $\mathcal{S}$ is not maximal.*

Proof. Since $P_{\mathcal{S}} = \sum_{i=1}^n M_{p_i} M_{p_i}^*$, the first defect space $\mathcal{D}_1 = \text{span}\{p_i : 1 \leq i \leq n\}$. Now the argument used to prove the previous theorem can be adapted to show that $\mathcal{S}$ is not maximal. □

Since $R_{\mathcal{S}}$ is a pure contractive d-tuple for a submodule $\mathcal{S}$, the adjoint of the Poisson kernel $K(R_{\mathcal{S}})$ in this case is a unique bounded linear operator $K(R_{\mathcal{S}})^* : H^2_d \otimes \mathcal{D}_1 \to H^2_d$ defined by taking linear and continuous extension of the following prescription:

$$p \otimes \xi \mapsto pD_{R_{\mathcal{S}}}\xi, \quad (p \in \mathbb{C}[\boldsymbol{z}], \xi \in \mathcal{D}_1).$$

The range of this map is precisely $\mathcal{S}$. A characterization for maximal submodules in terms of this operator is given next.

Theorem 4.5. *Let $\mathcal{S}$ be a submodule of H^2_d and $\Delta_{R_{\mathcal{S}}} < \infty$. Then the following are equivalent:*
(i) *$\mathcal{S}$ is maximal.*
(ii) $(\mathbb{C}[\boldsymbol{z}] \otimes \mathcal{D}_1) \cap \ker K(R_{\mathcal{S}})^* = \{0\}$, *where the operator $K(R_{\mathcal{S}})^*$ is as above.*

Proof. The proof follows from a slight modification of the argument given in the first remark after Theorem 3.9 as the tuple $R_{\mathcal{S}}$ in this case is a commuting tuple. □

In [Arv], Arveson introduced the notion of maximality for submodules of H^2_d $(d \geq 2)$ in terms of the energy sequence of the operator space generated by the restriction of the multiplication operators to the corresponding submodule. In his context H^2_d is the only maximal submodule of H^2_d, which is maximal in the present context as well. In the present consideration, we do not know any example of proper maximal submodule of H^2_d. We believe that any proper submodule of Drury-Arveson module is not maximal, but we do not have a proof as of yet.

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