SIMILARITY OF QUOTIENT HILBERT MODULES IN THE COWEN-DOUGLAS CLASS

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Dedicated to the memory of Ron Douglas

Abstract. In this paper, we consider the similarity and quasi-affinity problems for Hilbert modules in the Cowen-Douglas class associated with the complex geometric objects, the hermitian anti-holomorphic vector bundles and curvatures. Given a "simple" rank one Cowen-Douglas Hilbert module $\mathcal{M}$, we find necessary and sufficient conditions for a class of Cowen-Douglas Hilbert modules satisfying some positivity conditions to be similar to $\mathcal{M} \otimes \mathbb{C}^m$. We also show that under certain uniform bound condition on the anti-holomorphic frame, a Cowen-Douglas Hilbert module is quasi-affinity to a submodule of the free module $\mathcal{M} \otimes \mathbb{C}^m$.

1. Introduction

One of the most challenging problems in operator theory is to determine when two given bounded linear operators are similar. More precisely, let T and R be two bounded linear operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. When does there exists an invertible bounded linear map $X : \mathcal{H} \to \mathcal{K}$ such that $XT = RX$?

There are many fascinating subtleties connected with the similarity problem (see [13], [18], [19], [22], [23]). However, the problem becomes more tractable if one impose additional assumptions on the operators. In particular, there are several characterizations of operators similar to unitaries or isometries or even contractions.

In [24], Uchiyama proposed a characterization for contractions in the Cowen-Douglas class which are similar to the adjoint of the multiplication operators on the Hardy space with finite multiplicity. One of the main tools used in the work by Uchiyama is the structure of the tensor product bundle corresponding to a given hermitian holomorphic vector bundle. Later, Kwon and Treil [16] found some additional characterizations which involves the curvature, in the sense of Cowen-Douglas, and the Carleson measure [5] of the underlying operators (see also Douglas, Kwon and Treil [8] in the setting of n-hypercontractions [1]).

In the present study, we set up the similarity problem in a more general framework and provide some characterizations of operators in the Cowen-Douglas class which are similar to the adjoint of the multiplication operators on standard reproducing kernel Hilbert spaces of holomorphic functions (like weighted Bergman spaces). We prove that the earlier characterizations of operators similar to the adjoint of multiplication operators are valid beyond the

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class of contractions and n-hypercontractions. In particular, our results includes the similarity problem for the weighted Bergman spaces with not necessarily integer weights. Our framework is based on the $\frac{1}{K}$-calculus introduces by Arazy and Englis [4]. The similarity results of this paper are significantly more general than those obtained in [8], [16] and [24]. Moreover, the study of weighted Bergman spaces with not necessarily integer weights appears to be more fruitful and rewarding from analytic, geometric and representation theoretic point of views (for instance, see [14] and [17]).

We now summarize the content of this paper. In Section 2, we give a self-contained presentation of the theory of Cowen-Douglas Hilbert modules in the language of reproducing kernel Hilbert modules. The next section is devoted to assembling materials, like a projection formula, derivatives of holomorphic maps and curvature, which we will use in the sequel, from various sources. Here, however, our approach will be completely new. In Section 4, we develop the notion of Cowen-Douglas atoms. In Section 5, we obtain results concerning tensor product bundles and quotient modules. Section 6 deals with the quasi-affinity properties of Cowen-Douglas Hilbert modules. In Section 7, we relate the curvatures to the derivatives of holomorphic maps and obtain some similarity classification results. The final section lists some open problems.

2. Cowen-Douglas Hilbert Modules

In this section we introduce the basic concepts and known results related to the Cowen-Douglas class [6].

Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ is a module over $\mathbb{C}[z]$ in the following sense:

$$p \cdot f \mapsto p(T)f \qquad (p \in \mathbb{C}[z], f \in \mathcal{H}).$$

The above module is usually called the *Hilbert module* over $\mathbb{C}[z]$ (see [10]).

Note that a Hilbert module $\mathcal{H}$ over $\mathbb{C}[z]$ is uniquely determined by the underlying operator T via the module multiplication operator by the coordinate function z:

$$M_z f := z \cdot f = Tf, \qquad (f \in \mathcal{H}),$$

and vice versa. We say that $\mathcal{H}$ is *contractive* if

$$I_{\mathcal{H}} - M_z M_z^* \geq 0.$$

We denote the space of all bounded linear operators from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$ by $\mathcal{B}(\mathcal{H}, \mathcal{K})$, and by $\mathcal{B}(\mathcal{H})$ if $\mathcal{K} = \mathcal{H}$.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert modules over $\mathbb{C}[z]$. Then $M \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be a module map if $M(p \cdot f) = p \cdot (Mf)$ for all $p \in \mathbb{C}[z]$ and $f \in \mathcal{H}_1$.

Now we recall the definition of the Cowen-Douglas class [6].

Definition 2.1. *Let m be a positive integer, and let Ω be a domain in $\mathbb{C}$. The Cowen-Douglas class on Ω of rank m, denoted by $B_m(\Omega)$ is the set of all Hilbert modules $\mathcal{H}$ over $\mathbb{C}[z]$ such that*

(i) $\sigma(M_z^*) \subseteq \Omega$,

(ii) $ran(M_z - wI_{\mathcal{H}})^* = \mathcal{H}$ for all $w \in \Omega$,

(iii) dim ker$(M_z - wI_{\mathcal{H}})^* = m$ *for all* $w \in \Omega$, *and*
(iv) $\overline{\mathit{span}}\{\mathit{ker}(M_z - wI_{\mathcal{H}})^* : w \in \Omega\} = \mathcal{H}$.

A Hilbert module $\mathcal{H}$ is said to be a *Cowen-Douglas Hilbert module* if $\mathcal{H} \in B_m(\Omega)$ for some positive integer m.

A Cowen-Douglas Hilbert module $\mathcal{H}$ in $B_m(\Omega)$ defines a hermitian anti-holomorphic vector bundle $E_{\mathcal{H}}$ over Ω where

$$E_{\mathcal{H}} = \{(w, h) \in \Omega \times \mathcal{H} : M_z^* h = \bar{w} h\},$$

with the projection map $\pi_{\mathcal{H}} : E_{\mathcal{H}} \to \Omega$ defined by $\pi_{\mathcal{H}}(w, h) = h$ for all $w \in \Omega$ and $h \in \mathcal{H}$. More precisely, $E_{\mathcal{H}}$ is the anti-holomorphic vector bundle implemented by the anti-holomorphic map $w \mapsto E_{\mathcal{H}}(w) := \ker\,(M_z - wI_{\mathcal{H}})^*$, the pull-back bundle of the Grassmannian $GF(m, \mathcal{H})$ (see [6]) and hence locally at each point $w \in \Omega$, there exists anti-holomorphic $\mathcal{H}$-valued functions $\{\gamma_{i,w} : 1 \leq i \leq m\}$ such that

$$\text{span}\{\gamma_{i,w} : 1 \leq i \leq m\} = \ker(M_z - w)^*.$$

Also it follows from a theorem of Grauert [11] that $\gamma_{i,w}$ can be defined on all of Ω.

The rigidity theorem (Theorem 2.2 in [6]) states that a pair of Hilbert modules $\mathcal{H}$ and $\tilde{\mathcal{H}}$ in $B_m(\Omega)$ are unitarily equivalent if and only if the corresponding hermitian anti-holomorphic vector bundles $E_{\mathcal{H}}$ and $E_{\tilde{\mathcal{H}}}$ are equivalent.

We now briefly recall the notion of a reproducing kernel Hilbert module, which in turn is closely related to the Cowen-Douglas Hilbert modules. Let $\mathcal{E}$ be a Hilbert space. A Hilbert module $\mathcal{H} \subseteq \mathcal{O}(\Omega, \mathcal{E})$, where $\mathcal{O}(\Omega, \mathcal{E})$ is the space of $\mathcal{E}$-valued holomorphic functions on Ω, is said to be a *reproducing kernel Hilbert module* if

(i) the evaluation map $ev_w : \mathcal{H} \to \mathcal{E}$, $w \in \Omega$, defined by $ev_w(f) = f(w)$, $f \in \mathcal{H}$, is bounded, and

(ii) the module multiplication operator M_z is given by the multiplication operator by the coordinate function z.

Let $\mathcal{H} \in B_m(\Omega)$ with an anti-holomorphic frame $\{\gamma_{i,w} : 1 \leq i \leq m\}$ of $E_{\mathcal{H}}$, and let $G(z, w)$ be the corresponding Gram matrix

$$G(z, w) = (\langle \gamma_{j,w}, \gamma_{i,z} \rangle_{\mathcal{H}})_{m \times m}, \qquad (z, w \in \Omega).$$

Define $K : \Omega \times \Omega \to \mathcal{M}_m(\mathbb{C})$ by

$$K(z, w) = G(z, w) \qquad (z, w \in \Omega).$$

Then K is a positive definite kernel, and the corresponding reproducing kernel Hilbert space $\mathcal{H}_K \subseteq \mathcal{O}(\Omega, \mathbb{C}^m)$ is a reproducing kernel Hilbert module. Define $X : \mathcal{H} \to \mathcal{H}_K$ by

$$(Xh)(w) = (\langle h, \gamma_{1,w} \rangle_{\mathcal{H}}, \ldots, \langle h, \gamma_{m,w} \rangle_{\mathcal{H}}) \in \mathbb{C}^m \qquad (w \in \Omega).$$

It follows that X is a unitary and

$$XM_z = M_z X.$$

Also by the definition of K we have

$$K(\cdot, w)\eta = (\langle \gamma_{j,w}, \gamma_{i,z} \rangle_{\mathcal{H}})\eta,$$

and

$$\langle f(w), \eta\rangle_{\mathbb{C}^m} = \langle f, K(\cdot, w)\eta\rangle_{\mathcal{H}_K},$$

for all $w \in \Omega, \eta \in \mathbb{C}^m$, and $f \in \mathcal{H}_K$. Moreover, the evaluation operator $ev_w : \mathcal{H}_K \to \mathbb{C}^m$, $w \in \Omega$, satisfies

$$\langle ev_w^* \eta, f\rangle_{\mathcal{H}_K} = \langle \eta, f(w)\rangle_{\mathbb{C}^m} = \langle K(\cdot, w)\eta, f\rangle_{\mathcal{H}_K},$$

for $\eta \in \mathbb{C}^m$ and $f \in \mathcal{H}_K$. In particular,

$$ev_w^* \eta = K(\cdot, w)\eta,$$

and hence

$$K(z, w) = ev_z \circ ev_w^*,$$

for all $z, w \in \Omega, \eta \in \mathbb{C}^m$. Therefore, we have prove the following (see [2], [7]):

THEOREM 2.2. *Let $\mathcal{H} \in B_m(\Omega)$, and let $\{\gamma_{i,w}\}_{i=1}^m$ be an anti-holomorphic frame of $\mathcal{H}$. If*

$$K(z, w) = (\langle \gamma_{j,w}, \gamma_{i,z}\rangle_{\mathcal{H}})_{m\times m} \qquad (z, w \in \Omega),$$

then $\mathcal{H}_K$ is a reproducing kernel Hilbert module. Moreover, M_z on $\mathcal{H}$ and M_z on $\mathcal{H}_K$ are unitarily equivalent.

By virtue of the above theorem, from now on, we will often use the representation $\mathcal{H}_K$ of $\mathcal{H}$ in $B_m(\Omega)$.

3. Derivatives of Holomorphic Maps and Curvatures

The purpose of this section is to study the curvatures, a projection formula for eigenspace bundles and a trace-curvature formula in terms of Hilbert Schmidth norm of derivatives of eigenspace bundles. Most of the results of this section are known. However, our method of proofs are more geometric and explicit.

Let $\mathcal{H} \in B_m(\Omega)$, and let $\{\gamma_{i,w} : 1 \le i \le m\}$ be an anti-holomorphic frame of $E_{\mathcal{H}}$. The curvature matrix of $E_{\mathcal{H}}$ is given by

$$\mathcal{K}_{\mathcal{H}}(w) = -\overline{\partial}[G^{-1}(\bar{w})\partial G(\bar{w})] \qquad (w \in \Omega),$$

where G is the Gram matrix given by (see Section 2)

$$G(w) = (\langle \gamma_{j,w}, \gamma_{i,w}\rangle_{\mathcal{H}})_{i,j=1}^m = K(w, w) \qquad (w \in \Omega).$$

If $E_{\mathcal{H}}$ is a line bundle then it follows that

$$\mathcal{K}_{\mathcal{H}}(w) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log \|\gamma_w\|^2 \qquad (w \in \Omega).$$

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let $T : \mathcal{H} \to \mathcal{K}$ be a left invertible operator. Then

$$L = (T^*T)^{-1}T^*,$$

is a left inverse of T, and hence

$$Q = TL,$$

is an orthogonal projection of $\mathcal{K}$ onto ran T, that is,

$$Q = P_{\text{ran } T}.$$

Now let $\mathcal{H} \in B_m(\Omega)$. Then applying the above observations to $\Gamma(w) = ev_w^* : \mathbb{C}^m \to \mathcal{H}_K$, $w \in \Omega$, we have the following useful result.

THEOREM 3.1. *(Projection Formula) Let $\mathcal{H} \in B_m(\Omega)$, $\Gamma(w) = ev_w^*$, and let*

$$G(w) = \Gamma(w)^*\Gamma(w),$$

for all $w \in \Omega$. Then

$$P_{ker(M_z - w)^*} = \Gamma(w)G(w)^{-1}\Gamma(w)^*,$$

and

$$ker(M_z - w)^* = \ ran\ ev_w^*.$$

For simplicity of notation, we denote

$$\Pi_{\mathcal{H}}(w) = \Pi(w) = P_{\ker\ (M_z - w)^*} = \Gamma(w)G(w)^{-1}\Gamma(w)^* \qquad (w \in \Omega).$$

The above theorem should be compared with the results of Curto and Salinas (see, for example, Theorem 2.2 in [7]).

The following result provides a useful relation between curvature and the derivatives of the orthogonal projection-valued map Π.
We first define an infinitely differentiable function $\overline{\Pi}$ as follows

$$\overline{\Pi}(w) = \Pi(\bar{w}) \qquad (w \in \Omega).$$

Then

$$\overline{\Pi}(w) = P_{\ker\ (M_z^* - w)} = \Gamma(\bar{w})G(\bar{w})^{-1}\Gamma^*(\bar{w}) \qquad (w \in \Omega)$$

Also define $\overline{\Gamma}$ as

$$\overline{\Gamma}(w) = \Gamma(\bar{w}) \qquad (w \in \Omega).$$

Then we have:

THEOREM 3.2. *If $\mathcal{H} \in B_m(\Omega)$, then*

$$\bar{\partial}\overline{\Pi}(w)\partial\overline{\Pi}(w) = -\overline{\Gamma}(w)[\mathcal{K}_{\mathcal{H}}(w)G^{-1}(\bar{w})]\overline{\Gamma}^*(w) \qquad (w \in \Omega).$$

Proof. Note that

$$\partial\overline{\Pi}(w) = [(\partial\overline{\Gamma}(w))G^{-1}(\bar{w}) + \overline{\Gamma}(w)(\partial G^{-1})(\bar{w})]\overline{\Gamma}^*(w),$$

and so

$$\begin{aligned}
(\bar{\partial}\overline{\Pi})(w)(\partial\overline{\Pi})(w) &= \overline{\Gamma}(w)[(\bar{\partial}G^{-1}(\bar{w}))\overline{\Gamma}^*(w) + G^{-1}(\bar{w})(\bar{\partial}\overline{\Gamma}^*)(w)][(\partial\overline{\Gamma}(w))G^{-1}(\bar{w}) + \\
&\quad \overline{\Gamma}(w)(\partial G^{-1})(\bar{w})]\bar{\Gamma}^*(w) \\
&= \overline{\Gamma}(w)[(\bar{\partial}G^{-1}(\bar{w}))\overline{\Gamma}^*(w)(\partial\overline{\Gamma}(w))G^{-1}(\bar{w}) + (\bar{\partial}G^{-1}(\bar{w}))\overline{\Gamma}^*(w)\overline{\Gamma}(w)(\partial G^{-1}(\bar{w})) + \\
&\quad G^{-1}(\bar{w})(\bar{\partial}\overline{\Gamma}^*)(\partial\overline{\Gamma}(w))G^{-1}(\bar{w}) + G^{-1}(\bar{w})(\bar{\partial}\overline{\Gamma}^*(w))\overline{\Gamma}(w)(\partial G^{-1})(\bar{w})]\overline{\Gamma}^*(w) \\
&= \overline{\Gamma}(w)[(\bar{\partial}G^{-1}(\bar{w}))(\partial G(\bar{w}))G^{-1}(\bar{w}) + (\bar{\partial}G^{-1}(\bar{w}))G(\bar{w})(\partial G^{-1}(\bar{w})) \\
&\quad + G^{-1}(\bar{w})(\partial\bar{\partial}G(\bar{w}))G^{-1}(\bar{w}) + G^{-1}(\bar{w})(\bar{\partial}G(\bar{w}))(\partial G^{-1})(\bar{w})]\Gamma^*(\bar{w}) \\
&= \overline{\Gamma}(w)\{[(\bar{\partial}G^{-1})(\bar{w})(\partial G)(\bar{w}) + G^{-1}(\bar{w})(\partial\bar{\partial}G)(\bar{w})]G^{-1}(\bar{w}) + \\
&\quad [(\bar{\partial}G^{-1})(\bar{w})G(\bar{w}) + G^{-1}(\bar{w})(\bar{\partial}G)(\bar{w})](\partial G^{-1})(\bar{w})\}\overline{\Gamma}^*(w).
\end{aligned}$$

Then

$$(\bar{\partial}\overline{\Pi})(w)(\partial\overline{\Pi})(w) = \overline{\Gamma}(w)[(\bar{\partial}G^{-1})(\partial G) + G^{-1}(\partial\bar{\partial}G)](\bar{w})G^{-1}(\bar{w})\overline{\Gamma}^*(w),$$

because

$$(\bar{\partial}G^{-1})G + G^{-1}(\bar{\partial}G) = \bar{\partial}(G^{-1}G) = \bar{\partial}(I) = 0.$$

Therefore

$$\begin{aligned}(\bar{\partial}\overline{\Pi})(w)(\partial\overline{\Pi})(w) &= \overline{\Gamma}(w)[(\bar{\partial}G^{-1})(\partial G) + G^{-1}(\partial\bar{\partial}G)](\bar{w})G^{-1}(\bar{w})\overline{\Gamma}^*(w)\\ &= \overline{\Gamma}(w)[\bar{\partial}(G^{-1}(\bar{w})\partial G(\bar{w}))]G^{-1}(\bar{w})\overline{\Gamma}^*(w)\\ &= -\overline{\Gamma}(w)[\mathcal{K}_{\mathcal{H}}(w)G^{-1}(\bar{w})]\overline{\Gamma}^*(w).\end{aligned}$$

This completes the proof. ■

As a consequence, we have the following equality:

Corollary 3.3. *Let $\mathcal{H} \in B_m(\Omega)$. Then*

$$\|\partial\overline{\Pi}(w)\|_2^2 = -\mathit{trace}\ \mathcal{K}_{\mathcal{H}}(w),$$

where $\|\partial\overline{\Pi}(w)\|_2^2$ is the Hilbert-Schmidt norm of $\partial\overline{\Pi}(w)$ and $w \in \Omega$.

Proof. Clearly $\mathcal{K}_{\mathcal{H}}(w)$, $w \in \Omega$, is a finite rank operator on $\mathcal{H}$. Hence, in particular, $\mathcal{K}_{\mathcal{H}}(w)$, $w \in \Omega$, is in trace class. From this, it follows that

$$\begin{aligned}\text{trace }(\overline{\Gamma}(w)[\mathcal{K}_{\mathcal{H}}(w)G^{-1}(\bar{w})]\overline{\Gamma}^*(w)) &= \text{trace }([\mathcal{K}_{\mathcal{H}}(w)G^{-1}(\bar{w})]\overline{\Gamma}^*(w)\overline{\Gamma}(w))\\ &= \text{trace }\mathcal{K}_{\mathcal{H}}(w).\end{aligned}$$

Furthermore, Theorem 3.2 shows that

$$\|\partial\overline{\Pi}(w)\|_2^2 = \text{trace }((\bar{\partial}\overline{\Pi})(\partial\overline{\Pi})) = -\text{trace }((\overline{\Gamma}(w)[\mathcal{K}_{\mathcal{H}}(w)G^{-1}(\bar{w})]\overline{\Gamma}^*(w))).$$

Thus

$$\|\partial\overline{\Pi}(w)\|_2^2 = -\text{trace }\mathcal{K}_{\mathcal{H}}(w),$$

for all $w \in \Omega$. This completes the proof. ■

The above result is due to the first author and Hou and Kwon [12] (for the Hardy and the weighted Bergman spaces, see Lemma 1.7 in [16] and Lemma 3.3 in [8], respectively). However, the present proof is more geometric and simple.

4. Cowen-Douglas Atoms

In this section, we introduce the concept of a Cowen-Douglas atom, a special but large class of rank one Cowen-Douglas Hilbert modules over $\mathbb{D}$. Moreover, a Cowen-Douglas atom admits $\frac{1}{K}$-calculus in the sense of Arazy and Englis [4]. Our presentation of $\frac{1}{K}$-calculus is restricted to one variable. For more details, we refer the readers to the work by Arazy and Englis [4].

Definition 4.1. *A Hilbert module $\mathcal{M} \in B_1(\mathbb{D})$ is said to be a Cowen-Douglas atom if*

(i) the set of polynomials $\mathbb{C}[z]$ is dense in $\mathcal{M}$,

(ii) there exists a sequence of polynomials $\{p_l(z,\bar{w})\}_l \subseteq \mathbb{C}[z,\bar{w}]$ *such that*

$$p_l(z,\bar{w}) \to \frac{1}{k_{\mathcal{M}}(z,w)},$$

as $l \to \infty$ *and for all* $z, \bar{w} \in \mathbb{D}$*, where* $k_{\mathcal{M}}$ *is the kernel function of* $\mathcal{M}$ *(see Section 2),*
(iii) $sup_l \|p_l(M_z, M_z^*)\| < \infty$*, and*
(iv) $\{M_z\}' = \{M_\varphi : \varphi \in H^\infty(\mathbb{D})\}$.

Appealing to Theorem 1.6 of [4], a Cowen-Douglas atom $\mathcal{M}$ admits a $\frac{1}{K}$-calculus. Here we do not intend to define the $\frac{1}{K}$-calculus but spell out the required properties of such concept in the present set up. We again refer the reader to [4] for details.

Note that by condition (i) in the above definition and the Gram-Schmidth orthogonalization process, for a Cowen-Douglas atom $\mathcal{M}$ there exists a sequence of orthonormal basis of polynomials $\{q_l(z) : l \geq 0\}$ such that

$$k_{\mathcal{M}}(z,w) = \sum_{l\geq 0} q_l(z)\overline{q_l(w)}. \tag{4.1}$$

We henceforth assume $\mathcal{M}$ *to be a fixed Cowen-Douglas atom with the sequence of polynomials* $\{p_l(z,\bar{w})\}$ *as in (ii) of Definition 4.1 and the orthonormal basis as above with the kernel function identity (4.1).*

Natural examples of Cowen-Douglas atoms include the Hardy space and the weighted Bergman spaces (cf. [4]).

We turn now to define an analogue of the contractive Hilbert modules.

Definition 4.2. *A Hilbert module* $\mathcal{H}$ *over* $\mathbb{C}[z]$ *is said to be* $\mathcal{M}$*-contractive if*
(i) $sup_l \|p_l(M_z, M_z^*)\| < \infty$ *and*
(ii) $C_{\mathcal{H}} := WOT - \lim_{l\to\infty} p_l(M_z, M_z^*)$ *is a positive operator.*

The following lemma will be useful in the sequel.

Lemma 4.3. *Let* $\mathcal{M}$ *be a Cowen-Douglas atom, and let* $\mathcal{E}$ *be a Hilbert space. Also let* $\mathcal{Q} = (\mathcal{M} \otimes \mathcal{E})/\mathcal{S}$ *be an* $\mathcal{M}$*-contractive Hilbert module for some submodule* $\mathcal{S}$ *of* $\mathcal{M} \otimes \mathcal{E}$*. Then,*
(i) $C_{\mathcal{M}\otimes\mathcal{E}} = I_{\mathcal{M}} \otimes P_{\mathcal{E}}$*, and*
(ii) $C_{\mathcal{Q}} = P_{\mathcal{Q}} C_{\mathcal{M}\otimes\mathcal{E}} P_{\mathcal{Q}} = P_{\mathcal{Q}}(I_{\mathcal{M}} \otimes P_{\mathcal{E}})P_{\mathcal{Q}}$.

Proof. Let $z, w \in \mathbb{D}$, and let $x, y \in \mathcal{E}$. Then for all $l \geq 1$ we have

$$\begin{aligned}\langle p_l(M_z \otimes I_{\mathcal{E}}, M_z^* \otimes I_{\mathcal{E}})(k_{\mathcal{M}}(\cdot,w)\otimes x), k_{\mathcal{M}}(\cdot,z)\otimes y\rangle &= p_l(z,\bar{w})\langle k_{\mathcal{M}}(\cdot,w)\otimes x, k_{\mathcal{M}}(\cdot,z)\otimes y\rangle \\ &= p_l(z,\bar{w})k_{\mathcal{M}}(z,w)\langle x,y\rangle.\end{aligned}$$

This shows, by letting $l \to \infty$, that

$$\begin{aligned}\langle C_{\mathcal{M}\otimes\mathcal{E}}(k_{\mathcal{M}}(\cdot,w)\otimes x), k_{\mathcal{M}}(\cdot,z)\otimes y\rangle &= \frac{1}{k_{\mathcal{M}}(z,w)} k_{\mathcal{M}}(z,w)\langle x,y\rangle = \langle x,y\rangle \\ &= \langle (I_{\mathcal{M}}\otimes P_{\mathcal{E}})(k_{\mathcal{M}}(\cdot,w)\otimes x), k_{\mathcal{M}}(\cdot,z)\otimes y\rangle,\end{aligned}$$

and hence $C_{\mathcal{M}\otimes\mathcal{E}} = I_{\mathcal{M}} \otimes P_{\mathcal{E}}$. This completes the proof of part (i).

To prove (ii) we compute

$$p_l(P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}}, P_{\mathcal{Q}}(M_z^* \otimes I_{\mathcal{E}})|_{\mathcal{Q}}) = P_{\mathcal{Q}}(p_l(M_z \otimes I_{\mathcal{E}}, M_z^* \otimes I_{\mathcal{E}}))|_{\mathcal{Q}}.$$

Letting $l \to \infty$ in WOT, we deduce from part (i) that

$$C_{\mathcal{Q}} = P_{\mathcal{Q}}(I \otimes P_{\mathcal{E}})|_{\mathcal{Q}}.$$

This completes the proof of the lemma. ∎

We need the following analogue of $C_{\cdot 0}$ contractions [22].

Let A be a positive operator on a Hilbert space $\mathcal{H}$. Define $Q_{l,A} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ for all $l \geq 1$ by

$$Q_{l,A}(T) = I - \sum_{0 \leq j < l} q_j(T) A q_j(T)^* \qquad (T \in \mathcal{B}(\mathcal{H})).$$

An $\mathcal{M}$-contractive Hilbert module $\mathcal{H}$ is said to be *pure* if

$$\text{SOT} - \lim_{l \to \infty} Q_{l,C_{\mathcal{H}}}(M_z) = 0.$$

Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{H} = \mathcal{M} \otimes \mathcal{E}$. Observe that if $\mathcal{Q}$ is a quotient module of $\mathcal{H}$ then $\mathcal{Q}$ is a pure $\mathcal{M}$-contractive Hilbert module (see [4]).

5. Quotient modules and tensor product bundles

The aim of the present section is to prove that the hermitian anti-holomorphic vector bundle of a pure $\mathcal{M}$-contractive Hilbert module in $B_m(\mathbb{D})$ can be represented as the tensor product bundle of a hermitian anti-holomorphic line bundle and a rank m hermitian anti-holomorphic vector bundle.

We start by recalling a version of the model theorem due to Arazy and Englis (Corollary 3.2 in [4]).

Theorem 5.1. *(Arazy-Englis) Let $\mathcal{H}$ be a Hilbert module over $\mathbb{C}[z]$. Then $\mathcal{H}$ is a pure $\mathcal{M}$-contractive Hilbert module if and only if $\mathcal{H}$ is unitarily equivalent to a quotient module of $\mathcal{M} \otimes \mathcal{E}$ for some Hilbert space $\mathcal{E}$.*

The following proposition shows that an $\mathcal{M}$-contractive Hilbert modules in $B_m(\mathbb{D})$ is pure.

Proposition 5.2. *Let $\mathcal{H} \in B_m(\mathbb{D})$ be an $\mathcal{M}$-contractive Hilbert module. Then $\mathcal{H}$ is pure.*

Proof. Let $\{\gamma_{i,w} : 1 \leq i \leq m\}$ be an anti-holomorphic frame of $E_{\mathcal{H}}$ with

$$M_z^* \gamma_{i,w} = \bar{w} \gamma_{i,w},$$

for all $w \in \mathbb{D}$ and $1 \leq i \leq m$. Then for all $z, w \in \mathbb{D}$, and $1 \leq i, j \leq m$, we have

$$\begin{aligned}\langle C_{\mathcal{H}} \gamma_{i,w}, \gamma_{j,z} \rangle &= \lim_{l \to \infty} \langle p_l(M_z, M_z^*) \gamma_{i,w}, \gamma_{j,z} \rangle \\ &= (\lim_{l \to \infty} p_l(z, \bar{w})) \langle \gamma_{i,w}, \gamma_{j,z} \rangle \\ &= \frac{1}{k_{\mathcal{M}}(z,w)} \langle \gamma_{i,w}, \gamma_{j,z} \rangle,\end{aligned}$$

and hence

$$\begin{aligned}\langle Q_{l,C_{\mathcal{H}}}(M_z)\gamma_{i,w}, \gamma_{j,z}\rangle &= \langle \gamma_{i,w}, \gamma_{j,z}\rangle - \langle \sum_{t=0}^{l-1} q_t(M_z) C_{\mathcal{H}} q_t(M_z)^* \gamma_{i,w}, \gamma_{j,z}\rangle \\ &= \langle \gamma_{i,w}, \gamma_{j,z}\rangle - \langle \sum_{t=0}^{l-1} C_{\mathcal{H}} \overline{q_t(w)} \gamma_{i,w}, \overline{q_t(z)} \gamma_{j,z}\rangle \\ &= \langle \gamma_{i,w}, \gamma_{j,z}\rangle - (\sum_{t=0}^{l-1} q_t(z)\overline{q_t(w)}) \langle C_{\mathcal{H}} \gamma_{i,w}, \gamma_{j,z}\rangle \\ &= \langle \gamma_{i,w}, \gamma_{j,z}\rangle - (\sum_{t=0}^{l-1} q_t(z)\overline{q_t(w)}) \frac{1}{k_{\mathcal{M}}(z,w)} \langle \gamma_{i,w}, \gamma_{j,z}\rangle \\ &= (1 - (\sum_{t=0}^{l-1} q_t(z)\overline{q_t(w)}) \frac{1}{k_{\mathcal{M}}(z,w)}) \langle \gamma_{i,w}, \gamma_{j,z}\rangle \\ &\to 0 \text{ as } l \to \infty.\end{aligned}$$

From this we deduce that $Q_{l,C_{\mathcal{H}}}(M_z) \to 0$ in SOT. This concludes the proof. ∎

Let $\mathcal{H} \in B_m(\mathbb{D})$ be an $\mathcal{M}$-contractive module. As an application of the previous proposition and Theorem 5.1, $\mathcal{H}$ can be realized as

$$\mathcal{H} \cong \mathcal{Q} := (\mathcal{M} \otimes \mathcal{E})/\mathcal{S},$$

for some Hilbert space $\mathcal{E}$ and submodule $\mathcal{S}$ of $\mathcal{M} \otimes \mathcal{E}$. That is,

$$0 \to \mathcal{S} \to \mathcal{M} \otimes \mathcal{E} \to \mathcal{H} \to 0.$$

Therefore, an $\mathcal{M}$-contractive Hilbert module $\mathcal{H} \in B_m(\mathbb{D})$ can be realized as a quotient module $\mathcal{Q}$ of $\mathcal{M} \otimes \mathcal{E}$ for some coefficient space $\mathcal{E}$. In this representation, the module map M_z on $\mathcal{H}$ is identified with the compressed multiplication operator $P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}}$. Moreover,

$$P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})^*|_{\mathcal{Q}} = (M_z \otimes I_{\mathcal{E}})^*|_{\mathcal{Q}}.$$

In the rest of this paper we will assume the quotient module representations of the class of pure $\mathcal{M}$-contractive Hilbert modules in $B_m(\mathbb{D})$.

Also, often we will identify a Cowen-Douglas atom $\mathcal{M}$ with the reproducing kernel Hilbert module $\mathcal{H}_{k_{\mathcal{M}}}$ (see Section 2) with section

$$w \mapsto k_{\mathcal{M}}(\cdot, w) \qquad (w \in \mathbb{D}).$$

Now we are in a position to prove the main result of this section.

THEOREM 5.3. *Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{Q}$ is a quotient module of $\mathcal{M} \otimes \mathcal{E}$. Then $\mathcal{Q} \in B_m(\mathbb{D})$ if and only if there exists a rank m hermitian anti-holomorphic vector bundle V over $\mathbb{D}$ such that*

$$E_{\mathcal{Q}} \cong E_{\mathcal{M}} \otimes V.$$

Moreover, in this case

$$\mathcal{K}_{\mathcal{Q}} = \mathcal{K}_{\mathcal{M}} + \mathcal{K}_V.$$

Proof. The sufficient part of the statement is trivial, so we only have to prove the necessary part. Let $\{\gamma_{i,w} : 1 \leq i \leq m\}$ be an anti-holomorphic frame of $E_{\mathcal{Q}}$ such that

$$M_z^* \gamma_{i,w} = \bar{w} \gamma_{i,w},$$

for all $1 \leq i \leq m$ and $w \in \mathbb{D}$. Then for all $l \geq 1$ we have

$$p_l(M_z, M_z^*) \gamma_{i,w} = p_l(z, \bar{w}) \gamma_{i,w}.$$

Letting $l \to \infty$ in WOT, and applying Lemma 4.3 we have

$$\frac{1}{k_{\mathcal{M}}(\cdot, w)} \gamma_{i,w} = C_{\mathcal{Q}} \gamma_{i,w} = P_{\mathcal{Q}}(I \otimes P_{\mathcal{E}}) \gamma_{i,w}.$$

Since

$$P_{\mathcal{Q}}(I \otimes P_{\mathcal{E}}) \gamma_{i,w} = \gamma_{i,w}(0),$$

we have

$$\frac{1}{k_{\mathcal{M}}(\cdot, w)} \gamma_{i,w} = \gamma_{i,w}(0).$$

Therefore

$$\gamma_{i,w} = k_{\mathcal{M}}(\cdot, w) \otimes \gamma_{i,w}(0) = k_{\mathcal{M}}(\cdot, w) \otimes v_{i,w}, \tag{5.1}$$

where $v_{i,w} := \gamma_{i,w}(0)$ for all $1 \leq i \leq m$ and $w \in \mathbb{D}$. Let V be the anti-holomorphic curve over $\mathbb{D}$ with

$$V(w) = \text{span}\ \{v_{i,w} : 1 \leq i \leq m\} \qquad (w \in \mathbb{D}).$$

Then we conclude that $E_{\mathcal{Q}} \cong E_{\mathcal{M}} \otimes V$.
Finally, let G_V be the Gram matrix corresponding to the frame $\{v_{i,w}\}$ of E_V. Then

$$\begin{aligned}
\mathcal{K}_{E_{\mathcal{Q}}}(w) &= -\overline{\partial}[G_{E_{\mathcal{M}}}^{-1}(\bar{w}) \partial G_{E_{\mathcal{M}}}(\bar{w})] \\
&= -\overline{\partial}[\frac{1}{\|k_{\mathcal{M}}(\cdot, w)\|^2} G_V^{-1}(\bar{w}) \partial \{\|k_{\mathcal{M}}(\cdot, w)\|^2 G_V(\bar{w})\}] \\
&= -\overline{\partial}[\frac{1}{\|k_{\mathcal{M}}(\cdot, w)\|^2} G_V^{-1}(\bar{w}) \{\partial(\|k_{\mathcal{M}}(\cdot, w)\|^2) G_V(\bar{w}) + \|k_{\mathcal{M}}(\cdot, w)\|^2 \partial G_V(\bar{w})\}] \\
&= -\overline{\partial}[\frac{1}{\|k_{\mathcal{M}}(\cdot, w)\|^2} \partial(\|k_{\mathcal{M}}(\cdot, w)\|^2) + G_V^{-1}(\bar{w}) \partial(G_V(\bar{w}))] \\
&= \mathcal{K}_{E_{\mathcal{M}}}(w) + \mathcal{K}_V(w),
\end{aligned}$$

for all $w \in \mathbb{D}$. This concludes the proof of the theorem. ∎

In particular, we have the following useful result.

Corollary 5.4. *Let $\mathcal{H} \in B_m(\mathbb{D})$ be a pure $\mathcal{M}$-contractive Hilbert module. Then there exists a hermitian anti-holomorphic vector bundle V of rank m over $\mathbb{D}$ such that*

$$E_{\mathcal{H}} \cong E_{\mathcal{M}} \otimes V.$$

Moreover

$$\Pi_{\mathcal{H}} = \Pi_{\mathcal{M}} \otimes \Pi_V,$$

and for each $w \in \mathbb{D}$,

$$\|\partial \Pi_{\mathcal{H}}(w)\|_2^2 = m\|\partial \Pi_{\mathcal{M}}(w)\|_2^2 + \|\partial \Pi_V(w)\|_2^2 = m|\mathcal{K}_{\mathcal{M}}(w)| + \|\partial \Pi_V(w)\|_2^2.$$

Proof. First two conclusions follows from Theorem 5.3. For the remaining parts, we follow the same line of arguments as in [16] or [8]. Since

$$\partial \Pi_{\mathcal{Q}}(w) = \partial(\Pi_{\mathcal{M}}(w) \otimes \Pi_V(w)) = \partial \Pi_{\mathcal{M}}(w) \otimes \Pi_V(w) + \Pi_{\mathcal{M}}(w) \otimes \partial \Pi_V(w),$$

we have that

$$\begin{aligned}\|\partial \Pi_{\mathcal{Q}}(w)\|_2^2 =& tr([\partial \Pi_{\mathcal{M}}(w) \otimes \Pi_V(w)][\partial \Pi_{\mathcal{M}}(w) \otimes \Pi_V(w)]^*) + 2\text{Real}\{tr([\partial \Pi_{\mathcal{M}}(w) \otimes \Pi_V(w)]^* \\ &[\Pi_{\mathcal{M}}(w) \otimes \partial \Pi_V(w)])\} + tr([\Pi_{\mathcal{M}}(w) \otimes \partial \Pi_V(w)]^*[\Pi_{\mathcal{M}}(w) \otimes \partial \Pi_V(w)]).\end{aligned}$$

Notice that $\overline{\partial} \Pi_{\mathcal{M}}(w) \Pi_{\mathcal{M}}(w) = 0$ and hence the middle term in the last expression vanishes. Therefore,

$$\begin{aligned}\|\partial \Pi_{\mathcal{Q}}(w)\|_2^2 &= \|\partial \Pi_{\mathcal{M}}(w) \otimes \Pi_V(w)\|_2^2 + \|\Pi_{\mathcal{M}}(w) \otimes \partial \Pi_V(w)\|_2^2 \\ &= \|\partial \Pi_{\mathcal{M}}(w)\|_2^2 \|\Pi_V(w)\|_2^2 + \|\Pi_{\mathcal{M}}(w)\|_2^2 \|\partial \Pi_V(w)\|_2^2 \\ &= m|\mathcal{K}_{\mathcal{M}}(w)| + \|\partial \Pi_V(w)\|_2^2,\end{aligned}$$

where the last equality follows from Corollary 3.3. This completes the proof. ∎

6. Quasi-affinity

In this section we discuss the issue of quasi-affinity of Hilbert modules in the Cowen-Douglas class $B_m(\mathbb{D})$. We begin with the definition of quasi-affinity.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert modules. Then we say that $\mathcal{H}$ is *quasi-affine* to $\mathcal{K}$, and denote by $\mathcal{H} \prec \mathcal{K}$, if there exists a module map $X : \mathcal{H} \to \mathcal{K}$ such that X is one-to-one and has dense range.

Theorem 6.1. *Let $\mathcal{H} \in B_m(\mathbb{D})$ be a pure $\mathcal{M}$-contractive Hilbert module and $\{\gamma_{i,w}\}_{i=1}^m$ be an anti-holomorphic frame of $E_{\mathcal{H}}$ such that*

$$\sup_{w \in \mathbb{D}} \left(\frac{\|\gamma_{i,w}\|}{\|k_{\mathcal{M}}(\cdot, w)\|}\right) < \infty,$$

for all $i = 1, \ldots, m$. Then

(i) there exists a one-to-one module map $X : \mathcal{H} \to \mathcal{M} \otimes \mathbb{C}^m$, and

(ii) $\mathcal{H} \prec \mathcal{S}$ for some submodule $\mathcal{S} \subseteq \mathcal{M} \otimes \mathbb{C}^m$.

Proof. Identifying $\mathcal{H}$ with $\mathcal{Q} = (\mathcal{M} \otimes \mathcal{E})/\mathcal{S}$ for some submodule $\mathcal{S}$ of $\mathcal{M} \otimes \mathcal{E}$, we let

$$\gamma_{i,w} = k_{\mathcal{M}}(\cdot, w) \otimes v_{i,w},$$

for each $i = 1, \ldots, m$. Set

$$\delta := \sup_{w \in \mathbb{D}} \left(\frac{\|\gamma_{i,w}\|}{\|k_{\mathcal{M}}(\cdot, w)\|}\right) = \sup_{w \in \mathbb{D}} \|v_{i,w}\| < \infty.$$

For each $z \in \mathbb{D}$, define $\Theta(z) \in \mathcal{B}(\mathcal{E}, \mathbb{C}^m)$ by

$$\Theta(z)\eta = (\langle \eta, v_{1,z} \rangle_{\mathcal{E}}, \ldots, \langle \eta, v_{m,z} \rangle_{\mathcal{E}}) \in \mathbb{C}^m.$$

Then

$$\|\Theta(z)\eta\|^2 = \sum_{i=1}^m |\langle \eta, v_{i,z}\rangle_{\mathcal{E}}|^2 \leq \|\eta\|^2 \sum_{i=1}^m \|v_{i,w}\|^2 \leq m\delta^2 \|\eta\|^2,$$

for all $\eta \in \mathcal{E}$. Consequently, $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathbb{C}^m)}(\mathbb{D})$. Furthermore, for $f \in \mathcal{S} = \mathcal{Q}^\perp$ and $w \in \mathbb{D}$ we have

$$\begin{aligned}(\Theta f)(w) &= \Theta(w) f(w) = (\langle f(w), v_{1,w}\rangle_{\mathcal{E}}, \ldots, \langle f(w), v_{m,w}\rangle_{\mathcal{E}})\\ &= (\langle f, k_{\mathcal{M}}(\cdot, w) \otimes v_{1,w}\rangle_{\mathcal{M}\otimes\mathcal{E}}, \ldots, \langle f, k_{\mathcal{M}}(\cdot, w) \otimes v_{m,w}\rangle_{\mathcal{M}\otimes\mathcal{E}})\\ &= (\langle f, \gamma_{1,w}\rangle_{\mathcal{M}\otimes\mathcal{E}}, \ldots, \langle f, \gamma_{m,w}\rangle_{\mathcal{M}\otimes\mathcal{E}})\\ &= 0.\end{aligned}$$

Hence, $M_\Theta \mathcal{S} = \{0\}$. Next we define $X : \mathcal{Q} \to \mathcal{M} \otimes \mathbb{C}^m$ by

$$Xf = M_\Theta f,$$

for all $f \in \mathcal{Q}$. Then

$$\begin{aligned}XP_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}} &= M_\Theta P_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}}\\ &= M_\Theta (M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}}\\ &= (M_z \otimes I_{\mathbb{C}^m}) M_\Theta|_{\mathcal{Q}},\end{aligned}$$

that is,

$$XP_{\mathcal{Q}}(M_z \otimes I_{\mathcal{E}})|_{\mathcal{Q}} = (M_z \otimes I_{\mathbb{C}^m})X,$$

and hence, X is a module map. To prove that X is one-to-one, or equivalently, that X^* has dense range, we compute

$$\langle \Theta(w)^* e_i, \eta\rangle_{\mathcal{E}} = \langle e_i, \Theta(w)\eta\rangle_{\mathbb{C}^m} = \langle e_i, \sum_{j=1}^m \langle \eta, v_{j,w}\rangle_{\mathcal{E}}\rangle_{\mathbb{C}^m} = \langle v_{i,w}, \eta\rangle_{\mathcal{E}},$$

for all $w \in \mathbb{D}$, $\eta \in \mathcal{E}$ and $i = 1, \ldots, m$. Therefore

$$\Theta(w)^* e_i = v_{i,w},$$

and hence

$$\begin{aligned}X^*(k_{\mathcal{M}}(\cdot, w) \otimes e_i) &= P_{\mathcal{Q}} M_\Theta^* (k_{\mathcal{M}}(\cdot, w) \otimes e_i)\\ &= P_{\mathcal{Q}}(k_{\mathcal{M}}(\cdot, w) \otimes \Theta(w)^* e_i)\\ &= (k_{\mathcal{M}}(\cdot, w) \otimes v_{i,w})\\ &= P_{\mathcal{Q}} \gamma_{i,w}\\ &= \gamma_{i,w}.\end{aligned}$$

Hence, X is one-to-one. This proves part (i).

Part (ii) follows from part (i) and by considering $\mathcal{S}$ as the range closure of X. ■

In the anti-holomorphic vector bundle language, the above result can be stated as follows : Suppose there exist an anti-holomorphic bundle map $\Phi : E_{\mathcal{M}\otimes\mathbb{C}^m} \to E_{\mathcal{H}}$ and $\delta > 0$ such that

$$\|\Phi(w)\eta_w\|_{\mathcal{H}} \leq \delta \|\eta_w\|_{E_{\mathcal{M}\otimes\mathbb{C}^m}(w)},$$

for all $\eta_w \in E_{\mathcal{M}\otimes\mathbb{C}^m}(w)$ and $w \in \mathbb{D}$. Then $\mathcal{H}$ is quasi-affine to a submodule of $\mathcal{M} \otimes \mathcal{E}$.

One might expect that the submodule $\mathcal{S}$ in the above result is the entire free module $\mathcal{M}\otimes\mathbb{C}^m$. However, such results are closely related with the issue of the Beurling-Lax-Halmos type theorem for the Cowen-Douglas atoms. In particular, for $\mathcal{M} = H^2(\mathbb{D})$ the submodule $\mathcal{S}$ is unitarily equivalent with the Hardy module with the same multiplicity as the rank of the map $\Theta(w)$ which is m. Consequently, the conclusion is stronger for any $H^2(\mathbb{D})$-contractive module, that is, $\mathcal{H}$ is quasi-affine to the Hardy module $H^2(\mathbb{D}) \otimes \mathbb{C}^m$ (see [24]). We point out that even the Bergman module is quite subtle [3] for this consideration.

7. Similarity

The purpose of this section is to relate the similarity problem with the curvatures of Cowen-Douglas Hilbert modules. Recall that a Hilbert module $\mathcal{H}_1$ is said to be similar to a Hilbert module $\mathcal{H}_2$, denoted by $\mathcal{H}_1 \sim_s \mathcal{H}_2$, if there exists an invertible module map X from $\mathcal{H}_1$ to $\mathcal{H}_2$.

We begin by generalizing a result by Uchiyama on similarity of a contractive Hilbert modules to the Hardy module of finite multiplicity (see Theorem 3.8 in [24]).

Theorem 7.1. *Let $\mathcal{H} \in B_m(\mathbb{D})$ be an $\mathcal{M}$-contractive Hilbert module. Then $\mathcal{H}$ is similar to $\mathcal{M} \otimes \mathbb{C}^m$ if and only if there exists an anti-holomorphic pointwise invertible bundle map $\Phi : E_{\mathcal{M}\otimes\mathbb{C}^m} \to E_{\mathcal{H}}$ and $\delta > 0$ such that*

$$\frac{1}{\delta}\|\eta_w\|_{E_{\mathcal{M}\otimes\mathbb{C}^m}(w)} \leq \|\Phi(w)\eta_w\|_{\mathcal{H}} \leq \delta\|\eta_w\|_{E_{\mathcal{M}\otimes\mathbb{C}^m}(w)},$$

for all $\eta_w \in E_{\mathcal{M}\otimes\mathbb{C}^m}(w)$ and $w \in \mathbb{D}$.

Proof. Let $X : \mathcal{H} \to \mathcal{M} \otimes \mathbb{C}^m$ be an invertible module map. Then $\gamma_{i,w} := X^*(k_{\mathcal{M}}(\cdot, w) \otimes e_i)$ is the required anti-holomorphic frame of $E_{\mathcal{H}}$.
For the converse, we proceed as in the proof of Theorem 6.1. We first, consider an anti-holomorphic frame $\{\gamma_{i,w}\}_{i=1}^m = \{k_{\mathcal{M}}(\cdot, w) \otimes v_{i,w}\}_{i=1}^m$ of $E_{\mathcal{H}}$ and define $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathbb{C}^m)}(\mathbb{D})$ by

$$\Theta(w)\eta = (\langle \eta, v_{1,w}\rangle_{\mathcal{E}}, \ldots, \langle \eta, v_{m,w}\rangle_{\mathcal{E}}),$$

for all $\eta \in \mathcal{E}$ and $w \in \mathbb{D}$. Now

$$\|\Theta(w)^*x\| = \|\sum_{i=1}^m x_i v_{i,w}\| = \frac{1}{\|k_{\mathcal{M}}(\cdot, w)\|}\|\sum_{i=1}^m x_i\gamma_{i,w}\|,$$

and hence

$$\|\Theta(w)^*x\| \geq \delta\|x\|,$$

for all $x \in \mathbb{C}^m$ and $w \in \mathbb{D}$. Hence Θ is right invertible (cf. Proposition 3.7 in [24]). In particular,

$$\text{ran} M_\Theta = \mathcal{M} \otimes \mathbb{C}^m,$$

and since

$$M_\Theta\mathcal{S} = \{0\},$$

the module map $X : \mathcal{Q} \to \mathcal{M} \otimes \mathbb{C}^m$ defined by

$$Xf = \Theta f \qquad (f \in \mathcal{Q}),$$

is the required similarity. ■

Now, we are ready to formulate the following similarity result for pure $\mathcal{M}$-contractive Hilbert modules. Applying our result to the case where $\mathcal{H}$ is the Hardy module, or a weighted Bergman module, we recover the results of Kwon and Treil [16], and Kwon, Treil and Douglas [8].

Theorem 7.2. *Let $\mathcal{H} \in B_m(\mathbb{D})$ be a pure $\mathcal{M}$-contractive Hilbert module. Then the following statements are equivalent:*

(i) $\mathcal{H} \sim_s \mathcal{M} \otimes \mathbb{C}^m$.

(ii) There exists an anti-holomorphic pointwise invertible bundle map $\Phi : E_{\mathcal{M}\otimes\mathbb{C}^m} \to E_{\mathcal{H}}$ and $\delta > 0$ such that

$$\frac{1}{\delta}\|\eta_w\|_{E_{\mathcal{M}\otimes\mathbb{C}^m}(w)} \leq \|\Phi(w)\eta_w\|_{\mathcal{H}} \leq \delta\|\eta_w\|_{E_{\mathcal{M}\otimes\mathbb{C}^m}(w)},$$

for all $\eta_w \in E_{\mathcal{M}\otimes\mathbb{C}^m}(w)$ and $w \in \mathbb{D}$.

(iii) There exists a bounded solution φ defined on $\mathbb{D}$ to the Poisson equation

$$\Delta\varphi = trace\mathcal{K}_{\mathcal{M}\otimes\mathbb{C}^m} - trace\mathcal{K}_{\mathcal{H}}.$$

Proof. The equivalence of (i) and (ii) is Theorem 7.1.

(ii) implies (iii): We note that

$$E_{\mathcal{M}\otimes\mathbb{C}^m}(w) = \ker(M_z - w)^* = k_{\mathcal{M}}(\cdot, w) \otimes \mathbb{C}^m,$$

and

$$E_{\mathcal{H}}(w) = \ker(M_z - w)^* = k_{\mathcal{M}}(\cdot, w) \otimes V(w).$$

Consequently, for a given bundle equivalence Φ from $E_{\mathcal{M}\otimes\mathbb{C}^m}$ to $E_{\mathcal{H}}$ there exists a one-to-one bounded anti-holomorphic map $\Gamma : \mathbb{D} \to \mathcal{B}(\mathbb{C}^m, \mathcal{E})$ such that

$$\Phi(w)(k_{\mathcal{M}}(\cdot, w) \otimes \eta) = k_{\mathcal{M}}(\cdot, w) \otimes \Gamma(w)\eta,$$

or, equivalently,

$$\Phi^{-1}(w)(k_{\mathcal{M}}(\cdot, w) \otimes \Gamma(w)\eta) = k_{\mathcal{M}}(\cdot, w) \otimes \eta,$$

and

$$V(w) = \mathrm{ran}\Gamma(w),$$

for all $\eta \in \mathbb{C}^m$ and $w \in \mathbb{D}$. Set

$$F(w) = \Gamma(\overline{w}).$$

Since

$$\frac{1}{\delta}\|\eta_w\|_{E_{\mathcal{M}\otimes\mathbb{C}^m}(w)} \leq \|\Phi(w)\eta_w\|_{\mathcal{H}} \leq \delta\|\eta_w\|_{E_{\mathcal{M}\otimes\mathbb{C}^m}(w)} \qquad (w \in \mathbb{D}), \tag{7.1}$$

by letting

$$\eta_w = k_{\mathcal{M}}(\cdot, w) \otimes \eta \qquad (w \in \mathbb{D}),$$

we have

$$\frac{1}{\delta}\|k_{\mathcal{M}}(\cdot,w)\otimes\eta\|_{E_{\mathcal{M}\otimes\mathbb{C}^m}(w)} \le \|k_{\mathcal{M}}(\cdot,w)\otimes\Gamma(w)\eta\|_{\mathcal{H}} \le \delta\|k_{\mathcal{M}}(\cdot,w)\otimes\eta\|_{E_{\mathcal{M}\otimes\mathbb{C}^m}(w)},$$

that is

$$\frac{1}{\delta^2}\|k_{\mathcal{M}}(\cdot,w)\|^2\|\eta\|^2 \le \|k_{\mathcal{M}}(\cdot,w)\|^2\|\Gamma(w)\eta\|^2_{\mathcal{E}} \le \delta^2\|k_{\mathcal{M}}(\cdot,w)\|^2\|\eta\|^2,$$

and so

$$\frac{1}{\delta^2}\|\eta\|^2 \le \langle\Gamma^*(w)\Gamma(w)\eta,\eta\rangle \le \delta^2\|\eta\|^2, \tag{7.2}$$

for all $\eta\in\mathbb{C}^m$. Writing $c=\delta^2$, the above inequalities yields

$$c^{-1}I \le F^*F \le cI. \tag{7.3}$$

Claim: *Let* $\overline{\Pi}_V(w)$ *be the orthogonal projection of* $\mathcal{E}$ *onto* $V(\overline{w})$. *Then* $\|\partial\overline{\Pi}_V(w)\|_2 \le c^{\frac{1}{2}}\|F'(w)\|_2$.

Indeed, as

$$\overline{\Pi}_V = F(F^*F)^{-1}F^*,$$

we have

$$\overline{\Pi}_V F = F(F^*F)^{-1}F^*F = F.$$

Then by a direct calculation, we have that

$$\partial\overline{\Pi}_V F = (I-\overline{\Pi}_V)F', \tag{7.4}$$

and

$$\partial\overline{\Pi}_V\overline{\Pi}_V = \partial\overline{\Pi}_V.$$

This yields

$$\begin{aligned}(I-\overline{\Pi}_V)F'(F^*F)^{-1}F^* &= \partial\overline{\Pi}_V F(F^*F)^{-1}F^*\\ &= \partial\overline{\Pi}_V\overline{\Pi}_V\\ &= \partial\overline{\Pi}_V.\end{aligned}$$

By (7.3) we have

$$\begin{aligned}\|\partial\overline{\Pi}_V\|_2 &= \|(I-\overline{\Pi}_V)F'(F^*F)^{-1}F^*\|_2\\ &\le \|I-\overline{\Pi}_V\|\cdot\|F'(F^*F)^{-1}F^*\|_2\\ &\le \|F'(F^*F)^{-1}F^*\|_2\\ &\le \|(F^*F)^{-1}F^*\|\cdot\|F'\|_2\\ &= \|(F^*F)^{-1}F^*F(F^*F)^{-1}\|^{\frac{1}{2}}\cdot\|F'\|_2\\ &= \|(F^*F)^{-1}\|^{\frac{1}{2}}\cdot\|F'\|_2\\ &\le c^{\frac{1}{2}}\|F'\|_2.\end{aligned}$$

Therefore the claim does hold, as required.
Finally, by Corollaries 3.3 and 5.4, we have

$$trace\mathcal{K}_{\mathcal{M}\otimes\mathbb{C}^m}(w) - trace\mathcal{K}_{\mathcal{H}}(w) = \|\partial\overline{\Pi}_V(w)\|_2^2 \le c\|F'(w)\|_2^2.$$

Set

$$\varphi_1(w) = \|cF(w)\|_2^2.$$

It follows that

$$\Delta\varphi_1(w) = c\|F'(w)\|_2^2,$$

and hence

$$\Delta\varphi_1 \geq trace\mathcal{K}_{\mathcal{M}\otimes\mathbb{C}^m} - trace\mathcal{K}_{\mathcal{H}}.$$

Let

$$\mathcal{G}_f(\lambda) := \frac{2}{\pi}\iint_{\mathbb{D}} \ln|\frac{w-\lambda}{1-\bar{\lambda}w}|f(w)dxdy,$$

be the Green potential to the solution of $\Delta u = f(\lambda)$. Then

$$\mathcal{G}_{\Delta\varphi_1} \leq \mathcal{G}_{trace\mathcal{K}_{\mathcal{M}\otimes\mathbb{C}^m} - trace\mathcal{K}_{\mathcal{H}}} \leq 0.$$

Set

$$\varphi = \mathcal{G}_{trace\mathcal{K}_{\mathcal{M}\otimes\mathbb{C}^m} - trace\mathcal{K}_{\mathcal{H}}}.$$

Then φ is bounded and

$$\Delta\varphi = trace\mathcal{K}_{\mathcal{M}\otimes\mathbb{C}^m} - trace\mathcal{K}_{\mathcal{H}}.$$

(iii) implies (i): We use Theorem 0.2 in [21] to get a bounded anti-holomorphic projection $\Theta(w)$ onto $\mathrm{ran}\Pi_V(w)$. Let Θ_i be the inner part of the inner-outer factorization of Θ. Then it follows that the Toeplitz operator T_{Θ_i} is invertible and the required similarity operator with (see [8] or [16] for more details)

$$T_{\Theta_i}E_{\mathcal{M}\otimes\mathbb{C}^m}(w) = E_{\mathcal{H}}(w), w \in \mathbb{D}.$$

■

It is of interest to note the following consequence of Theorem 7.2:
Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be two Cowen-Douglas atoms, and let $\mathcal{H}, \tilde{\mathcal{H}} \in B_m(\mathbb{D})$ be $\mathcal{M}$-contractive and $\tilde{\mathcal{M}}$ Hilbert modules, respectively. Let V and $\tilde{V}$ be the corresponding hermitian anti-holomorphic vector bundles such that $E_{\mathcal{H}} \cong E_{\mathcal{M}} \otimes V$ and $E_{\tilde{\mathcal{H}}} \cong E_{\tilde{\mathcal{M}}} \otimes \tilde{V}$ (see Theorem 5.3).
Now if $\mathcal{H}$ is similar to $\mathcal{M}\otimes\mathbb{C}^m$ then by Corollary 5.4 and part (iii) of the previous theorem, we have

$$\Delta\varphi(w) = \|\overline{\partial}\Pi_V(w)\|_2^2,$$

for some bounded subharmonic function on $\mathbb{D}$. Another application of Corollary 5.4 and part (iii) of the previous theorem to $\tilde{\mathcal{H}}$ yields the similarity of $\tilde{\mathcal{H}}$ to $\tilde{\mathcal{M}}\otimes\mathbb{C}^m$. Therefore, we have the following result.

Corollary 7.3. *Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be two Cowen-Douglas atoms, and let V be a rank m hermitian anti-holomorphic vector bundle over $\mathbb{D}$. Then the pure $\mathcal{M}$-contractive Hilbert module $\mathcal{H}$ corresponding to the hermitian anti-holomorphic vector bundle $E_{\mathcal{M}}\otimes V$ is similar to $\mathcal{M}\otimes\mathbb{C}^m$ if and only if the $\tilde{\mathcal{M}}$-contractive Hilbert module $\tilde{\mathcal{H}}$ is similar to $\tilde{\mathcal{M}}\otimes\mathbb{C}^m$, where $\tilde{\mathcal{H}} \in B_m(\mathbb{D})$ is the Hilbert module corresponding to the hermitian anti-holomorphic vector bundle $E_{\tilde{\mathcal{M}}}\otimes V$.*

The above result is a generalization of Corollary 4.5 (restricted to the Cowen-Douglas atoms) in [9] where the quotient module representations are assumed to be the orthocomplements of the submodules implemented by left invertible multipliers. Moreover, the free modules associated to the quotient modules are also assumed to be of finite rank.

Let $\mathcal{H} \in B_m(\mathbb{D})$ be a pure $\mathcal{M}$-contractive Hilbert module such that $E_{\mathcal{H}} \cong E_{\mathcal{M}} \otimes V$ and $V(w) \subseteq \mathcal{E}$, $w \in \mathbb{D}$. Let $\overline{\Pi}_V(\mathrm{w})$ be the orthogonal projection of $\mathcal{E}$ onto $V(\bar{w})$. In the following theorem, we prove that $\mathcal{H}$ is similar to $\mathcal{M} \otimes \mathbb{C}^m$ if and only if $\overline{\Pi}_V(\mathrm{w})$ can factor as

$$\overline{\Pi}_V(w) = F(w)G(w) \qquad (w \in \mathbb{D}),$$

for some $F \in H^\infty_{B(\mathcal{E}_*,\mathcal{E})}(\mathbb{D})$ and $G \in L^\infty_{B(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ with *ran* $F(w) = V(w)$, $w \in \mathbb{D}$.

THEOREM 7.4. *Let $\mathcal{E}$ and $\mathcal{E}_*$ be two Hilbert spaces, let $\mathcal{H} \in B_m(\mathbb{D})$ be a pure $\mathcal{M}$-contractive Hilbert module such that*

$$E_{\mathcal{H}} \cong E_{\mathcal{M}} \otimes V,$$

where $V(w) \subseteq \mathcal{E}$. Assume that $\overline{\Pi}_V(w)$ is the orthogonal projection of $\mathcal{E}$ onto $V(\bar{w})$. Then

$$\mathcal{H} \sim_s \mathcal{M} \otimes \mathbb{C}^m$$

if and only if there exist $F \in H^\infty_{B(\mathcal{E}_*,\mathcal{E})}(\mathbb{D})$ and $G \in L^\infty_{B(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ such that

$$ran\ F(w) = V(\bar{w}) \qquad (w \in \mathbb{D})$$

and

$$\overline{\Pi}_V(w) = F(w)G(w) \qquad (w \in \mathbb{D}).$$

Proof. First we prove that the conditions are sufficient. Note that

$$ranF(w) = V(\bar{w}) = ran\overline{\Pi}_V(w),$$

implies that

$$\overline{\Pi}_V(w)F(w) = F(w) \qquad (w \in \mathbb{D}).$$

Note also that (cf. (7.1))

$$\partial\overline{\Pi}_V(w)F(w) = (1 - \overline{\Pi}_V(w))F'(w).$$

and (cf. Lemma 2.2, [21])

$$\partial\overline{\Pi}_V(w) = \partial\overline{\Pi}_V(w)\overline{\Pi}_V(w),$$

for all $w \in \mathbb{D}$. We therefore have

$$\begin{aligned} \partial\overline{\Pi}_V(w) &= \partial\overline{\Pi}_V(w)\overline{\Pi}_V(w) \\ &= \partial\overline{\Pi}_V(w)F(w)G(w) \\ &= (1 - \overline{\Pi}_V(w))F'(w)G(w) \end{aligned}$$

Then for $M = \|G\|_\infty > 0$, it follows that

$$\begin{aligned} \|\partial\overline{\Pi}_V(w)\|_2 &= \|(I - \overline{\Pi}_V(w))F'(w)G(w)\|_2 \\ &\leq \|I - \overline{\Pi}_V(w)\| \cdot \|F_2'(w)G(w)\|_2 \\ &\leq \|G(w)\| \cdot \|F'(w)\|_2 \\ &\leq \|G\|_\infty \|F'(w)\|_2 \\ &= M\|F'(w)\|_2, \end{aligned}$$

for all $w \in \mathbb{D}$. Now as in the proof of Theorem 7.2, we set the bounded subharmonic function ψ_1 as $\|M^{\frac{1}{2}}F_2\|_2^2$. Then

$$traceK_{\mathcal{H}_1} - traceK_{\mathcal{M}\otimes\mathbb{C}^m} = \|\partial\overline{\Pi}_{V_1}(w)\|_2^2 \leq \Delta\psi_1,$$

and hence by $(iii) \implies (i)$ of Theorem 7.2, we have that

$$\mathcal{H} \sim_s \mathcal{M} \otimes \mathbb{C}^m.$$

Conversely, assume that $\mathcal{H} \sim_s \mathcal{M} \otimes \mathbb{C}^m$. By $(i) \implies (ii)$ of Theorem 7.2, there exist an anti-holomorphic point-wise invertible bundle map $\Phi : E_{\mathcal{M}\otimes\mathbb{C}^m} \to E_{\mathcal{H}}$ and a one-one bounded anti-holomorphic map $\Gamma : \mathbb{D} \to \mathcal{B}(\mathbb{C}^m, \mathcal{E})$ such that

$$\Phi(w)(k_{\mathcal{M}}(\cdot, w) \otimes \eta) = k_{\mathcal{M}}(\cdot, w) \otimes \Gamma(w)\eta,$$

for all $\eta \in \mathbb{C}^m$ and $w \in \mathbb{D}$. We have therefore

$$\Phi^{-1}(w)(k_{\mathcal{M}}(\cdot, w) \otimes \Gamma(w)\eta) = k_{\mathcal{M}}(\cdot, w) \otimes \eta,$$

and

$$V(w) = \text{ran}\Gamma(w),$$

for all $\eta \in \mathbb{C}^m$ and $w \in \mathbb{D}$. Let

$$F(w) = \Gamma(\overline{w}),$$

and

$$G(w) = (F^*(w)F(w))^{-1}F^*(w),$$

for all $w \in \mathbb{D}$. Then by (7.2), there exists $\delta > 0$ such that

$$\delta^{-1} \leq \|F\| \leq \delta,$$

and

$$\|G\| \leq \delta^3.$$

Since Γ is anti-holomorphic, it follows that $F \in H^\infty_{B(\mathbb{C}^m,\mathcal{E})}(\mathbb{D})$ and $G \in L^\infty_{B(\mathcal{E},\mathbb{C}^m)}(\mathbb{D})$. Moreover

$$\begin{aligned}\overline{\Pi}_V(w) &= F(w)(F^*(w)F(w))^{-1}F^*(w)\\ &= F(w)G(w)\end{aligned}$$

for all $w \in \mathbb{D}$. This completes the proof of the result. ∎

8. Concluding remarks

A number of questions and directions remain to be explored, including the similarity problem for the Dirichlet module (however, see [15]). We point out that the notion of the Cowen-Douglas atom does not cover the Dirichlet module (see [20]).

Some of the results of this paper can be generalized in the several variables set up. However, one of the key ideas to achieve results of full strength is closely related to the corona problem in several variables (see [21]).

Another interesting question relates the quasi-affinity of the Cowen-Douglas Hilbert modules. For the Hardy space, quasi-affinity to a submodule of a Hardy module is as same as the Hardy module it self. It is not known under what additional condition on the frame, that module will be quasi-affine to a Cowen-Douglas atom of finite multiplicity.

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