BEURLING TYPE INVARIANT SUBSPACES OF COMPOSITION OPERATORS

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Abstract. Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$, let $H^2$ denote the Hardy space on $\mathbb{D}$ and let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic self map of $\mathbb{D}$. The composition operator $C_\varphi$ on $H^2$ is defined by

$$(C_\varphi f)(z) = f(\varphi(z)) \quad (f \in H^2, \ z \in \mathbb{D}).$$

Denote by $S(\mathbb{D})$ the set of all functions that are holomorphic and bounded by one in modulus on $\mathbb{D}$, that is

$$S(\mathbb{D}) = \{ \psi \in H^\infty(\mathbb{D}) : \|\psi\|_\infty := \sup_{z \in \mathbb{D}} |\psi(z)| \leq 1 \}.$$ The elements of $S(\mathbb{D})$ are called Schur functions. The aim of this paper is to answer the following question concerning invariant subspaces of composition operators: Characterize $\varphi$, holomorphic self maps of $\mathbb{D}$, and inner functions $\theta \in H^\infty(\mathbb{D})$ such that the Beurling type invariant subspace $\theta H^2$ is an invariant subspace for $C_\varphi$. We prove the following result:

$$C_\varphi(\theta H^2) \subseteq \theta H^2$$

if and only if

$$\frac{\theta \circ \varphi}{\theta} \in S(\mathbb{D}).$$

This classification also allows us to recover or improve some known results on Beurling type invariant subspaces of composition operators.

1. Introduction

The invariant subspace problem [8], one of the most important open problems in linear analysis, asks if every bounded linear operator on a separable Hilbert space has a non-trivial closed invariant subspace. This problem has an equivalent form which turns it into a more concrete function theoretic problem. To be more specific, let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$, let $H^2$ denote the Hardy space on $\mathbb{D}$ and let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic self map of $\mathbb{D}$. The composition operator $C_\varphi$ on $H^2$ is defined by $C_\varphi f = f \circ \varphi$, that is

$$(C_\varphi f)(z) = f(\varphi(z)),$$

for all $f \in H^2$ and $z \in \mathbb{D}$. Littlewood’s subordination principle [17] implies that $C_\varphi$ is a bounded operator on $H^2$ and

$$\|C_\varphi\| \leq \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}.$$
In [15, 16], Nordgren, Rosenthal and Wintrobe proved that the classical invariant subspace problem has a positive solution if and only if any minimal invariant subspace for $C_\varphi$ that acts on $H^2$ and is induced by a hyperbolic automorphism $\varphi$ of $\mathbb{D}$ is one-dimensional. Therefore, classification of closed invariant subspaces of $C_\varphi$ is far too difficult to tackle in its full generality (however, see [14] for a complete description of invariant subspaces of $C_\varphi$ where $\varphi$ is a parabolic non-automorphism).

The situation changes if we consider shift invariant closed subspaces of $H^2$ instead of an arbitrary closed subspace of $H^2$. In this context, the problem of joint invariant subspaces of $C_\varphi$ and $M_z$ on $H^2$ (here $M_z$ denote the shift operator or the multiplication operator induced by the coordinate function $z$ on $H^2$) was considered by Cowen and Wahl [5] and Matache [13] (also see Mahvidi [11] and Jones [9]).

The joint invariant subspace problem of Cowen and Wahl and Matache introduces a lot of additional structure of holomorphic self maps and inner functions. Indeed, the notion of inner functions arose as a result of the representations of shift invariant subspaces of the Hardy space. Recall that an inner function is a function $\theta \in H^2$ whose radial limits have modulus one a.e. on $\partial \mathbb{D}$. A classical result of A. Beurling [1] classifies the invariant subspaces of the shift operator as follows:

**Beurling's Theorem:** Let $\mathcal{S} \neq \{0\}$ be a closed subspace of $H^2$. Then $\mathcal{S}$ is invariant under $M_z$ if and only if there exists an inner function $\theta$ (unique up to a scalar factor of unit modulus) such that

$$\mathcal{S} = \theta H^2.$$

Among many other results, Matache [13] proved that for every holomorphic self map $\varphi$ of $\mathbb{D}$ there exists a non-trivial $M_z$-invariant closed subspace $\mathcal{S} \subsetneq H^2$ (depending on $\varphi$) such that $C_\varphi \mathcal{S} \subseteq \mathcal{S}$ (also see Theorem 3.5 for a new proof).

Typical results and proofs in this direction (including the ones mentioned above) often involves analytic properties of $\varphi$ like (Denjoy-Wolff) fixed points and derivative of $\varphi$ at fixed points. However, due to the complex classificational structure of (bi-)holomorphic self maps of $\mathbb{D}$, most known results are case-specific. But, from a more general point of view, we prove the following result: Let $\varphi$ be a holomorphic self map of $\mathbb{D}$, and let $\theta \in H^\infty(\mathbb{D})$ be an inner function. Then, the Beurling type invariant subspace $\theta H^2$ is invariant under $C_\varphi$ (that is, $C_\varphi(\theta H^2) \subseteq \theta H^2$) if and only if

$$\frac{\theta \circ \varphi}{\theta} \in \mathcal{S}(\mathbb{D}).$$

Here $\mathcal{S}(\mathbb{D})$ denote the set of all functions that are holomorphic and bounded by one in modulus on $\mathbb{D}$, that is

$$\mathcal{S}(\mathbb{D}) = \{ \psi \in H^\infty(\mathbb{D}) : \|\psi\|_\infty := \sup_{z \in \mathbb{D}} |\psi(z)| \leq 1 \}.$$

The set $\mathcal{S}(\mathbb{D})$ is known as the *Schur class* and the elements of $\mathcal{S}(\mathbb{D})$ are called *Schur functions*.

The proof of the above result, as presented in Section 2, is a simple application of Riesz factorization theorem for $H^2$ functions. Moreover, it is curious to note that several variants of the above result have been used, implicitly, in a number of constructions and proofs in the existing literature (see for instance [5, 9, 11, 13]). In Section 3, we present this point of
view by recovering and improving some known results. We also outline some rather direct
applications of our approach and present a number of examples and counter-examples.

In the final section, we point out and correct an error in a corollary of Jones [9]. On
the contrary to the claim of Part 1 of [9, Corollary 1], in Theorem 4.1 we prove that for a
parabolic automorphism \( \varphi \) of \( \mathbb{D} \), the closed subspace \( B_z H^2 \) is invariant under \( C_\varphi \), where \( B_z \) is the Blaschke product corresponding to the orbit \( \{ \varphi_m(z) \}_{m \geq 0} \) and \( z \in \mathbb{D} \) (here \( \varphi_m \) denotes the composition of \( \varphi \) with itself \( m \) times).

For general theory of composition operators on \( H^2 \) we refer the reader to Cowen [2] and
the books by Cowen and MacCluer [3] and Shapiro [17].

2. Invariant subspaces

We begin by recalling basic facts about Hardy space and bounded holomorphic functions
on \( \mathbb{D} \) and refer the reader to Duren [7, Chapter 2] for a more detailed exposition.

Let \( \mathcal{O}(\mathbb{D}) \) denote the set of all holomorphic functions on \( \mathbb{D} \). We define the Hardy space \( H^2 \) as the set of all functions \( f \in \mathcal{O}(\mathbb{D}) \) such that

\[
\| f \|_2 := \sup_{0 \leq r < 1} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, dt \right)^{\frac{1}{2}} < \infty.
\]

It is well known (due to Fatou’s theorem) that for \( f \in H^2 \), the radial limit

\[
\tilde{f}(e^{it}) := \lim_{r \to 1^-} f(re^{it}),
\]

exists almost everywhere and \( \tilde{f} \in L^2(\partial \mathbb{D}) \) (with respect to the Lebesgue measure on \( \partial \mathbb{D} \)).

In what follows, we will identify \( f \) with \( \tilde{f} \) and regard \( H^2 \) as the closed subspace of \( L^2(\partial \mathbb{D}) \). Therefore

\[
H^2 = \overline{\mathbb{C} z^{\infty}} L^2(\partial \mathbb{D}),
\]

and

\[
\langle f, g \rangle_{H^2} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{g(e^{it})} \, dt \quad (f, g \in H^2).
\]

The space \( H^\infty(\mathbb{D}) \) consists of all bounded functions \( \psi \in \mathcal{O}(\mathbb{D}) \). Clearly \( H^\infty(\mathbb{D}) \subseteq H^2 \), and

\( H^\infty(\mathbb{D}) \) is a Banach algebra with respect to the uniform norm. Therefore, \( \mathcal{S}(\mathbb{D}) \) is the closed unit ball of \( H^\infty(\mathbb{D}) \). It is also worth noting that (cf. [12, Corollary 1.1.24])

\[
H^2 \cap L^\infty(\partial \mathbb{D}) = H^\infty(\mathbb{D}).
\]

Recall again that a function \( \theta \in \mathcal{O}(\mathbb{D}) \) is said to be an inner function if \( |\theta(z)| \leq 1 \) for all \( z \in \mathbb{D} \) (in particular, \( \theta \in H^\infty(\mathbb{D}) \)) and its radial limit \( |\theta(e^{it})| = 1 \) a.e. on \( \partial \mathbb{D} \). Every inner function \( \theta \) can be factored into a Blaschke product and a singular inner function. That is

\[
\theta = BS,
\]

where the Blaschke product

\[
B(z) = z^m \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \cdot \frac{a_n - z}{1 - \overline{a_n}z} \quad (z \in \mathbb{D}),
\]
for some non-negative integer $m$, is constructed from the zeros of $\theta$ and the singular inner factor

$$S(z) = c \exp \left( - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t) \right) \quad (z \in \mathbb{D}),$$

for some unimodular constant $c$ and positive measure $\mu$ supported on a set of Lebesgue measure zero, has no zeros in $\mathbb{D}$. Along the same line, Riesz factorization theorem is enormously useful [7, Theorem 2.5]:

**Theorem 2.1. (Riesz factorization theorem)** Let $f$ be a non-zero function in $H^2$. Then there exist a Blaschke product $B$ and a function $g \in H^2$ such that $g(z) \neq 0$ for all $z \in \mathbb{D}$ and

$$f = Bg.$$  

Moreover, if $f \in H^\infty(\mathbb{D})$, then $g \in H^\infty(\mathbb{D})$ and $\|f\|_\infty = \|g\|_\infty$.

It is worth noticing that every Blaschke product is an inner function.

Denote by $Z(f)$ the zero set of a holomorphic function $f \in \mathcal{O}(\mathbb{D})$. The multiplicity (or, order) of $w \in Z(f)$ will be denoted by $\text{mult}_f(w)$.

We now return to invariant subspaces of composition operators. Throughout this article, $\varphi$ will denote a holomorphic self map of $\mathbb{D}$ and $\theta$ will denote an inner function in $H^\infty(\mathbb{D})$.

Suppose $C_\varphi(\theta H^2) \subseteq \theta H^2$. Then there exists $f \in H^2$ such that

$$C_\varphi(\theta 1) = \theta \circ \varphi = \theta f.$$  

This yields

$$Z(\theta) \subseteq Z(\theta \circ \varphi),$$

or equivalently

$$\varphi(Z(\theta)) \subseteq Z(\theta).$$

More generally, the following easy-to-see remarks adds additional illustration of the concept of zero sets.

**Remark 2.2.** (1) If $\theta H^2$ is an invariant subspace for $C_\varphi$, then

$$\text{mult}_\theta(\alpha) \leq \text{mult}_{\theta \circ \varphi}(\alpha),$$

for all $\alpha \in Z(\theta)$.

(2) The quotient $\frac{\theta \circ \varphi}{\theta}$ defines a holomorphic function on $\mathbb{D}$ if and only if

$$\text{mult}_\theta(\alpha) \leq \text{mult}_{\theta \circ \varphi}(\alpha),$$

for all $\alpha \in Z(\theta)$.

The first inequality is merely a necessary condition for $\theta H^2$ to be $C_\varphi$-invariant and is not a sufficient condition. A converse of the first remark will be discussed in the next section (see Corollary 3.2). Moreover, it is equally evident that the problem of determining effective sufficient conditions, in terms of zero sets of holomorphic functions, is more elusive for zero-free holomorphic functions (like singular inner functions).

Now we are ready to present the central result of this paper.

**Theorem 2.3.** The following statements are equivalent:
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(a) $\theta H^2$ is an invariant subspace for $C_\varphi$.

(b) $\frac{\theta \circ \varphi}{\theta} \in S(D)$.

Proof. (a) $\Rightarrow$ (b): Suppose $\theta H^2$ is an invariant subspace for $C_\varphi$. By Remark 2.2, we see that

$$\frac{\theta \circ \varphi}{\theta} \in O(D).$$

Since $\theta \circ \varphi \in \theta H^2$, there exists $f \in H^2$ such that

$$\theta \circ \varphi = \theta f.$$

It follows that

$$f = \frac{\theta \circ \varphi}{\theta} \in H^2.$$

Now by Theorem 2.1, there exist a function $g_1 \in H^\infty(D)$ and a Blaschke product $B_1$ (note that $B_1(z) \equiv 1$ if $Z(\theta) = \emptyset$) such that $g_1(z) \neq 0$ for all $z \in D$ and

$$\theta = B_1 g_1.$$

Since $Z(\theta) \subseteq Z(\theta \circ \varphi)$, again by Theorem 2.1, there exist a function $g_2 \in H^\infty(D)$ and a Blaschke product $B_2$ such that $g_2(z) \neq 0$ for all $z \in D$ and

$$\theta \circ \varphi = B_1 B_2 g_2.$$

Since $g_2 \in H^\infty(D)$ and $\|B_1 B_2\|_\infty = 1$, as $B_1 B_2$ is an inner function, it follows that

$$\|g_2\|_\infty = \|B_1 B_2 g_2\|_\infty = \|\theta \circ \varphi\|_\infty \leq 1.$$

Observe

$$f = \frac{\theta \circ \varphi}{\theta} = \frac{B_2 g_2}{g_1} \in H^2.$$

As $|g_1(e^{it})| = 1$ a.e., by taking the radial limit of both sides, we get

$$|f(e^{it})| = \left| \frac{g_2(e^{it}) B_2(e^{it})}{g_1(e^{it})} \right| = |g_2(e^{it})| \text{ a.e.}$$

Hence $f \in H^\infty(D)$ and $\|f\|_\infty = \|g_2\|_\infty \leq 1$. Therefore $f \in S(D)$.

(b) $\Rightarrow$ (a): Suppose $\frac{\theta \circ \varphi}{\theta} \in S(D)$. Then, there exists $f \in S(D)$ such that

$$\theta \circ \varphi = \theta f.$$

Suppose $h \in H^2$. Then

$$C_\varphi(\theta h) = (\theta \circ \varphi)(h \circ \varphi) = \theta f(h \circ \varphi).$$

On the other hand,

$$h \circ \varphi \in H^2,$$

since $C_\varphi$ is bounded. As $f \in H^\infty(D)$, we have $f(h \circ \varphi) \in H^2$ and hence $C_\varphi(\theta h) \in \theta H^2$. This completes the proof of the theorem. \qed
It is worth noting that the above proof depends on the Riesz factorization theorem on the Hardy space $H^2$. Thus, the above classification result is also valid for $H^p$ spaces on $\mathbb{D}$.

Now we proceed to prove a bounded extension problem. Recall that the Hardy space $H^2$ is also a reproducing kernel Hilbert space corresponding to the Szegö kernel

$$K(z, w) = (1 - z\bar{w})^{-1} \quad (z, w \in \mathbb{D}).$$

For each $w \in \mathbb{D}$, denote by $K(\cdot, w) \in H^2$ the kernel function at $w$:

$$\left(K(\cdot, w)\right)(z) = K(z, w) \quad (z \in \mathbb{D}).$$

The Szegö kernel has the following reproducing property:

$$f(w) = \langle f, K(\cdot, w) \rangle,$$

for all $f \in H^2$ and $w \in \mathbb{D}$. By using this property, one readily checks that

$$M_\psi^* K(\cdot, w) = \overline{\psi(w)} K(\cdot, w),$$

and

$$C_\varphi^* K(\cdot, w) = K(\cdot, \varphi(w)),$$

for all $w \in \mathbb{D}$ and $\psi \in H^\infty(\mathbb{D})$. Now we observe that $C_\varphi(\theta H^2) \subseteq \theta H^2$ if and only if

$$\text{ran } (C_\varphi M_\theta) \subseteq \text{ran } M_\theta,$$

which is, by Douglas range inclusion theorem [6, Theorem 1], equivalent to

$$C_\varphi M_\theta = M_\theta X,$$

or equivalently

$$X^* M_\theta^* = M_\theta^* C_\varphi^*,$$

for some bounded linear operator $X$ on $H^2$. Evaluating each side of the equation by the kernel function $K(\cdot, w)$, $w \in \mathbb{D}$, we get

$$X^* \left(\overline{\theta(w)} K(\cdot, w)\right) = \overline{\theta(\varphi(w))} K(\cdot, \varphi(w)).$$

After noting that $\{K(\cdot, w) : w \in \mathbb{D}\}$ is a total set in $H^2$, we have the following corollary to Theorem 2.3:

**Corollary 2.4.** The following statements are equivalent:

(a) $\theta H^2$ is an invariant subspace for $C_\varphi$.

(b) The map

$$A \left(\overline{\theta(w)} K(\cdot, w)\right) = \overline{\theta(\varphi(w))} K(\cdot, \varphi(w)) \quad (w \in \mathbb{D}),$$

extends to a bounded linear operator on $H^2$.

(c) $\frac{\theta \circ \varphi}{\theta} \in \mathcal{S}(\mathbb{D})$. 

3. Examples and Applications

We begin with a simple example about zero sets of $\varphi$ and $\theta \circ \varphi$.

**Example 3.1.** Let $\alpha \in \mathbb{D} \setminus \{0\}$ and suppose

$$\varphi(z) = \frac{\alpha - z}{1 - \alpha z} \quad (z \in \mathbb{D}).$$

Clearly, $\varphi$ is a holomorphic self map of $\mathbb{D}$ (in fact, $\varphi$ is an inner function and an element of the disc automorphism $\text{Aut}(\mathbb{D})$). Then $\theta = z^2 \varphi$ is an inner function. Since $(\varphi \circ \varphi)(z) = z$ for all $z \in \mathbb{D}$, it follows that

$$\theta \circ \varphi = z \varphi^2.$$

Hence

$$Z(\theta) = Z(\theta \circ \varphi) = \{0, \alpha\}.$$

On the other hand, if $\theta H^2$ is invariant under $C_\varphi$, then $C_\varphi \theta = \theta \circ \varphi \in \theta H^2$, that is

$$z \varphi^2 \in z^2 \varphi H^2.$$

Hence $\varphi \in zH^2$ implies $\varphi(0) = 0$, which contradicts the fact that $\varphi(0) = \alpha$. This allows us to conclude that $\theta H^2$ is not invariant under $C_\varphi$. Here we also note that

$$\text{mult}_\theta(0) = 2 > 1 = \text{mult}_{\theta \circ \varphi}(0),$$

and hence, the same conclusion directly follows from Remark 2.2.

Next, we apply Theorem 2.3 to a concrete example. Consider the atomic singular inner function

$$\theta(z) = \exp \left( \frac{z + 1}{z - 1} \right),$$

and suppose

and $\varphi(z) = z^2$,

for all $z \in \mathbb{D}$. Clearly $Z(\theta) = \emptyset = Z(\theta \circ \varphi)$. Moreover

$$\frac{\theta \circ \varphi}{\theta} = \exp \left( \frac{z^2 + 1}{z^2 - 1} - \frac{z + 1}{z - 1} \right) = \exp \left( \frac{2z}{1 - z^2} \right),$$

and so

$$\left( \frac{\theta \circ \varphi}{\theta} \right)(z) \to \infty,$$

as $z \to 1$ along the real axis. Hence

$$\frac{\theta \circ \varphi}{\theta} \notin H^\infty(\mathbb{D}).$$

By Theorem 2.3, this implies that $\theta H^2$ is not invariant under $C_\varphi$. However, see [5, Corollary 7] and also Theorem 3.4.

The above example also shows that the converse of part (1) of Remark 2.2 is not true in general. However, the converse holds for Blaschke products:

**Corollary 3.2.** Let $B$ be a Blaschke product and let $\varphi$ be a holomorphic self map of $\mathbb{D}$. Then the following statements are equivalent:
(1) $BH^2$ is invariant under $C_{\varphi}$.
(2) $\text{mult}_B(w) \leq \text{mult}_{B\circ \varphi}(w)$ for all $w$ in $Z(B)$.

Proof. In view of part (1) of Remark 2.2, we only need show (2) $\Rightarrow$ (1). In this case, by Theorem 2.1, there exist a Blaschke product $B_2$ and a function $g_2 \in H^\infty(\mathbb{D})$ such that $g_2(z) \neq 0$ for all $z \in \mathbb{D}$ and

$$B \circ \varphi = B B_2 g_2.$$ 

Arguing as in the proof of Theorem 2.3, we see that $g_2 \in S(\mathbb{D})$ and hence

$$\frac{B \circ \varphi}{B} = B_2 g_2 \in S(\mathbb{D}).$$

Therefore, by Theorem 2.3, $BH^2$ is invariant under $C_{\varphi}$. \qed

We refer to Cowen and Wahl [5, Lemma 8] for a particular case (where $\varphi$ is a non-constant and non-elliptic automorphism) of the above result. Also the special case of inner functions $\varphi$ is due to Jones [9, Lemma 1].

Now we briefly review the notion of fixed points of holomorphic self maps. Let $\varphi$ be a holomorphic self map of $\mathbb{D}$ and let $w \in \overline{\mathbb{D}}$. We say that $w$ is a fixed point [3, page 50] of $\varphi$ if

$$\lim_{r \to 1_-} \varphi(rw) = w.$$ 

By a well known result [3, page 51], if $w \in \partial \mathbb{D}$ is a fixed point of $\varphi$, then

$$\varphi'(w) = \lim_{r \to 1_-} \varphi'(rw),$$

exists as a positive real number or $+\infty$. Now let $\varphi$ be an automorphism of $\mathbb{D}$. We say that $\varphi$ is:

(1) \textit{elliptic} if it has one fixed point in $\mathbb{D}$ and one outside $\overline{\mathbb{D}}$,
(2) \textit{hyperbolic} if it has two distinct fixed points on $\partial \mathbb{D}$, and
(3) \textit{parabolic} if there is one fixed point of multiplicity two on $\partial \mathbb{D}$.

Next we recall the \textit{Denjoy-Wolff theorem}: Let $\varphi$ be a holomorphic self map of $\mathbb{D}$. If $\varphi$ is not an elliptic automorphism, then there exists $w \in \overline{\mathbb{D}}$ such that $\varphi_n$ (the composition of $\varphi$ with itself $n$ times) converges to the constant function $w$ uniformly on compact subsets of $\mathbb{D}$. Moreover, $\varphi(w) = w$ and (i) $|\varphi'(w)| < 1$ if $w \in \mathbb{D}$, and (ii) $0 < \varphi'(w) \leq 1$ if $w \in \partial \mathbb{D}$.

The point $w$ is referred to as the \textit{Denjoy-Wolff point} of $\varphi$. In connection with the notion of Denjoy-Wolff point and Denjoy-Wolff theorem, we refer the interested reader to [3, Chapter 2] (also see [4, 18]).

We now proceed to shift invariant subspaces generated by atomic singular inner functions. Here we aim at presenting a result by Cowen and Wahl [5, Theorem 6] in our setting. However, the technical part of the present proof is similar to the one in [5], and we will directly borrow some terminology and a computation from [5]. First we recall the following lemma (see [5, Lemma 4]). The proof is an application of Julia’s Lemma.

\textbf{Lemma 3.3.} \textit{Let $\varphi$ be a holomorphic self map of $\mathbb{D}$, $\varphi(1) = 1$ and let $\varphi'(1) \leq 1$. Then}

$$\text{Re} \left( \frac{\varphi(z) + 1}{\varphi(z) - 1} - \frac{z + 1}{z - 1} \right) \leq 0 \quad (z \in \mathbb{D}).$$
Theorem 3.4. Let \( \varphi \) be a holomorphic self map of \( \mathbb{D} \) and let \( \alpha > 0 \). Consider the atomic singular inner function \( \theta(z) = e^{\alpha \left( \frac{z+1}{z-1} \right)} \), \( z \in \mathbb{D} \). Then the following statements are equivalent:

(a) \( \varphi(1) = 1 \) and \( \varphi'(1) \leq 1 \), that is, 1 is the Denjoy-Wolff point of \( \varphi \).

(b) \( \frac{\theta \circ \varphi}{\theta} \in \mathcal{S}(\mathbb{D}) \).

Proof. (a) \( \Rightarrow \) (b): If 1 is the Denjoy-Wolff point of \( \varphi \), then by Lemma 3.3, it follows that

\[
\left| \frac{\theta(\varphi(z))}{\theta(z)} \right| = \left| e^{\alpha \left( \frac{\varphi(z)+1}{\varphi(z)-1} - \frac{z+1}{z-1} \right)} \right| = e^{\alpha \text{Re} \left( \frac{\varphi(z)+1}{\varphi(z)-1} - \frac{z+1}{z-1} \right)} \leq e^0 = 1,
\]

for all \( z \in \mathbb{D} \). We have then

\[
\frac{\theta \circ \varphi}{\theta} \in \mathcal{S}(\mathbb{D}).
\]

(b) \( \Rightarrow \) (a): Suppose (b) holds. Then, there exists \( M > 0 \) such that

\[
\text{Re} \left( \frac{\varphi(z)+1}{\varphi(z)-1} - \frac{z+1}{z-1} \right) \leq M,
\]

or equivalently,

\[
\text{Re} \left( \frac{1+z}{1-z} \right) \leq M + \text{Re} \left( \frac{1+\varphi(z)}{1-\varphi(z)} \right),
\]

for all \( z \in \mathbb{D} \). Then the left side of the above inequality approaches \( \infty \) as \( z \to 1^- \) along the real axis, and thus

\[
\varphi(1) = \lim_{r \to 1^-} \varphi(r) = 1.
\]

By proceeding along the same lines as the final part of the proof of [5, Theorem 6] we infer that 1 is the Denjoy-Wolff point of \( \varphi \). This completes the proof of the result.

As a consequence, we recover the result of Cowen and Wahl [5, Corollary 7] stating that \( e^{\alpha \left( \frac{z+1}{z-1} \right)} \mathcal{H}^2 \) is invariant under \( C_\varphi \) if and only if \( \varphi(1) = 1 \) and \( \varphi'(1) \leq 1 \) (that is, 1 is the Denjoy-Wolff point of \( \varphi \)).

We turn now to a remarkable theorem, due to Matache [13], that given a holomorphic self map \( \varphi \) of \( \mathbb{D} \), there exists an inner function \( \theta \in \mathcal{H}^\infty \) such that \( \theta \mathcal{H}^2 \subseteq \mathcal{H}^2 \) and

\[
C_\varphi(\theta \mathcal{H}^2) \subseteq \theta \mathcal{H}^2.
\]

This is one of the main results of the paper [13]. Here, we reprove Matache’s result. However, our proof is somewhat shorter and simpler.

But before presenting the result, we recall the notion of invariant subspace lattices of operators and make one additional useful observation: For a bounded linear operator \( T \) on a Hilbert space \( \mathcal{H} \) we denote by \( \text{Lat} \ T \) the lattice of \( T \), that is, the set of all closed invariant subspaces of \( T \).

Now, let \( \varphi \) is a holomorphic self map of \( \mathbb{D} \) and let \( a \in \partial \mathbb{D} \). Define \( \omega \) and \( \psi \), holomorphic self maps of \( \mathbb{D} \), by

\[
\omega(z) = \overline{a}z \quad \text{and} \quad \psi = \omega \circ \varphi \circ \omega^{-1},
\]

for all \( z \in \mathbb{D} \). It is easy to see that \( a \) is the Denjoy-Wolff point of \( \varphi \) if and only if 1 is the Denjoy-Wolff point of \( \psi \). Moreover, if \( \theta \) is an inner function, then \( \theta \mathcal{H}^2 \in \text{Lat} \ C_\psi \) if and only if
(by Theorem 2.3) \( \theta \circ \psi = \theta g \) for some \( g \in S(\mathbb{D}) \). On the other hand, \( \theta \circ \psi = \theta \circ (\omega \circ \varphi \circ \omega^{-1}) \) and \( g \circ \omega \in S(\mathbb{D}) \). Hence
\[
(\theta \circ \omega) \circ \varphi = \theta g \circ \omega = (\theta \circ \omega)(g \circ \omega),
\]
implies, again by Theorem 2.3, that \( (\theta \circ \omega)H^2 \in \operatorname{Lat} C_\varphi \). In summary, we have the following:
(i) \( \theta H^2 \in \operatorname{Lat} C_\psi \) if and only if \( (\theta \circ \omega)H^2 \in \operatorname{Lat} C_\varphi \), and (ii) \( a \) is the Denjoy-Wolff point of \( \varphi \) if and only if 1 is the Denjoy-Wolff point of \( \psi \).

**Theorem 3.5.** If \( \varphi \) is a holomorphic self map of \( \mathbb{D} \), then there exists a non-zero closed subspace \( S \subseteq H^2 \) such that
\[
S \in \operatorname{Lat} C_\varphi \cap \operatorname{Lat} M_z.
\]

**Proof.** Suppose \( \varphi \) has a fixed point \( \alpha \) in \( \mathbb{D} \). Consider the inner function (Blaschke factor)
\[
\theta(z) = \frac{\alpha - z}{1 - \bar{\alpha}z} \quad (z \in \mathbb{D}).
\]
Clearly, \( \alpha \) is also a zero of \( \varphi \) with multiplicity at least one, and so Corollary 3.2, we have \( C_\varphi(\theta H^2) \subseteq \theta H^2 \).

Finally, suppose \( \varphi \) does not have any fixed point in \( \mathbb{D} \). Then the Denjoy-Wolff point \( a \) of \( \varphi \) must necessarily lie on \( \partial \mathbb{D} \), and so by Theorem 3.4 (along with the remark above), \( e^{\alpha \left( \frac{z + \bar{a}}{z - \bar{a}} \right)}H^2 \) is invariant under \( C_\varphi \) for all \( \alpha > 0 \). This completes the proof of the theorem. \( \square \)

In the case of elliptic automorphisms of \( \mathbb{D} \), Theorem 2.3 is more definite:

**Theorem 3.6.** Let \( \theta \) be an inner function and \( \varphi \) be an elliptic automorphism of \( \mathbb{D} \). Then the following statements are equivalent:
(a) \( \theta H^2 \) is invariant under \( C_\varphi \).
(b) \( \frac{\theta \circ \varphi}{\theta} \) is unimodular constant.

Moreover, in this case, if \( w \in \mathbb{D} \) is the unique fixed point of \( \varphi \), then
\[
\frac{\theta \circ \varphi}{\theta} = \begin{cases} 
(\varphi'(w))^{\text{mult}_\theta(w)} & \text{if } w \in Z(\theta) \\
1 & \text{otherwise.}
\end{cases}
\]

**Proof.** Suppose \( \theta H^2 \) is invariant under \( C_\varphi \). By Theorem 2.3, there exists \( f \in S(\mathbb{D}) \) such that \( f = \frac{\theta \circ \varphi}{\theta} \). Suppose \( w \in \mathbb{D} \) is the unique fixed point of \( \varphi \). Define
\[
b_w(z) = \frac{w - z}{1 - \bar{w}z} \quad (z \in \mathbb{D}).
\]
Now, if \( w \in Z(\theta) \), then there exists an inner function \( \theta_1 \) such that \( \theta_1(w) \neq 0 \) and
\[
\theta(z) = (b_w(z))^{\text{mult}_\theta(w)} \theta_1(z) \quad (z \in \mathbb{D}).
\]
Using this we get
\[
f = \left( \frac{b_w \circ \varphi}{b_w} \right)^{\text{mult}_\theta(w)} \frac{\theta_1 \circ \varphi}{\theta_1}.
\]
On the other hand
\[ \lim_{z \to w} \frac{b_w \circ \varphi}{b_w} = \varphi'(w), \]
and \( \varphi(w) = w \) implies that
\[ f(w) = \varphi'(w)^{\text{mult}_\theta(w)}. \]
But, since \( \varphi \) is an elliptic automorphism, we have that \( |\varphi'(w)| = 1 \), and hence \( |f(w)| = 1 \).
Then the maximum modulus principle implies that \( f \equiv \varphi'(w)^{\text{mult}_\theta(w)} \). Clearly, if \( \theta(w) \neq 0 \), then \( f \equiv f(w) = 1 \).
The converse part follows directly from Theorem 2.3.

\[ \square \]

The same proof yields the following result:

**Corollary 3.7.** Let \( \varphi \) be a holomorphic self map of \( \mathbb{D} \) and let \( w \in \mathbb{D} \) be the fixed point of \( \varphi \). Let \( \theta \) be an inner function and suppose that \( \theta(w) \neq 0 \). Then \( \theta H^2 \) is invariant under \( C_\varphi \) if and only if \( \theta \circ \varphi = \theta \).

Now we prove a more definite result on non-automorphic holomorphic self maps.

**Corollary 3.8.** Let \( \varphi \) be a non-automorphic and holomorphic self map of \( \mathbb{D} \) and let \( w \in \mathbb{D} \) be the fixed point of \( \varphi \). Let \( \theta \) be an inner function and suppose that \( \theta(w) \neq 0 \). Then \( \theta H^2 \) is invariant under \( C_\varphi \) if and only if \( \theta \) is an unimodular constant. In particular, if \( \theta \) is a singular inner function, then \( \theta H^2 \) cannot be invariant under \( C_\varphi \).

**Proof.** Suppose \( \theta H^2 \) is invariant under \( C_\varphi \). By Corollary 3.7, \( \theta \circ \varphi = \theta \), and hence
\[ \theta \circ \varphi_m = \theta, \]
for all \( m \geq 1 \) (here \( \varphi_m \) denote the composition of \( \varphi \) with itself \( m \) times). Since \( \varphi_m \) converges uniformly to the constant function \( w \) on every compact subset of \( \mathbb{D} \), it follows that \( \theta \equiv \theta(w) \).
Since \( \theta \) is an inner function, we see that \( \theta(w) \) is a unimodular constant. The converse part again follows from Theorem 2.3. \[ \square \]

**4. On Corollary 1 and Concluding remarks**

We are mainly concerned here with Part 1 of [9, Corollary 1]: “If \( \varphi \) is a parabolic automorphism then \( \text{Lat} C_\varphi \) contains no non-trivial \( BH^p \).” This claim is incorrect. Indeed, on the contrary, we prove the following (as always \( \varphi_m \) denotes the composition of \( \varphi \) with itself \( m \) times):

**Theorem 4.1.** If \( \varphi \) be a parabolic automorphism of \( \mathbb{D} \), then (i) every orbit of \( \varphi \) is Blaschke summable, and (ii) for each \( z \in \mathbb{D} \) we have
\[ B_z H^2 \in \text{Lat} C_\varphi, \]
where \( B_z \) is the Blaschke product corresponding to the orbit \( \{ \varphi_m(z) \}_{m \geq 0} \).

**Proof.** Let \( \varphi \) be a parabolic automorphism of \( \mathbb{D} \). Suppose
\[ \omega(z) = \frac{1 + z}{1 - z} \quad (z \in \mathbb{D}). \]
Then \( w \) is a conformal map from \( \mathbb{D} \) onto the right half-plane \( \mathbb{H} \). Note that
\[
\omega^{-1}(s) = \frac{s - 1}{s + 1} \quad (s \in \mathbb{H}).
\]
Set
\[
\sigma = \omega \circ \varphi \circ \omega^{-1}.
\]
Then there exists a non-zero real number \( b \) such that
\[
\sigma(s) = s + ib \quad (s \in \mathbb{H}),
\]
by the Linear-Fractional Model Theorem (cf. [3, Section 2.4]). On the other hand,
\[
\varphi_m = \omega^{-1} \circ \sigma_m \circ \omega,
\]
for all \( m \), and hence
\[
1 - |\varphi_m(z)|^2 = 1 - \left| \omega^{-1}(\sigma_m(\omega(z))) \right|^2
\]
\[
= 1 - \frac{|\sigma_m(\omega(z)) - 1|^2}{|\sigma_m(\omega(z)) + 1|^2}
\]
\[
= \frac{4 \Re \left( \sigma_m(\omega(z)) \right)}{|\sigma_m(\omega(z)) + 1|^2},
\]
for all \( z \in \mathbb{D} \). Now we fix \( z \in \mathbb{D} \) and let \( \omega(z) = u + iv \). Then
\[
\sigma_m(\omega(z)) = \omega(z) + imb = u + i(mb + v),
\]
for all \( m \). It follows that
\[
|\sigma_m(\omega(z)) + 1|^2 = (1 + u)^2 + (mb + v)^2,
\]
and hence
\[
1 - |\varphi_m(z)|^2 = \frac{4u}{(mb + v)^2 + (1 + u)^2} \sim \frac{4u}{b^2m^2},
\]
for large \( m \). Therefore
\[
\sum_m 1 - |\varphi_m(z)|^2 < \infty \quad (z \in \mathbb{D}).
\]
Hence \( |\varphi_m(z)| \geq |\varphi_m(z)|^2 \) for all \( m \) yields that
\[
\sum_m 1 - |\varphi_m(z)| < \infty,
\]
that is, the orbit \( \{\varphi_m(z)\}_{m \geq 0} \) of \( \varphi \) at \( z \in \mathbb{D} \) is Blaschke summable. The second part follows from the first and Corollary 3.2. This completes the proof of the theorem. \( \square \)

From the above proof it is now evident that the estimate
\[
“1 - |\varphi_n(z)|^2 \sim \frac{c}{n},
\]
in the proof of Part 2 of [9, Lemma 3] is incorrect.
To conclude, we remark that a Schur function always admits a fractional linear transformation representations in the following sense: Given \( \varphi \in \mathcal{S}(\mathbb{D}) \), there exist a Hilbert space \( \mathcal{H} \) and a unitary (isometry/co-isometry/contractive) matrix
\[
U = \begin{bmatrix} a & B \\ C & D \end{bmatrix} : \mathbb{C} \oplus \mathcal{H} \rightarrow \mathbb{C} \oplus \mathcal{H},
\]
such that
\[
\varphi(z) = a + zB(I - zD)^{-1}C \quad (z \in \mathbb{D}).
\]
This point of view has proved extremely fruitful in understanding the structure of composition operators (cf. [10]). In the context of Theorem 2.3, a number of questions arise naturally at this point. For instance, a natural question arises as to whether one can relate the fractional linear transformations of \( \varphi \) and \( \theta \) with the fractional linear transformation of \( \frac{\theta \circ \varphi}{\theta} \). We hope to return to this theme in future work.

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