APPLICATIONS OF HILBERT MODULE APPROACH TO MULTIVARIABLE OPERATOR THEORY

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ABSTRACT. A commuting n-tuple $(T_1, \ldots, T_n)$ of bounded linear operators on a Hilbert space $\mathcal{H}$ associate a Hilbert module $\mathcal{H}$ over $\mathbb{C}[z_1, \ldots, z_n]$ in the following sense:

$$\mathbb{C}[z_1, \ldots, z_n] \times \mathcal{H} \to \mathcal{H}, \qquad (p, h) \mapsto p(T_1, \ldots, T_n)h,$$

where $p \in \mathbb{C}[z_1, \ldots, z_n]$ and $h \in \mathcal{H}$. A companion survey provides an introduction to the theory of Hilbert modules and some (Hilbert) module point of view to multivariable operator theory. The purpose of this survey is to emphasize algebraic and geometric aspects of Hilbert module approach to operator theory and to survey several applications of the theory of Hilbert modules in multivariable operator theory. The topics which are studied include: generalized canonical models and Cowen-Douglas class, dilations and factorization of reproducing kernel Hilbert spaces, a class of simple submodules and quotient modules of the Hardy modules over polydisc, commutant lifting theorem, similarity and free Hilbert modules, left invertible multipliers, inner resolutions, essentially normal Hilbert modules, localizations of free resolutions and rigidity phenomenon.

This article is a companion paper to "An Introduction to Hilbert Module Approach to Multivariable Operator Theory".

CONTENTS

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1. Introduction

The main motivation of Hilbert module approach to (multivariable) operator theory is fourfold: (1) elucidating role of Brown-Douglas-Fillmore theory (1973) to operator theory, (2) complex geometric interpretation of (a class of) reproducing kernel Hilbert spaces in the sense of Cowen-Douglas class (1978), (3) Hormandar's algebraic approach, in the sense of Koszul complex, to corona problem (1967) and (4) Taylor's notion of joint spectrum (1970), again in the sense of Koszul complex, in operator theory and function theory.

The general topics for this article is to survey several applications of complex geometry and commutative algebra, with a view of (Hilbert) module approach, to multivariable operator theory.

It is hoped that the formalism and observations presented here will provide better understanding of the problems in operator theory in a more general framework. The underlying idea of this survey is to:
(i) Study generalized canonical models and make connections between the multipliers and the quotient modules on one side, and the hermitian anti-holomorphic vector bundles and curvatures on the other side (see Section 2).
(ii) Determine when a quasi-free Hilbert module can be realized as a quotient module of a reproducing kernel Hilbert module (see Section 3).
(iii) Analyze Beurling type representation of (a class of) submodules and quotient modules of $H^2(\mathbb{D}^n)$, $n > 1$ (see Section 4).

(iv) Determine when a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ is similar to a quasi-free (or reproducing kernel) Hilbert module (see Section 5).
(v) Analyze similarity problem for generalized canonical models corresponding to corona pairs in $H^\infty(\mathbb{D})$ (see Section 6).
(vi) Analyze free resolutions of Hilbert modules and corresponding localizations and to relate with the Taylor's joint spectrum (see Section 7).
(vii) Study the rigidity properties, that is, to determine the lattice of submodules of a reproducing kernel Hilbert module, up to unitarily equivalence (see Section 8).
(viii) Determine when a Hilbert module is small, that is, when a (reproducing kernel) Hilbert module is essentially normal (see Section 9).

Notations and Conventions: (i) $\mathbb{N}$ = Set of all natural numbers including 0. (ii) $n \in \mathbb{N}$ and $n \geq 1$, unless specifically stated otherwise. (iii) $\mathbb{N}^n = \{\boldsymbol{k} = (k_1, \ldots, k_n) : k_i \in \mathbb{N}, i = 1, \ldots, n\}$. (iv) $\mathbb{C}^n$ = the complex n-space. (v) Ω : Bounded domain in $\mathbb{C}^n$. (vi) $\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$. (vii) $z^{\boldsymbol{k}} = z_1^{k_1} \cdots z_n^{k_n}$. (viii) $\mathcal{H}, \mathcal{K}, \mathcal{E}, \mathcal{E}_*$: Hilbert spaces. (ix) $\mathcal{B}(\mathcal{H}, \mathcal{K})$ = the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. (x) $T = (T_1, \ldots, T_n)$, n-tuple of commuting operators. (xi) $T^{\boldsymbol{k}} = T_1^{k_1} \cdots T_n^{k_n}$. (xii) $\mathbb{C}[\boldsymbol{z}] = \mathbb{C}[z_1, \ldots, z_n]$. (xiii) $\mathbb{D}^n = \{\boldsymbol{z} : |z_i| < 1, i = 1, \ldots, n\}$, $\mathbb{B}^n = \{\boldsymbol{z} : \|\boldsymbol{z}\|_{\mathbb{C}^n} < 1\}$. (xiv) $H^2_{\mathcal{E}}(\mathbb{D})$: $\mathcal{E}$-valued Hardy space over $\mathbb{D}$.

Throughout this note all Hilbert spaces are over the complex field and separable. Also for a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$ will be denoted by $P_{\mathcal{S}}$.

This article is a companion paper to "An Introduction to Hilbert Module Approach to Multivariable Operator Theory" (see [Sa14a]).

2. Generalized canonical models in the Cowen-Douglas class

Let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces and $\mathcal{H} \in B_1^*(\Omega)$. Moreover, assume $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathcal{H})$. Then the quotient module $\mathcal{H}_\Theta = \mathcal{H} \otimes \mathcal{E}_* / \Theta(\mathcal{H} \otimes \mathcal{E})$ is called the *generalized canonical model* associated with $\mathcal{H}$ and Θ. In other words, a generalized canonical model can be obtained by the resolution

$$\cdots \longrightarrow \mathcal{H} \otimes \mathcal{E} \xrightarrow{M_\Theta} \mathcal{H} \otimes \mathcal{E}_* \xrightarrow{\pi_\Theta} \mathcal{H}_\Theta \longrightarrow 0.$$

This is a generalization of Sz.-Nagy-Foias notion of canonical model (see Section 4 in [Sa14a]) to quotient modules of Hilbert modules.

Let $\mathcal{H} \in B_1^*(\mathbb{D})$ be a contractive Hilbert module over $A(\mathbb{D})$. Then $\mathcal{H}$ is in $C_{\cdot 0}$ class and the characteristic function $\Theta_{\mathcal{H}}$, in the sense of Sz.-Nagy and Foias, is a complete unitary invariant (see Section 4 in [Sa14a]). On the other hand, the curvature, in the sense of Cowen and Douglas, is another complete unitary invariant. A very natural question then arises: whether the characteristic function is connected with the curvature of the canonical model of $\mathcal{H}$.

One can formulate the above problem in a more general framework by replacing the Hardy module with a Hilbert module in $B_1^*(\Omega)$. More precisely, let $\mathcal{H} \in B_1^*(\Omega)$ and $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathcal{H})$. Suppose the quotient module $\mathcal{H}_\Theta = \mathcal{H} \otimes \mathcal{E}_* / \Theta(\mathcal{H} \otimes \mathcal{E})$ is in $B_m^*(\Omega)$. Does there exists any connection between the multipliers and curvature corresponding to the hermitian anti-holomorphic vector bundle $E_{\mathcal{H}_\Theta}$?

The purpose of this section is to study generalized canonical models and make connections between the multipliers and the quotient modules on one side, and the hermitian anti-holomorphic vector bundles and curvatures on the other side. Results concerning similarity and unitarily equivalence will be derived from these connections. The final subsection of this section will discuss some quotient modules of the familiar Hardy and weighted Bergman modules over $A(\mathbb{D})$ and trace basic facts about unitary equivalence and curvature equality.

2.1. **Generalized canonical models in $B^*_m(\Omega)$.** Generalized canonical models yields a deeper understanding of many issues in the study of Hilbert modules. However, the present approach we will assume only finite dimensional coefficient spaces with left invertible multiplier:

$$0 \longrightarrow \mathcal{H} \otimes \mathbb{C}^p \xrightarrow{M_\Theta} \mathcal{H} \otimes \mathbb{C}^q \xrightarrow{\pi_\Theta} \mathcal{H}_\Theta \longrightarrow 0,$$

where $p, q \in \mathbb{N}$ and $q > p$.

Theorem 2.1. *Let $1 \leq p < q$ and $\mathcal{H}_\Theta$ be a generalized canonical model corresponding to $\mathcal{H} \in B^*_1(\Omega)$ and a left invertible $\Theta \in \mathcal{M}_{\mathcal{B}(\mathbb{C}^p, \mathbb{C}^q)}(\mathcal{H})$. Then*

(1) $\mathcal{H}_\Theta \in B^*_{q-p}(\Omega)$, and

(2) $V^*_\Theta(\boldsymbol{w}) = (ran\ \Theta(\boldsymbol{w}))^\perp = ker\ \Theta(\boldsymbol{w})^*$ defines a hermitian anti-holomorphic vector bundle

$$V^*_\Theta = \coprod_{\boldsymbol{w} \in \Omega} V_\Theta(\boldsymbol{w})^*,$$

over Ω such that

$$E^*_{\mathcal{H}_\Theta} \cong E^*_{\mathcal{H}} \otimes V^*_\Theta.$$

In particular, if $q = p + 1$ then $\mathcal{H}_\Theta \in B^*_1(\Omega)$ and V_Θ is a line bundle.

Proof. Localizing the short exact sequence of Hilbert modules

$$0 \longrightarrow \mathcal{H} \otimes \mathbb{C}^p \xrightarrow{M_\Theta} \mathcal{H} \otimes \mathbb{C}^q \xrightarrow{\pi_\Theta} \mathcal{H}_\Theta \longrightarrow 0,$$

at $\boldsymbol{w} \in \Omega$, that is, taking quotients by $I_{\boldsymbol{w}} \cdot (\mathcal{H} \otimes \mathbb{C}^p)$, $I_{\boldsymbol{w}} \cdot (\mathcal{H} \otimes \mathbb{C}^q)$, and $I_{\boldsymbol{w}} \cdot \mathcal{H}_\Theta$, respectively, one obtain the following exact sequence (see Theorem 5.12 in [DoPa89])

$$\mathbb{C}_{\boldsymbol{w}} \otimes \mathbb{C}^p \xrightarrow{I_{\mathbb{C}_{\boldsymbol{w}}} \otimes \Theta(\boldsymbol{w})} \mathbb{C}_{\boldsymbol{w}} \otimes \mathbb{C}^q \xrightarrow{\pi_\Theta(\boldsymbol{w})} \mathcal{H}_\Theta / I_{\boldsymbol{w}} \cdot \mathcal{H}_\Theta \longrightarrow 0.$$

Since $\dim[\text{ran } \Theta(\boldsymbol{w})] = p$ for all $\boldsymbol{w} \in \Omega$, it follows that $\dim\Big[\ker\ \pi_\Theta(\boldsymbol{w})\Big] = p$, and thus

$$\dim\ \Big[\mathcal{H}_\Theta / I_{\boldsymbol{w}} \cdot \mathcal{H}_\Theta\Big] = \dim\ \Big[\mathcal{H}_\Theta \Big/ \Big(\sum_{i=1}^n (M_{z_i} - w_i I_{\mathcal{H}}) \mathcal{H}_\Theta \Big)\Big] = q - p,$$

that is,

$$\dim\Big[\bigcap_{i=1}^n \ker\ (M_{z_i} - w_i I_{\mathcal{H}})^*|_{\mathcal{H}_\Theta}\Big] = q - p,$$

for all $\boldsymbol{w} \in \Omega$.

The next step is to prove the following equality

$$\bigvee_{w \in \Omega} \{\ker\ (M_z - w I_{\mathcal{H}})^* \otimes \ker\ \Theta(w)^*\} = (\mathcal{H} \otimes \mathbb{C}^q) \ominus \text{ran } M_\Theta.$$

For simplicity of notation, assume that $q = p+1$. The proof of the general case is essentially the same as the one presented below (or see Theorem 3.3 in [DoKKSa14]). To this end, let $\{e_i\}_{i=1}^{p+1}$ be the standard orthonormal basis for $\mathbb{C}^{p+1}$ and let Δ_Θ be the formal determinant

$$\Delta_\Theta(\boldsymbol{w}) = \det \begin{bmatrix} e_1 & \theta_{1,1}(\boldsymbol{w}) & \cdots & \theta_{1,p}(\boldsymbol{w}) \\ \vdots & \vdots & \vdots & \vdots \\ e_{p+1} & \theta_{p+1,1}(\boldsymbol{w}) & \cdots & \theta_{p+1,p}(\boldsymbol{w}) \end{bmatrix} \in \mathbb{C}^{p+1},$$

where $\Theta(\boldsymbol{w}) = (\theta_{i,j}(\boldsymbol{w}))$ and $\boldsymbol{w} \in \Omega$. Since $\Theta(\boldsymbol{w})$ has a left inverse $\Psi(\boldsymbol{w})$, it follows that rank $\Theta(\boldsymbol{w}) = l$, and hence $\Delta_\Theta(\boldsymbol{w}) \neq 0$ for all $\boldsymbol{w} \in \Omega$. Set $\gamma_{\boldsymbol{w}} := k_{\boldsymbol{w}} \otimes \overline{\Delta_\Theta(\boldsymbol{w})} \neq 0$ for all $\boldsymbol{w} \in \Omega$, where $k_{\boldsymbol{w}}$ is any non-zero vector in $E_{\mathcal{H}}^*(\boldsymbol{w}) \subseteq \mathcal{H}$ and $\overline{\Delta_\Theta(\boldsymbol{w})}$ is the complex conjugate of $\Delta_\Theta(\boldsymbol{w})$ relative to the basis $\{e_i\}_{i=1}^{p+1}$. Moreover, consider the inner product of $\gamma_{\boldsymbol{w}}$ with

$$M_\Theta \begin{bmatrix} h_1 \\ \vdots \\ h_l \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^p \theta_{1,j} h_j \\ \vdots \\ \sum_{j=1}^p \theta_{p+1,j} h_j \end{bmatrix} \in \mathcal{H} \otimes \mathbb{C}^{p+1},$$

for $\{h_i\}_{i=1}^p \subseteq \mathcal{H}$. Evaluating the resulting functions at $\boldsymbol{w} \in \Omega$, one can conclude that these functions are the sum of the products of $h_i(\boldsymbol{w})$ with coefficients equal to the determinants of matrices with repeated columns and hence

$$\langle M_\Theta \begin{bmatrix} h_1 \\ \vdots \\ h_p \end{bmatrix}, \gamma_{\boldsymbol{w}} \rangle = 0.$$

Thus, $\gamma_{\boldsymbol{w}} \perp \operatorname{ran} M_\Theta$ for all $\boldsymbol{w} \in \Omega$. Also, it is easy to see that

$$(M_{z_i}^* \otimes I_{\mathbb{C}^{p+1}}) \gamma_{\boldsymbol{w}} = \bar{w}_i \gamma_{\boldsymbol{w}},$$

for $\boldsymbol{w} \in \Omega$ and for all $i = 1, \ldots, n$, so that

$$\bigcap_{i=1}^n \ker\, (M_{z_i} \otimes I_{\mathbb{C}^{p+1}} - w_i I_{\mathcal{H} \otimes \mathbb{C}^{p+1}})^*|_{\mathcal{H}_\Theta} = \mathbb{C} \cdot \gamma_{\boldsymbol{w}},$$

for all $\boldsymbol{w} \in \Omega$.

The next step is to prove that $\bigvee_{\boldsymbol{w} \in \Omega} k_{\boldsymbol{w}} \otimes \overline{\Delta_\Theta(w)} = \mathcal{H}_\Theta$. For all $g = \sum_{i=1}^{p+1} g_i \otimes e_i \in \mathcal{H} \otimes \mathbb{C}^{p+1}$ with $g \perp \gamma_{\boldsymbol{w}}$ for every $\boldsymbol{w} \in \Omega$, one must exhibit the representation $g_i(\boldsymbol{w}) = \sum_{j=1}^p \eta_j(\boldsymbol{w}) \theta_{ij}(\boldsymbol{w})$ for $i = 1, ..., p+1$, where the $\{\eta_j\}_{j=1}^p$ are functions in $\mathcal{H}$. Fix $\boldsymbol{w}_0 \in \Omega$. The assumption $\langle g, \gamma_{\boldsymbol{w}_0} \rangle = 0$ implies that

$$\det \begin{bmatrix} g_1(\boldsymbol{w}_0) & \theta_{1,1}(\boldsymbol{w}_0) & \cdots & \theta_{1,p}(\boldsymbol{w}_0) \\ \vdots & \vdots & \vdots & \vdots \\ g_{p+1}(\boldsymbol{w}_0) & \theta_{p+1,1}(\boldsymbol{w}_0) & \cdots & \theta_{p+1,p}(\boldsymbol{w}_0) \end{bmatrix} = 0. \tag{2.1}$$

Now view the matrix

$$\Theta(\boldsymbol{w}_0) = \begin{bmatrix} \theta_{1,1}(\boldsymbol{w}_0) & \cdots & \theta_{1,p}(\boldsymbol{w}_0) \\ \vdots & \vdots & \vdots \\ \theta_{p+1,1}(\boldsymbol{w}_0) & \cdots & \theta_{p+1,l}(\boldsymbol{w}_0) \end{bmatrix}$$

as the coefficient matrix of a linear system of $(p+1)$ equations in p unknowns. Since rank $\Theta(\boldsymbol{w}_0) = p$, some principal minor (which means taking some p rows) has a non-zero determinant. Hence, using Cramer's rule, one can uniquely solve for $\{\eta_j(\boldsymbol{w}_0)\}_{j=1}^p \subseteq \mathbb{C}^p$, at least for these p rows. But by (2.1), the solution must also satisfy the remaining equation. Hence we obtain the $\{\eta_j(\boldsymbol{w}_0)\}_{j=1}^p \subseteq \mathbb{C}^p$ and define

$$\xi(\boldsymbol{w}_0) = \sum_{j=1}^p \eta_j(\boldsymbol{w}_0) \otimes e_j,$$

so that

$$g(\boldsymbol{w}_0) = \Theta(\boldsymbol{w}_0)\xi(\boldsymbol{w}_0),$$

for each $\boldsymbol{w}_0 \in \Omega$. After doing this for each $\boldsymbol{w} \in \Omega$, we use the left inverse $\Psi(\boldsymbol{w})$ for $\Theta(\boldsymbol{w})$ to obtain

$$\xi(\boldsymbol{w}) = (\Psi(\boldsymbol{w})\Theta(\boldsymbol{w}))\xi(\boldsymbol{w}) = \Psi(\boldsymbol{w})(\Theta(\boldsymbol{w})\xi(\boldsymbol{w})) = \Psi(\boldsymbol{w})g(\boldsymbol{w}) \in \mathcal{H} \otimes \mathbb{C}^p.$$

Consequently, $\{\eta_j\}_{j=1}^p \subseteq \mathcal{H}$ and $\bigvee_{\boldsymbol{w}\in\Omega} \gamma_{\boldsymbol{w}} = \mathcal{H}_\Theta$.
Lastly, the closed range property of $\mathcal{H}_\Theta$ follows from that of $\mathcal{H}$. In particular, since the column operator $M_z^* - \bar{w} I_{\mathcal{H}}$ (see Definition 3.1 in [Sa14a]) acting on $\mathcal{H} \otimes \mathbb{C}^{l+1}$ has closed range and a finite dimensional kernel, it follows that restricting it to the invariant subspace $\mathcal{H}_\Theta \subseteq \mathcal{H} \otimes \mathbb{C}^{p+1}$ yields an operator with closed range and hence $\mathcal{H}_\Theta \in B_1^*(\Omega)$. ∎

The above result allows one to construct a wide range of Cowen-Douglas Hilbert modules over domains in $\mathbb{C}^n$.

2.2. **Curvature equality.** The following is a very useful equality for the class of generalized canonical models.

THEOREM 2.2. *Let $1 \leq p < q$ and $\mathcal{H}_\Theta$ be a generalized canonical model corresponding to $\mathcal{H} \in B_1^*(\Omega)$ and a left invertible $\Theta \in \mathcal{M}_{\mathcal{B}(\mathbb{C}^p,\mathbb{C}^q)}(\mathcal{H})$. Then*

$$\mathcal{K}_{E^*_{\mathcal{H}_\Theta}} - \mathcal{K}_{E^*_{\mathcal{H}}} = \mathcal{K}_{V^*_\Theta}.$$

Proof. To establish the curvature formula, first recall that the formula for the curvature of the Chern connection on an open subset $U \subseteq \Omega$ for a hermitian anti-holomorphic vector bundle is $\bar\partial[G^{-1}\partial G]$, where G is the Gramian for an anti-holomorphic frame $\{f_i\}_{i=1}^{q-p}$ for the vector bundle on U (cf. [CuSal84]). Assume that U is chosen so that the $\{k_{\boldsymbol{w}}\}$ for $\boldsymbol{w} \in \Omega$ can be chosen to be an anti-holomorphic function on U. Denoting by G_Θ the Gramian for the frame $\{k_{\boldsymbol{w}} \otimes f_i(\boldsymbol{w})\}_{i=1}^{q-p}$, $G_\Theta(\boldsymbol{w})$ equals the $(q-p)\times(q-p)$ matrix

$$G_\Theta(\boldsymbol{w}) = \big(\langle k_{\boldsymbol{w}} \otimes f_i(\boldsymbol{w}), k_{\boldsymbol{w}} \otimes f_j(\boldsymbol{w})\rangle\big)_{i,j=1}^{q-p} = \|k_{\boldsymbol{w}}\|^2 \big(\langle f_i(\boldsymbol{w}), f_j(\boldsymbol{w})\rangle\big)_{i,j=1}^{q-p} = \|k_{\boldsymbol{w}}\|^2 G_f(\boldsymbol{w}),$$

where G_f is the Gramian for the anti-holomorphic frame $\{f_i(\boldsymbol{w})\}_{i=1}^{q-p}$ for V_Θ^*. Then

$$\begin{aligned}\bar\partial[G_\Theta^{-1}(\partial G_\Theta)] &= \bar\partial[\frac{1}{\|k_{\boldsymbol{w}}\|^2}G_f^{-1}(\partial(\|k_{\boldsymbol{w}}\|^2 G_f))]\\ &= \bar\partial[\frac{1}{\|k_{\boldsymbol{w}}\|^2}G_f^{-1}(\partial(\|k_{\boldsymbol{w}}\|^2)G_f + \|k_{\boldsymbol{w}}\|^2\partial G_f)]\\ &= \bar\partial[\frac{1}{\|k_{\boldsymbol{w}}\|^2}\partial(\|k_{\boldsymbol{w}}\|^2) + G_f^{-1}\partial G_f]\\ &= \bar\partial[\frac{1}{\|k_{\boldsymbol{w}}\|^2}\partial(\|k_{\boldsymbol{w}}\|^2)] + \bar\partial[G_f^{-1}\partial G_f].\end{aligned}$$

Hence, expressing these matrices in terms of the respective frames and using the fact that the coordinates of a bundle and of its dual can be identified using the basis given by the frame, one has

$$\mathcal{K}_{E^*_{\mathcal{H}_\Theta}}(\boldsymbol{w}) - \mathcal{K}_{E^*_{\mathcal{H}}}(\boldsymbol{w}) \otimes I_{V^*_\Theta(\boldsymbol{w})} = I_{E^*_{\mathcal{H}}(\boldsymbol{w})} \otimes \mathcal{K}_{V^*_\Theta}(\boldsymbol{w}),$$

for all $\boldsymbol{w} \in U$. Since the coordinate free formula does not involve U, this completes the proof. ■

Based on Theorems 2.1 and 2.2, one can say that the isomorphism of quotient Hilbert modules is independent of the choice of the basic Hilbert module "building blocks" from which they were created.

Corollary 2.3. *Let $\mathcal{H}, \tilde{\mathcal{H}} \in B_1^*(\Omega)$ and $\Theta_1, \Theta_2 \in \mathcal{M}_{\mathcal{B}(\mathbb{C}^p,\mathbb{C}^q)}(\mathcal{H}) \cap \mathcal{M}_{\mathcal{B}(\mathbb{C}^p,\mathbb{C}^q)}(\tilde{\mathcal{H}})$ are left invertible with inverse in $\mathcal{M}_{\mathcal{B}(\mathbb{C}^q,\mathbb{C}^p)}(\mathcal{H}) \cap \mathcal{M}_{\mathcal{B}(\mathbb{C}^q,\mathbb{C}^p)}(\tilde{\mathcal{H}})$. Then $\mathcal{H}_{\Theta_1}$ is isomorphic to $\mathcal{H}_{\Theta_2}$ if and only if $\tilde{\mathcal{H}}_{\Theta_1}$ is isomorphic to $\tilde{\mathcal{H}}_{\Theta_2}$.*

Proof. The statement is obvious from the tensor product representations $E^*_{\mathcal{H}_{\Theta_i}} \cong E^*_{\mathcal{H}} \otimes V^*_{\Theta_i}$ and $E^*_{\tilde{\mathcal{H}}_{\Theta_i}} \cong E^*_{\tilde{\mathcal{H}}} \otimes V^*_{\Theta_i}$, for $i = 1, 2$; that is, isomorphic as hermitian anti-holomorphic bundles, and the result that $\mathcal{K}_{E^*_{\mathcal{H}_{\Theta_1}}} = \mathcal{K}_{E^*_{\mathcal{H}_{\Theta_2}}}$ if and only if $\mathcal{K}_{V^*_{\Theta_1}} = \mathcal{K}_{V^*_{\Theta_2}}$ as two forms. ■

In what follows, $\bigtriangledown^2$ denotes the Laplacian

$$\bigtriangledown^2 = 4\partial\bar\partial = 4\bar\partial\partial.$$

Theorem 2.4. *Let $\mathcal{H} \in B_1^*(\Omega)$ and $\Theta_1, \Theta_2 \in \mathcal{M}_{\mathcal{B}(\mathbb{C}^l,\mathbb{C}^{l+1})}(\mathcal{H})$ are left invertible multipliers. Then the quotient Hilbert modules $\mathcal{H}_{\Theta_1}$ and $\mathcal{H}_{\Theta_2}$ are isomorphic if and only if*

$$\bigtriangledown^2 log\, \|\Delta_{\Theta_1}\| = \bigtriangledown^2 log\, \|\Delta_{\Theta_2}\|,$$

where Δ_{Θ_i} is an anti-holomorphic cross section of $V^*_{\Theta_i}$ and $i = 1, 2$.

Proof. Choose a cross section $k_{\boldsymbol{w}}$ so that $k_{\boldsymbol{w}} \otimes \overline{\Delta_{\Theta_i}(\boldsymbol{w})}$, $i = 1, 2$, are anti-holomorphic local cross-sections of $E^*_{\mathcal{H}_{\Theta_1}}$ and $E^*_{\mathcal{H}_{\Theta_2}}$, respectively, over some open subset $U \subseteq \Omega$. Since every $\boldsymbol{w}_0 \in \Omega$ is contained in such an open subset U of Ω, by rigidity theorem [CoDo78] (or Theorem 3.2 in [Sa14a]), it follows that $\mathcal{H}_{\Theta_1} \cong \mathcal{H}_{\Theta_2}$ if and only if

$$\mathcal{K}_{E^*_{\mathcal{H}_{\Theta_1}}}(\boldsymbol{z}) = \mathcal{K}_{E^*_{\mathcal{H}_{\Theta_2}}}(\boldsymbol{z}),$$

for every $\boldsymbol{z} \in \Omega$ or, equivalently,

$$\bigtriangledown^2 \log \|\Delta_{\Theta_1}\| = \bigtriangledown^2 \log \|\Delta_{\Theta_2}\|,$$

by Theorem 2.2. This completes the proof. ∎

2.3. **Examples and applications.** The purpose of this subsection is to describe a class of simple examples of generalized canonical models in $B_1^*(\mathbb{D})$.

Let $\Theta \in H^\infty_{\mathcal{B}(\mathbb{C},\mathbb{C}^2)}(\mathbb{D})$ so that

$$\Theta(z) = \begin{bmatrix} \theta_1(z) \\ \theta_2(z) \end{bmatrix},$$

and $\theta_1, \theta_2 \in H^\infty(\mathbb{D})$ and $z \in \mathbb{D}$. Θ is said to satisfy the *corona condition* if there exists an $\epsilon > 0$ such that $|\theta_1(z)|^2 + |\theta_2(z)|^2 > \epsilon$ for all $z \in \mathbb{D}$ (see Section 5).

For the rest of this subsection, fix a corona pair $\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \in H^\infty_{\mathcal{B}(\mathbb{C},\mathbb{C}^2)}(\mathbb{D})$ and use the notation $\mathcal{H}$ to denote the Hardy, the Bergman, or a weighted Bergman module over $\mathbb{D}$. Consider the generalized canonical model $\mathcal{H}_\Theta$ corresponding to the exact sequence of Hilbert modules:

$$0 \longrightarrow \mathcal{H} \otimes \mathbb{C} \xrightarrow{M_\Theta} \mathcal{H} \otimes \mathbb{C}^2 \xrightarrow{\pi_\Theta} \mathcal{H}_\Theta \longrightarrow 0,$$

where the first map M_Θ is $M_\Theta f = \theta_1 f \otimes e_1 + \theta_2 f \otimes e_2$ and the second map π_Θ is the quotient Hilbert module map.

Note that by taking the kernel functions for $H^2(\mathbb{D})$ and $L^2_{a,\alpha}(\mathbb{D})$ as an anti-holomorphic cross section of bundles an easy computation shows that

$$\mathcal{K}_{E^*_{H^2(\mathbb{D})}}(z) = -\frac{1}{(1-|z|^2)^2},$$

and

$$\mathcal{K}_{E^*_{L^2_{a,\alpha}(\mathbb{D})}}(z) = -\frac{2+\alpha}{(1-|z|^2)^2}.$$

The following is immediate consequence of Theorems 2.1 and 2.2.

THEOREM 2.5. *For* $\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ *satisfying the corona condition,* $\mathcal{H}_\Theta \in B_1^*(\mathbb{D})$ *and*

$$\mathcal{K}_{E^*_{\mathcal{H}_\Theta}}(w) = \mathcal{K}_{E^*_{\mathcal{H}}}(w) - \frac{1}{4} \bigtriangledown^2 \, log\,(|\theta_1(w)|^2 + |\theta_2(w)|^2). \qquad (w \in \mathbb{D}) \tag{2.2}$$

THEOREM 2.6. *Let* $\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ *and* $\Phi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$ *satisfy the corona condition. The quotient Hilbert modules* $\mathcal{H}_\Theta$ *and* $\mathcal{H}_\Phi$ *are isomorphic if and only if*

$$\bigtriangledown^2 log \frac{|\theta_1(z)|^2 + |\theta_2(z)|^2}{|\varphi_1(z)|^2 + |\varphi_2(z)|^2} = 0. \qquad (z \in \mathbb{D})$$

Proof. Since $\mathcal{H}_\Theta, \mathcal{H}_\Phi \in B_1^*(\mathbb{D})$, they are isomorphic if and only if $\mathcal{K}_{E^*_{\mathcal{H}_\Theta}}(w) = \mathcal{K}_{E^*_{\mathcal{H}_\Phi}}(w)$ for all $w \in \mathbb{D}$. But note that (2.2) and an analogous identity for Φ hold, where the θ_i are replaced with the φ_i. Since both Θ and Φ satisfy the corona condition, the result then follows. ∎

THEOREM 2.7. *Suppose that* $\Theta = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$ *and* $\Phi = \begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}$ *satisfy the corona condition. The quotient Hilbert modules* $(L^2_{a,\alpha}(\mathbb{D}))_\Theta$ *and* $(L^2_{a,\beta}(\mathbb{D}))_\Phi$ *are isomorphic if and only if* $\alpha = \beta$ *and*

$$\bigtriangledown^2 log\,\frac{|\theta_1(z)|^2 + |\theta_2(z)|^2}{|\varphi_1(z)|^2 + |\varphi_2(z)|^2} = 0. \qquad (z \in \mathbb{D})$$

Proof. Since

$$\mathcal{K}_{E^*_{(L^2_{a,\alpha}(\mathbb{D}))_\Theta}}(w) = -\frac{2+\alpha}{(1-|w|^2)^2} - \frac{1}{4}\bigtriangledown^2 \log\left(|\theta_1(w)|^2 + |\theta_2(w)|^2\right),$$

and

$$\mathcal{K}_{E^*_{(L^2_{a,\beta}(\mathbb{D}))_\Phi}}(w) = -\frac{2+\beta}{(1-|w|^2)^2} - \frac{1}{4}\bigtriangledown^2 \log\left(|\varphi_1(w)|^2 + |\varphi_2(w)|^2\right),$$

by (2.2), one implication is obvious. For the other one, suppose that $(L^2_{a,\alpha}(\mathbb{D}))_\Theta$ is isomorphic to $(L^2_{a,\beta}(\mathbb{D}))_\Phi$ so that the curvatures coincide. Observe next that

$$\frac{4(\beta-\alpha)}{(1-|w|^2)^2} = \bigtriangledown^2 \log\frac{|\theta_1(w)|^2 + |\theta_2(w)|^2}{|\varphi_1(w)|^2 + |\varphi_2(w)|^2}.$$

Since a function f with $\bigtriangledown^2 f(z) = \frac{1}{(1-|z|^2)^2}$ for all $z \in \mathbb{D}$ is necessarily unbounded, one arrives at a contradiction, unless $\alpha = \beta$ (see Lemma 2.8 below). This is due to the assumption that the bounded functions Θ and Φ satisfy the corona condition. ■

LEMMA 2.8. *There is no bounded function* f *defined on the unit disk* $\mathbb{D}$ *that satisfies* $\bigtriangledown^2 f(z) = \frac{1}{(1-|z|^2)^2}$ *for all* $z \in \mathbb{D}$.

Proof. Suppose that such f exists. Since $\frac{1}{4}\bigtriangledown^2 [(|z|^2)^m] = \partial\bar\partial[(|z|^2)^m] = m^2(|z|^2)^{m-1}$ for all $m \in \mathbb{N}$, one see that for

$$g(z) := \frac{1}{4}\sum_{m=1}^{\infty} \frac{|z|^{2m}}{m} = -\frac{1}{4}\log\left(1-|z|^2\right),$$

$\bigtriangledown^2 g(z) = \frac{1}{(1-|z|^2)^2}$ for all $z \in \mathbb{D}$. Consequently, $f(z) = g(z) + h(z)$ for some harmonic function h. Since the assumption is that f is bounded, there exists an $M > 0$ such that $|g(z) + h(z)| \le M$ for all $z \in \mathbb{D}$. It follows that

$$\exp\left(h(z)\right) \le \exp\left(-g(z) + M\right) = (1-|z|^2)^{\frac{1}{4}} \exp\left(M\right),$$

and letting $z = re^{i\theta}$, we have $\exp\left(h(re^{i\theta})\right) \le (1-r^2)^{\frac{1}{4}}\exp\left(M\right)$. Thus $\exp\left(h(re^{i\theta})\right) \to 0$ uniformly as $r \to 1^-$, and hence $\exp h(z) \equiv 0$. This is due to the maximum modulus principle because $\exp h(z) = |\exp(h(z) + i\tilde{h}(z))|$, where $\tilde{h}$ is a harmonic conjugate for h. This leads to a contradiction, and the proof is complete. ■

THEOREM 2.9. *For* $\Theta = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$ *and* $\Phi = \begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}$ *satisfying the corona condition,* $(H^2(\mathbb{D}))_\Theta$ *cannot be isomorphic to* $(L^2_{a,\alpha}(\mathbb{D}))_\Phi$.

Proof. By identity (4.2), one can conclude that $(H^2)_\Theta$ is isomorphic to $(A^2_\alpha)_\Phi$ if and only if

$$\frac{4(1+\alpha)}{(1-|w|^2)^2} = \bigtriangledown^2 \log \frac{|\varphi_1(w)|^2 + |\varphi_2(w)|^2}{|\theta_1(w)|^2 + |\theta_2(w)|^2}.$$

But according to Lemma 4.6, this is impossible unless $\alpha = -1$. ∎

Further results and comments:

(1) Let $\mathcal{E}$ and $\mathcal{E}_*$ be two Hilbert spaces and $\Theta \in \mathcal{O}(\Omega, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$. One can define the holomorphic kernel and co-kernel bundles with fibers $\ker \Theta(\boldsymbol{w})$ and $\operatorname{coker} \Theta(\boldsymbol{w}) = \mathcal{E}_*/\Theta(\boldsymbol{w})\mathcal{E}$ for $\boldsymbol{w} \in \Omega$, respectively, whenever it make sense. Moreover, related Hilbert modules with $\mathcal{H} \in B^*_m(\Omega)$ can be defined for an arbitrary $m \geq 1$. Here consideration is restricted to the "simplest" case, when Θ is left invertible, and obtain some of the most "direct" possible results.

(2) Let $\mathcal{H} \in B^*_m(\mathbb{D})$ be a contractive Hilbert module over $A(\mathbb{D})$. Then one can prove that $\mathcal{H}$ is in the $C_{\cdot 0}$ class. In this case, the connection between the characteristic function $\Theta_{\mathcal{H}}$ and the curvature of the generalized canonical model, that is, the Sz.-Nagy-Foias canonical model $H^2_{\mathcal{D}_*}/\Theta_{\mathcal{H}} H^2_{\mathcal{D}}(\mathbb{D})$, was addressed earlier by Uchiyama in [U90]. His theory is instrumental in the study of generalized canonical models (cf. [KT09], [Sa13b]).

(3) All results presented in this section can be found in [DoKKSa12] and [DoKKSa14].

(4) In connection with this section, see also the work by Zhu [Zh00], Eschmeier and Schmitt [EsS14] and Kwon and Treil [KT09] and Uchiyama [U90] (see also [Sa13b]).

3. Dilation to quasi-free Hilbert modules

Recall that a Hilbert module $\mathcal{H}$ over $\mathbb{C}[z]$ is $C_{\cdot 0}$-contractive if and only if (see Section 4 in [Sa14a]) there exists a resolution of Hilbert modules

$$0 \longrightarrow \mathcal{F}_1 \stackrel{i}{\longrightarrow} \mathcal{F}_2 \stackrel{\pi}{\longrightarrow} \mathcal{H} \longrightarrow 0,$$

where $\mathcal{F}_i = H^2_{\mathcal{E}_i}(\mathbb{D})$ for some Hilbert spaces $\mathcal{E}_1$ and $\mathcal{E}_2$.

Now let $\mathcal{H}$ be a $C_{\cdot 0}$-contractive Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ (that is, $M_i \in C_{\dot{0}}$ for each i) and $n \geq 2$. If one attempts to obtain a similar resolution for $\mathcal{H}$, then one quickly runs into trouble. In particular, if $n > 2$ then Parrott's example [Pu94] shows that, in general, an isometric dilation need not exist. On the other hand, a pair of commuting contractions is known to have an isometric dilation [An63], that is, a resolution exists for contractive Hilbert module over $\mathbb{C}[z_1, z_2]$. However, such dilations are not necessarily unique, that is, one can not expect that $\mathcal{F}_2$ to be a free module $H^2(\mathbb{D}^2) \otimes \mathcal{E}_2$.

The purpose of this section is to study the following problem: Let $\mathcal{R} \subseteq \mathcal{O}(\Omega, \mathbb{C})$ be a reproducing kernel Hilbert module over $A(\Omega)$ and $\mathcal{M}$ be a quasi-free Hilbert module over $A(\Omega)$. Determine when $\mathcal{M}$ can be realized as a quotient module of the free module $\mathcal{R} \otimes \mathcal{E}$ for some coefficient space $\mathcal{E}$, that is, when $\mathcal{M}$ admits a free resolution

$$0 \longrightarrow \mathcal{S} \stackrel{i}{\longrightarrow} \mathcal{R} \otimes \mathcal{E} \stackrel{\pi}{\longrightarrow} \mathcal{M} \longrightarrow 0,$$

where $\mathcal{S}$ is a submodule of $\mathcal{R} \otimes \mathcal{E}$.

Another important motivation for studying dilation to quasi-free Hilbert modules is to develop some connections between free resolutions, positivity of kernel functions and factorizations of kernel functions. Our main tool is to establish a close relationship between the kernel functions for the Hilbert modules in an exact sequence using localization.

3.1. **Factorization of reproducing kernels.** Let $\mathcal{R} \subseteq \mathcal{O}(\Omega, \mathbb{C})$ be a reproducing kernel Hilbert space and $\mathcal{H}$ be a quasi-free Hilbert module of multiplicity m over $\mathbb{C}[\boldsymbol{z}]$ or $A(\Omega)$ and $\mathcal{E}$ a Hilbert space. Then $\mathcal{R} \otimes \mathcal{E}$ being a dilation of $\mathcal{H}$ is equivalent to the exactness of the sequence of Hilbert modules

$$0 \longrightarrow \mathcal{S} \xrightarrow{\ i\ } \mathcal{R} \otimes \mathcal{E} \xrightarrow{\ \pi\ } \mathcal{H} \longrightarrow 0,$$

where the second map is the inclusion i and the third map is the quotient map π which is a co-isometry. The aim of this subsection is to relate the existence of an $\mathcal{R} \otimes \mathcal{E}$-dilation of a reproducing kernel Hilbert module $\mathcal{H}_K$ to the positivity of the kernel function K.

Theorem 3.1. *Let $\mathcal{R} \subseteq \mathcal{O}(\Omega, \mathbb{C})$ be a reproducing kernel Hilbert module with the scalar kernel function k and $\mathcal{H}$ be a quasi-free Hilbert module of multiplicity m over $A(\Omega)$ or $\mathbb{C}[\boldsymbol{z}]$. Then $\mathcal{R} \otimes \mathcal{E}$ is a dilation of $\mathcal{H}$ for some Hilbert space $\mathcal{E}$, if and only if there is a holomorphic map $\pi_{\boldsymbol{z}} \in \mathcal{O}(\Omega, \mathcal{L}(\mathcal{E}, l^2_m))$ such that*

$$K_{\mathcal{H}}(\boldsymbol{z}, \boldsymbol{w}) = k(\boldsymbol{z}, \boldsymbol{w}) \pi_{\boldsymbol{z}} \pi^*_{\boldsymbol{w}}. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \Omega)$$

Proof. Let $\mathcal{R} \otimes \mathcal{E}$ be a dilation of $\mathcal{H}$, that is,

$$0 \to \mathcal{S} \to \mathcal{R} \otimes \mathcal{E} \to \mathcal{H} \to 0.$$

Localizing the above exact sequence of Hilbert modules at $\boldsymbol{z} \in \Omega$ one arrives at

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S} & \xrightarrow{\ i\ } & \mathcal{R} \otimes \mathcal{E} & \xrightarrow{\ \pi\ } & \mathcal{H} & \longrightarrow & 0 \\
 & & \Big\downarrow N_{\boldsymbol{z}} & & \Big\downarrow P_{\boldsymbol{z}} & & \Big\downarrow Q_{\boldsymbol{z}} & & \\
 & & \mathcal{S}/I_{\boldsymbol{z}}\mathcal{S} & \xrightarrow[i_{\boldsymbol{z}}]{} & (\mathcal{R} \otimes \mathcal{E})/I_{\boldsymbol{z}}(\mathcal{R} \otimes \mathcal{E}) & \xrightarrow[\pi_{\boldsymbol{z}}]{} & \mathcal{H}/I_{\boldsymbol{z}}\mathcal{H} & \longrightarrow & 0
\end{array}$$

which is commutative with exact rows for all $\boldsymbol{w}$ in Ω (see [DoPa89]). Here $N_{\boldsymbol{z}}, P_{\boldsymbol{z}}$ and $Q_{\boldsymbol{z}}$ are the quotient module maps. Since one can identify $\mathcal{H}/I_{\boldsymbol{z}}\mathcal{H}$ with l^2_m and $(\mathcal{R} \otimes \mathcal{E})/I_{\boldsymbol{z}}(\mathcal{R} \otimes \mathcal{E})$ with $\mathcal{E}$, the kernel functions of $\mathcal{H}$ and $\mathcal{R} \otimes \mathcal{E}$ are given by $Q_{\boldsymbol{z}} Q^*_{\boldsymbol{w}}$ and $P_{\boldsymbol{z}} P^*_{\boldsymbol{w}}$, respectively. Moreover, since $Q_{\boldsymbol{w}} \pi = \pi_{\boldsymbol{w}} P_{\boldsymbol{w}}$ for all $\boldsymbol{w} \in \Omega$, it follows that

$$Q_{\boldsymbol{z}} \pi \pi^* Q_{\boldsymbol{w}} = \pi_{\boldsymbol{z}} P_{\boldsymbol{z}} P^*_{\boldsymbol{w}} \pi^*_{\boldsymbol{w}}. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \Omega)$$

Using the fact that $\pi\pi^* = I_{\mathcal{H}}$ and $P_{\boldsymbol{z}} P^*_{\boldsymbol{w}} = k(\boldsymbol{z}, \boldsymbol{w}) \otimes I_{\mathcal{E}}$, one can now conclude that

$$Q_{\boldsymbol{z}} Q^*_{\boldsymbol{w}} = k(\boldsymbol{z}, \boldsymbol{w}) \pi_{\boldsymbol{z}} \pi^*_{\boldsymbol{w}}. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \Omega)$$

Conversely, let the kernel function of the quasi-free Hilbert module $\mathcal{H}$ has the factorization

$$K_{\mathcal{H}}(\boldsymbol{z}, \boldsymbol{w}) = k(\boldsymbol{z}, \boldsymbol{w}) \pi_{\boldsymbol{z}} \pi_{\boldsymbol{w}}^*, \qquad (\boldsymbol{z}, \boldsymbol{w} \in \Omega)$$

for some function $\pi : \Omega \to \mathcal{L}(\mathcal{E}, l^2_m)$. Note that if the function π satisfies the above equality then it is holomorphic on Ω. Define a linear map $X : \mathcal{H} \to \mathcal{R} \otimes \mathcal{E}$ so that

$$X Q_{\boldsymbol{z}}^* \eta = P_{\boldsymbol{z}}^* \pi_{\boldsymbol{z}}^* \eta. \qquad (\eta \in l^2_m)$$

It then follows that

$$\langle X(Q_{\boldsymbol{w}}^* \eta), X(Q_{\boldsymbol{z}}^* \zeta) \rangle = \langle P_{\boldsymbol{w}}^* \pi_{\boldsymbol{w}}^* \eta, P_{\boldsymbol{z}}^* \pi_{\boldsymbol{z}}^* \zeta \rangle = \langle \pi_{\boldsymbol{z}} P_{\boldsymbol{z}} P_{\boldsymbol{w}}^* \pi_{\boldsymbol{w}}^* \eta, \zeta \rangle = \langle Q_{\boldsymbol{z}} Q_{\boldsymbol{w}}^* \eta, \zeta \rangle = \langle Q_{\boldsymbol{w}}^* \eta, Q_{\boldsymbol{z}}^* \zeta \rangle,$$

for all $\eta, \zeta \in l^2_m$. Therefore, since $\{Q_{\boldsymbol{z}}^* \eta : \boldsymbol{z} \in \Omega, \eta \in l^2_m\}$ is a total set of $\mathcal{H}$, that X extends to a bounded isometric operator. Moreover, by the reproducing property of the kernel function, it follows that

$$M_{z_i}^* X(Q_{\boldsymbol{z}}^* \eta) = M_{z_i}^* P_{\boldsymbol{z}}^* (\pi_{\boldsymbol{z}}^* \eta) = \bar{z}_i P_{\boldsymbol{z}}^* \pi_{\boldsymbol{z}}^* \eta = \bar{z}_i X(Q_{\boldsymbol{z}}^* \eta) = X Q_{\boldsymbol{z}}^* (\bar{z}_i \eta) = X M_{z_i}^* (Q_{\boldsymbol{z}}^* \eta),$$

for all $1 \leq i \leq n$ and $\eta \in l^2_m$. Hence, $X \in \mathcal{B}(\mathcal{H}, \mathcal{R} \otimes \mathcal{E})$ is a co-module map. ■

The following result is an application of the previous theorem.

THEOREM 3.2. *Let $\mathcal{H}$ be a quasi-free Hilbert module of finite multiplicity and $\mathcal{R}$ be a reproducing kernel Hilbert module over $A(\Omega)$ (or over $\mathbb{C}[\boldsymbol{z}]$). Let k be the kernel function of $\mathcal{R}$. Then $\mathcal{R} \otimes \mathcal{E}$ is a dilation of $\mathcal{H}$ for some Hilbert space $\mathcal{E}$ if and only if*

$$K_{\mathcal{H}}(\boldsymbol{z}, \boldsymbol{w}) = k(\boldsymbol{z}, \boldsymbol{w}) \tilde{K}(\boldsymbol{z}, \boldsymbol{w}), \qquad (\boldsymbol{z}, \boldsymbol{w} \in \Omega)$$

for some positive definite kernel $\tilde{K}$ over Ω. Moreover, if k^{-1} is defined, then the above conclusion is true if and only if $k^{-1} K_{\mathcal{H}}$ is a positive definite kernel.

Proof. The necessary part follows from the previous theorem by setting $\tilde{K}(\boldsymbol{z}, \boldsymbol{w}) = \pi_{\boldsymbol{z}} \pi_{\boldsymbol{w}}^*$. To prove the sufficiency part, let $K_{\mathcal{H}} = k \cdot \tilde{K}$ for some positive definite kernel $\tilde{K}$. We let $\mathcal{H}(\tilde{K})$ be the corresponding reproducing kernel Hilbert space and set $\mathcal{E} = \mathcal{H}(\tilde{K})$. Let

$$\pi_{\boldsymbol{z}} = ev_{\boldsymbol{z}} \in \mathcal{B}(\mathcal{E}, l^2_m) \qquad (\boldsymbol{z} \in \Omega)$$

be the evaluation operator for the reproducing kernel Hilbert space $\mathcal{H}(\tilde{K})$. Then

$$\tilde{K}(\boldsymbol{z}, \boldsymbol{w}) = \pi_{\boldsymbol{z}} \pi_{\boldsymbol{w}}^*. \qquad (\boldsymbol{z}, \boldsymbol{w} \in \Omega)$$

Consequently, by the previous theorem it follows that $\mathcal{R} \otimes \mathcal{E}$ is a dilation of $\mathcal{H}$. ■

Note that the reproducing kernel Hilbert space corresponding to the kernel function $\tilde{K}$ is not necessarily a bounded module over $A(\Omega)$ or even over $\mathbb{C}[\boldsymbol{z}]$. If it is a bounded module, then one can identify $\mathcal{M}$ canonically with the Hilbert module tensor product, $\mathcal{R} \otimes_{\mathbb{C}[\boldsymbol{z}]} \mathcal{H}(\tilde{K})$, which yields an explicit representation of the co-isometry from the co-extension space $\mathcal{R} \otimes \mathcal{H}(\tilde{K})$ to $\mathcal{M}$.

3.2. **Hereditary functional calculus.** Let p be a polynomial in the $2n$ variables $\boldsymbol{z} = (z_1, \dots, z_n), \bar{\boldsymbol{w}} = (\bar{w}_1, \dots, \bar{w}_n)$, where the $\boldsymbol{z}$ -variables all commute and the $\bar{\boldsymbol{w}}$-variables all commute with no assumptions made about the relation of the $\boldsymbol{z}$ and $\bar{\boldsymbol{w}}$ variables. For any commuting n-tuple of operators $\mathbf{T} = (T_1, \dots, T_n)$, define the value of p at $\mathbf{T}$ using the hereditary functional calculus (following Agler [Ag82]):

$$p(T, T^*) = \sum_{\boldsymbol{k},\boldsymbol{l}} a_{\boldsymbol{k},\boldsymbol{l}} T^{\boldsymbol{k}} T^{*\boldsymbol{l}},$$

where $p(\boldsymbol{z}, \bar{\boldsymbol{w}}) = \sum_{\boldsymbol{k},\boldsymbol{l}} a_{\boldsymbol{k},\boldsymbol{l}} \boldsymbol{z}^{\boldsymbol{k}} \bar{\boldsymbol{w}}^{\boldsymbol{l}}$ and $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}^n$. Here, in the "non-commutative polynomial" $p(\boldsymbol{z}, \bar{\boldsymbol{w}})$, the "$\boldsymbol{z}$'s" are all placed on the left, while the "$\bar{\boldsymbol{w}}$'s" are placed on the right.

Let $\mathcal{R} \subseteq \mathcal{O}(\Omega, \mathcal{E})$ be an $\mathcal{E}$-valued reproducing kernel Hilbert module over Ω for some Hilbert space $\mathcal{E}$ and k be a positive definite kernel over Ω. Moreover, let

$$k^{-1}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\boldsymbol{k},\boldsymbol{l}} a_{\boldsymbol{k},\boldsymbol{l}} \boldsymbol{z}^{\boldsymbol{k}} \bar{\boldsymbol{w}}^{\boldsymbol{l}},$$

be a polynomial in $\boldsymbol{z}$ and $\bar{\boldsymbol{w}}$. Therefore, for the module multiplication operators on $\mathcal{R}$ one gets

$$k^{-1}(M, M^*) = \sum_{\boldsymbol{k},\boldsymbol{l}} a_{\boldsymbol{k},\boldsymbol{l}} M^{\boldsymbol{k}} M^{*\boldsymbol{l}}.$$

PROPOSITION 3.3. *Let $\mathcal{R} \subseteq \mathcal{O}(\Omega, \mathcal{E})$ be a reproducing kernel Hilbert module with kernel function $K_{\mathcal{R}}$. Moreover, let k be a positive definite function defined on Ω and $k^{-1}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\boldsymbol{k},\boldsymbol{l}} a_{\boldsymbol{k},\boldsymbol{l}} \boldsymbol{z}^{\boldsymbol{k}} \bar{\boldsymbol{w}}^{\boldsymbol{l}}$ be a polynomial in $\boldsymbol{z}$ and $\bar{\boldsymbol{w}}$. Then*

$$k^{-1}(M, M^*) \geq 0,$$

if and only if

$$(\boldsymbol{z}, \boldsymbol{w}) \mapsto k^{-1}(\boldsymbol{z}, \boldsymbol{w}) K_{\mathcal{R}}(\boldsymbol{z}, \boldsymbol{w}),$$

is a positive definite kernel on Ω.

Proof. For each $\boldsymbol{z}, \boldsymbol{w} \in \Omega$ and $\eta, \zeta \in \mathcal{E}$, as a result of the preceding identity,

$$\begin{aligned}
\langle k^{-1}(M, M^*) K_{\mathcal{R}}(\cdot, \boldsymbol{w})\eta, K_{\mathcal{R}}(\cdot, \boldsymbol{z})\zeta \rangle_{\mathcal{R}} &= \langle \Big(\sum_{\boldsymbol{k},\boldsymbol{l}} a_{\boldsymbol{k},\boldsymbol{l}} M^{\boldsymbol{k}} M^{*\boldsymbol{l}} \Big) K_{\mathcal{R}}(\cdot, \boldsymbol{w})\eta, K_{\mathcal{R}}(\cdot, \boldsymbol{z})\zeta \rangle_{\mathcal{R}} \\
&= \sum_{\boldsymbol{k},\boldsymbol{l}} a_{\boldsymbol{k},\boldsymbol{l}} \langle M^{*\boldsymbol{l}} K_{\mathcal{R}}(\cdot, \boldsymbol{w})\eta, M^{*\boldsymbol{k}} K_{\mathcal{R}}(\cdot, \boldsymbol{z})\zeta \rangle_{\mathcal{R}} \\
&= \sum_{\boldsymbol{k},\boldsymbol{l}} a_{\boldsymbol{k},\boldsymbol{l}} \boldsymbol{z}^{\boldsymbol{k}} \bar{\boldsymbol{w}}^{\boldsymbol{l}} \langle K_{\mathcal{R}}(\cdot, \boldsymbol{w})\eta, K_{\mathcal{R}}(\cdot, \boldsymbol{z})\zeta \rangle_{\mathcal{R}} \\
&= k^{-1}(\boldsymbol{z}, \boldsymbol{w}) \langle K_{\mathcal{R}}(\boldsymbol{z}, \boldsymbol{w})\eta, \zeta \rangle_{\mathcal{E}} \\
&= \langle k^{-1}(\boldsymbol{z}, \boldsymbol{w}) K_{\mathcal{R}}(\boldsymbol{z}, \boldsymbol{w})\eta, \zeta \rangle_{\mathcal{E}}.
\end{aligned}$$

Hence, for $\{\boldsymbol{z}_i\}_{i=1}^l \subseteq \Omega$ and $\{\eta_i\}_{i=1}^l \subseteq l^2_m$ and $l \in \mathbb{N}$ it follows that

$$\begin{aligned}\langle k^{-1}(M, M^*)(\sum_{i=1}^l K_{\mathcal{R}}(\cdot, \boldsymbol{z}_i)\eta_i), \sum_{j=1}^l K_{\mathcal{R}}(\cdot, \boldsymbol{z}_j)\eta_j\rangle_{\mathcal{R}} \\ &= \sum_{i,j=1}^l \langle k^{-1}(M, M^*)(K_{\mathcal{R}}(\cdot, \boldsymbol{z}_i)\eta_i), K_{\mathcal{R}}(\cdot, \boldsymbol{z}_j)\eta_j\rangle_{\mathcal{R}} \\ &= \sum_{i,j=1}^l \langle k^{-1}(\boldsymbol{z}_j, \boldsymbol{z}_i) K_{\mathcal{R}}(\boldsymbol{z}_j, \boldsymbol{z}_i)\eta_i, \eta_j\rangle_{\mathcal{E}} \\ &= \sum_{i,j=1}^l \langle (k^{-1} \circ K_{\mathcal{R}})(\boldsymbol{z}_j, \boldsymbol{z}_i)\eta_j, \eta_i\rangle_{\mathcal{E}}.\end{aligned}$$

Consequently, $k^{-1}(M, M^*) \geq 0$ if and only if $k^{-1}(\boldsymbol{z}, \boldsymbol{w})K_{\mathcal{M}}(\boldsymbol{z}, \boldsymbol{w})$ is a non-negative definite kernel. This completes the proof. ■

The following corollary is immediate.

Corollary 3.4. *Let $\mathcal{R} \subseteq \mathcal{O}(\Omega, \mathcal{E})$ be a reproducing kernel Hilbert module with kernel function $K_{\mathcal{R}}$. Moreover, let k be a positive definite function defined on Ω and $k^{-1}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\boldsymbol{k},\boldsymbol{l}} a_{\boldsymbol{k},\boldsymbol{l}} \boldsymbol{z}^{\boldsymbol{k}} \bar{\boldsymbol{w}}^{\boldsymbol{l}}$ be a polynomial in $\boldsymbol{z}$ and $\bar{\boldsymbol{w}}$. Then $k^{-1}(M, M^*) \geq 0$ if and only if $K_{\mathcal{R}}$ factorizes as*

$$K_{\mathcal{R}}(\boldsymbol{z}, \boldsymbol{w}) = k(\boldsymbol{z}, \boldsymbol{w})\tilde{K}(\boldsymbol{z}, \boldsymbol{w}), \qquad (\boldsymbol{z}, \boldsymbol{w} \in \Omega)$$

for some positive definite kernel $\tilde{K}$ on Ω.

The following dilation result is an application of Theorem 3.2 and Corollary 3.4.

Theorem 3.5. *Let $\mathcal{M}$ be a quasi-free Hilbert module over $A(\mathbb{D}^n)$ of multiplicity m and $\mathcal{H}_k$ be a reproducing kernel Hilbert module over $A(\mathbb{D}^n)$. Moreover, let $k^{-1}(\boldsymbol{z}, \boldsymbol{w}) = \sum_{\boldsymbol{k},\boldsymbol{l}} a_{\boldsymbol{k},\boldsymbol{l}} \boldsymbol{z}^{\boldsymbol{k}} \bar{\boldsymbol{w}}^{\boldsymbol{l}}$ be a polynomial in $\boldsymbol{z}$ and $\bar{\boldsymbol{w}}$. Then $\mathcal{H}_k \otimes \mathcal{F}$ is a dilation of $\mathcal{M}$ for some Hilbert space $\mathcal{F}$ if and only if $k^{-1}(M, M^*) \geq 0$.*

It is the aim of the present consideration to investigate the issue of uniqueness of the minimal isometric dilations of contractive reproducing kernel Hilbert modules. The proof is based on operator theory exploiting the fact that the co-ordinate multipliers define doubly commuting isometries.

Theorem 3.6. *Let $\mathcal{H}_k$ be a contractive reproducing kernel Hilbert module over $A(\mathbb{D}^n)$. Then $\mathcal{H}_k$ dilates to $H^2(\mathbb{D}^n) \otimes \mathcal{E}$ if and only if $\mathbb{S}^{-1}(M, M^*) \geq 0$ or, equivalently, $\mathbb{S}^{-1}k \geq 0$. Moreover, if such dilation exists, then the minimal one is unique.*

Proof. By virtue of Theorem 3.5, one only needs to prove the uniqueness of the minimal dilation. Let $\Pi_i : \mathcal{H}_k \to H^2(\mathbb{D}^n) \otimes \mathcal{E}_i$ be minimal isometric dilations of $\mathcal{H}_k$, that is,

$$H^2(\mathbb{D}^n) \otimes \mathcal{E}_i = \overline{\text{span}}\{M_z^{\boldsymbol{k}}(\Pi_i \mathcal{H}_k) : \boldsymbol{k} \in \mathbb{N}^n\},$$

for $i = 1, 2$. Define

$$V : H^2(\mathbb{D}^n) \otimes \mathcal{E}_1 \to H^2(\mathbb{D}^n) \otimes \mathcal{E}_2,$$

by

$$V(\sum_{|\boldsymbol{k}|\leq N} M_z^{\boldsymbol{k}}\Pi_1 f_{\boldsymbol{k}}) = \sum_{|\boldsymbol{\alpha}|\leq N} M_z^{\boldsymbol{k}}\Pi_2 f_{\boldsymbol{k}},$$

where $f_{\boldsymbol{k}} \in \mathcal{H}$ and $N \in \mathbb{N}$. Let $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}^n$ and define multi-indices $\tilde{\boldsymbol{k}}$ and $\tilde{\boldsymbol{l}}$ so that

$$\tilde{k}_i = \begin{cases} k_i - l_i & \text{for } k_i - l_i \geq 0 \\ 0 & \text{for } k_i - l_i < 0 \end{cases} \quad \text{and} \quad \tilde{l}_i = \begin{cases} l_i - k_i & \text{for } l_i - k_i \geq 0 \\ 0 & \text{for } l_i - k_i < 0 \end{cases}$$

Note that $k_i - l_i = \tilde{k}_i - \tilde{l}_i, \tilde{k}_i, \tilde{l}_i \geq 0$ and hence

$$M_z^{*\boldsymbol{l}} M_z^{\boldsymbol{k}} = M_z^{*\tilde{\boldsymbol{l}}} M_z^{\tilde{\boldsymbol{k}}} = M_z^{\tilde{\boldsymbol{k}}} M_z^{*\tilde{\boldsymbol{l}}}.$$

Therefore, for $i = 1, 2$, it follows that

$$\langle M_z^{\boldsymbol{k}}\Pi_i f_{\boldsymbol{k}}, M_z^{\boldsymbol{l}}\Pi_i f_{\boldsymbol{l}}\rangle = \langle M_z^{*\boldsymbol{l}} M_z^{\boldsymbol{k}}\Pi_i f_{\boldsymbol{k}}, \Pi_i f_{\boldsymbol{l}}\rangle = \langle M_z^{*\tilde{\boldsymbol{l}}}\Pi_i f_{\boldsymbol{k}}, M_z^{*\tilde{\boldsymbol{k}}}\Pi_i f_{\boldsymbol{l}}\rangle,$$

and, since Π_i is an co-module isometry, one gets

$$\langle M_z^{\boldsymbol{k}}\Pi_i f_{\boldsymbol{k}}, M_z^{\boldsymbol{l}}\Pi_i f_{\boldsymbol{l}}\rangle = \langle \Pi_i M_z^{*\tilde{\boldsymbol{l}}} f_{\boldsymbol{k}}, \Pi_i M_z^{*\tilde{\boldsymbol{k}}} f_{\boldsymbol{l}}\rangle = \langle M_z^{*\tilde{\boldsymbol{l}}} f_{\boldsymbol{k}}, M_z^{*\tilde{\boldsymbol{k}}} f_{\boldsymbol{l}}\rangle.$$

Hence V is well-defined and isometric and

$$V\Pi_1 = \Pi_2.$$

Moreover, since

$$\{\sum_{|\boldsymbol{k}|\leq N} M_z^{\boldsymbol{k}}\Pi_i f_{\boldsymbol{k}} : f_{\boldsymbol{k}} \in \mathcal{H}, N \in \mathbb{N}\}$$

is a total subset of $H^2(\mathbb{D}^n) \otimes \mathcal{E}_i$ for $i = 1, 2$, by minimality, V is a unitary module map and hence $V = I_{H^2(\mathbb{D}^n)} \otimes V_0$ for some unitary $V_0 \in \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$. Therefore, the minimal dilations Π_1 and Π_2 are unitarily equivalent, which concludes the proof. ∎

Corollary 3.7. *If $\mathcal{H}_k$ be a contractive reproducing kernel Hilbert space over $A(\mathbb{D}^n)$. Then the Hardy module $H^2(\mathbb{D}^n) \otimes \mathcal{E}$ is a dilation of $\mathcal{H}_k$ if and only if $\mathbb{S}_n^{-1}(M, M^*) \geq 0$ or, equivalently, if and only if $\mathbb{S}_n^{-1} k \geq 0$. Moreover, if an $H^2(\mathbb{D}^n) \otimes \mathcal{E}$ dilation exists, then the minimal one is unique.*

Proof. The necessary and sufficient part follows from Theorem 3.5. The uniqueness part follows from Theorem 3.6. ∎

The above proof will only work if the algebra is generated by functions for which module multiplication defines doubly commuting isometric operators which happens for the Hardy module on the polydisk. For a more general quasi-free Hilbert module $\mathcal{R}$, the maps X_i^* identify anti-holomorphic sub-bundles of the bundle $E_{\mathcal{R}} \otimes \mathcal{E}_i$, where $E_{\mathcal{R}}$ is the Hermitian holomorphic line bundle defined by $\mathcal{R}$. To establish uniqueness, some how one must extend this identification to the full bundles. Equivalently, one has to identify the holomorphic quotient bundles of $E_{\mathcal{R}} \otimes \mathcal{E}_1$, and $E_{\mathcal{R}} \otimes \mathcal{E}_2$ and must some how lift it to the full bundles. At this point it is not even obvious that the dimensions of $\mathcal{E}_1$ and $\mathcal{E}_2$ or the ranks of the bundles are equal. This seems to be an interesting question. Using results on exact sequences of bundles (cf. [GriHar94] and [We80]), one can establish uniqueness if $\dim \mathcal{E} = \operatorname{rank} E_{\mathcal{H}} + 1$.

Further results and comments:

(1) Most of the material in this section is based on the article [DoMiSa12].
(2) In [Ag85], [Ag82], [At87], [At90], [At92], [AEn03] and [AmEnMu02], Agler, Athavale, Ambrozie, Arazy, Englis and Muller pointed out that the dilation theory and operator positivity implemented by kernel functions are closely related to each other.
(3) Theorem 3.6 was proved by Douglas and Foias in [DoFo93] for the case of multiplicity one. More precisely, let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two submodules of $H^2(\mathbb{D}^n)$. Then $H^2(\mathbb{D}^n)/\mathcal{S}_1 \cong H^2(\mathbb{D}^n)/\mathcal{S}_2$ if and only if $\mathcal{S}_1 = \mathcal{S}_2$. This is a rigidity result concerning submodules of the Hardy module (see Section 8).
(4) Notice that any n-tuple of doubly commuting contractions on a functional Hilbert space over $A(\mathbb{D}^n)$ satisfies the hypothesis of Theorem 3.5. Consequently, one can recover the result of Sz.-Nagy and Foias (cf. [NaFo70a]) in this situation. In particular, $\mathcal{M}^n = \mathcal{M} \otimes \cdots \otimes \mathcal{M}$ always possesses a dilation to the Hardy module $H^2(\mathbb{D}^n) \otimes \mathcal{E}$, where $\mathcal{E}$ is some Hilbert space, if $\mathcal{M}$ is contractive Hilbert module. The contractivity condition implies that

$$K(\boldsymbol{z}, \boldsymbol{w}) = (1 - z_\ell \bar{w}_\ell)^{-1} Q_\ell(\boldsymbol{z}, \boldsymbol{w}), \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n)$$

for some positive definite kernel Q_ℓ and for each $\ell = 1, 2, \ldots, n$. Thus

$$K^n(\boldsymbol{z}, \boldsymbol{w}) = \mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w}) Q(\boldsymbol{z}, \boldsymbol{w}), \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n)$$

where $Q = \prod_{\ell=1}^n Q_\ell$. Thus the Hilbert module $\mathcal{M}^n$ corresponding to the positive definite kernel K^n is contractive and admits the kernel $\mathbb{S}_{\mathbb{D}^n}$ as a factor, as shown above. This shows that $\mathcal{M}^n$ has an isometric co-extension to $H^2_{\mathcal{Q}}(\mathbb{D}^n)$, where $\mathcal{Q}$ is the reproducing kernel Hilbert space for the kernel Q.

4. Hardy module over polydisc

This section begins by formulating a list of basic problems in commutative algebra. Let $\mathcal{M}$ be a module over $\mathbb{C}[z]$ and $\mathcal{M}^{\otimes n} := \mathcal{M} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{M}$, the n-fold vector space tensor product of $\mathcal{M}$. Then $\mathcal{M}^{\otimes n}$ is a module over $\mathbb{C}[z] \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}[z] \cong \mathbb{C}[\boldsymbol{z}]$. Here the module action on $\mathcal{M}^{\otimes n}$ is given by

$$(p_1 \otimes \cdots \otimes p_n) \cdot (f_1 \otimes \cdots \otimes f_n) \mapsto p_1 \cdot f_1 \otimes \cdots \otimes p_n \cdot f_n,$$

for all $\{p_i\}_{i=1}^n \subseteq \mathbb{C}[z]$ and $\{f_i\}_{i=1}^n \in \mathcal{M}_i$. Let $\{\mathcal{Q}_i\}_{i=1}^n$ be quotient modules of $\mathcal{M}$. Then

$$\mathcal{Q}_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{Q}_n, \tag{4.3}$$

is a quotient module of $\mathcal{M}^{\otimes n}$.

On the other hand, let $\mathcal{Q}$ be a quotient module and $\mathcal{S}$ a submodule of $\mathcal{M}_n$. One is naturally led to formulate the following problems:
(a) When is $\mathcal{Q}$ of the form (4.3)?
(b) When is $\mathcal{M}/\mathcal{S}$ of the form (4.3)?

Let now $\mathcal{M}$ be the Hardy space $H^2(\mathbb{D})$, the Hilbert space completion of $\mathbb{C}[z]$, and consider the analogous problem. The purpose of this section is to provide a complete answer to these questions when $\mathcal{M} = H^2(\mathbb{D})$. In particular, a quotient module $\mathcal{Q}$ of the Hardy module $H^2(\mathbb{D}^n) \cong H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$ is of the form

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n,$$

for n quotient modules $\{\mathcal{Q}_i\}_{i=1}^n$ of $H^2(\mathbb{D})$ if and only if $\mathcal{Q}$ is doubly commuting.

A quotient module $\mathcal{Q} \subseteq H^2(\mathbb{D}^n)$ is said to be *doubly commuting* if

$$C_{z_i} C_{z_j}^* = C_{z_j}^* C_{z_i}. \qquad (1 \leq i < j \leq n)$$

A submodule $\mathcal{S}$ is called *co-doubly commuting* if $\mathcal{S}^\perp \cong H^2(\mathbb{D}^n)/\mathcal{S}$ is doubly commuting quotient module.

4.1. **Submodules and Jordan blocks.** A closed subspace $\mathcal{Q} \subseteq H^2(\mathbb{D})$ is said to be a *Jordan block* of $H^2(\mathbb{D})$ if $\mathcal{Q}$ is a quotient module and $\mathcal{Q} \neq H^2(\mathbb{D})$ (see [NaFo70b], [NaFo70a]). By Beurling's theorem (see Corollary 5.4 in [Sa14a]), a closed subspace $\mathcal{Q}(\neq H^2(\mathbb{D}))$ is a quotient module of $H^2(\mathbb{D})$ if and only if the submodule $\mathcal{Q}^\perp$ is given by $\mathcal{Q}^\perp = \Theta H^2(\mathbb{D})$ for some inner function $\Theta \in H^\infty(\mathbb{D})$. In other words, the quotient modules and hence the Jordan blocks of $H^2(\mathbb{D})$ are precisely given by

$$\mathcal{Q}_\Theta := H^2(\mathbb{D})/\Theta H^2(\mathbb{D}),$$

for inner functions $\Theta \in H^\infty(\mathbb{D})$. Thus on the level of orthogonal projections, one gets

$$P_{\mathcal{Q}_\Theta} = I_{H^2(\mathbb{D})} - M_\Theta M_\Theta^* \quad \text{and} \quad P_{\Theta H^2(\mathbb{D})} = M_\Theta M_\Theta^*.$$

The following lemma is a variation on the theme of the isometric dilation theory of contractions.

Lemma 4.1. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{D})$ and $\mathcal{L} = ran(I_\mathcal{Q} - C_z C_z^*) = ran(P_\mathcal{Q} P_\mathbb{C} P_\mathcal{Q})$. Then $\mathcal{Q} = \vee_{l=0}^\infty P_\mathcal{Q} M_z^l \mathcal{L}$.*

Proof. The result is trivial if $\mathcal{Q} = \{0\}$. Let $\mathcal{Q} \neq \{0\}$, that is, $\mathcal{Q}^\perp$ is a proper submodule of $H^2(\mathbb{D})$, or equivalently, $1 \notin \mathcal{Q}^\perp$. Notice that

$$\bigvee_{l=0}^\infty P_\mathcal{Q} M_z^l \mathcal{L} \subseteq \mathcal{Q}.$$

Let now

$$f = \sum_{l=0}^\infty a_l z^l \in \mathcal{Q},$$

be such that $f \perp \vee_{l=0}^\infty P_\mathcal{Q} M_z^l \mathcal{L}$. It then follows that $f \perp P_\mathcal{Q} M_z^l P_\mathcal{Q} P_\mathbb{C} \mathcal{Q}$, or equivalently, $P_\mathbb{C} M_z^{*l} f \in \mathcal{Q}^\perp$ for all $l \geq 0$. Since $P_\mathbb{C} M_z^{*l} f = a_l \in \mathbb{C}$ and $1 \notin \mathcal{Q}^\perp$, it follows that $a_l = P_\mathbb{C} M_z^{*l} f = 0$ for all $l \geq 0$. Consequently, $f = 0$. This concludes the proof. ∎

4.2. **Reducing submodules.** The following result gives a characterization of M_{z_1}-reducing subspace of $H^2(\mathbb{D}^n)$.

Proposition 4.2. *Let $n > 1$ and $\mathcal{S}$ be a closed subspace of $H^2(\mathbb{D}^n)$. Then $\mathcal{S}$ is a $(M_{z_2}, \ldots, M_{z_n})$-reducing subspace of $H^2(\mathbb{D}^n)$ if and only if $\mathcal{S} = \mathcal{S}_1 \otimes H^2(\mathbb{D}^{n-1})$ for some closed subspace $\mathcal{S}_1$ of $H^2(\mathbb{D})$.*

Proof. Let $\mathcal{S}$ be a $(M_{z_2}, \ldots, M_{z_n})$-reducing closed subspace of $H^2(\mathbb{D}^n)$, that is, $M_{z_i} P_{\mathcal{S}} = P_{\mathcal{S}} M_{z_i}$ for all $2 \leq i \leq n$. Since

$$\sum_{0 \leq i_1 < \ldots < i_l \leq n, i_1, i_2 \neq 1} (-1)^l M_{z_{i_1}} \cdots M_{z_{i_l}} M^*_{z_{i_1}} \cdots M^*_{z_{i_l}} = (I_{H^2(\mathbb{D}^n)} - M_{z_2} M^*_{z_2}) \cdots (I_{H^2(\mathbb{D}^2)} - M_{z_n} M^*_{z_n})$$
$$= P_{H^2(\mathbb{D})} \otimes P_{\mathbb{C}} \otimes \cdots \otimes P_{\mathbb{C}},$$

we have that $(P_{H^2(\mathbb{D})} \otimes P_{\mathbb{C}} \otimes \cdots \otimes P_{\mathbb{C}}) P_{\mathcal{S}} = P_{\mathcal{S}} (P_{H^2(\mathbb{D})} \otimes P_{\mathbb{C}} \otimes \cdots \otimes P_{\mathbb{C}})$. Therefore, $P_{\mathcal{S}}(P_{H^2(\mathbb{D})} \otimes P_{\mathbb{C}} \otimes \cdots \otimes P_{\mathbb{C}})$ is an orthogonal projection and

$$P_{\mathcal{S}}(P_{H^2(\mathbb{D})} \otimes P_{\mathbb{C}} \otimes \cdots \otimes P_{\mathbb{C}}) = (P_{H^2(\mathbb{D})} \otimes P_{\mathbb{C}} \otimes \cdots \otimes P_{\mathbb{C}}) P_{\mathcal{S}} = P_{\tilde{\mathcal{S}}_1},$$

where $\tilde{\mathcal{S}}_1 := (H^2(\mathbb{D}) \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C}) \cap \mathcal{S}$. Let $\tilde{\mathcal{S}}_1 = \mathcal{S}_1 \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C}$, for some closed subspace $\mathcal{S}_1$ of $H^2(\mathbb{D})$. It then follows that

$$\mathcal{S} = \overline{\text{span}} \{ M^{l_2}_{z_2} \cdots M^{l_n}_{z_n} \tilde{\mathcal{S}}_1 : l_2, \ldots, l_n \in \mathbb{N} \} = \mathcal{S}_1 \otimes H^2(\mathbb{D}^{n-1}).$$

The converse part is immediate. This concludes the proof of the proposition. ■

It is well known that a closed subspace $\mathcal{M}$ of $H^2(\mathbb{D}^n)$ is M_{z_1}-reducing if and only if

$$\mathcal{M} = H^2(\mathbb{D}) \otimes \mathcal{E},$$

for some closed subspace $\mathcal{E} \subseteq H^2(\mathbb{D}^{n-1})$. The following key proposition is a generalization of this fact.

Proposition 4.3. *Let $\mathcal{Q}_1$ be a quotient module of $H^2(\mathbb{D})$ and $\mathcal{M}$ be a closed subspace of $\mathcal{Q} = \mathcal{Q}_1 \otimes H^2(\mathbb{D}^{n-1})$. Then $\mathcal{M}$ is a $P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}$-reducing subspace of $\mathcal{Q}$ if and only if*

$$\mathcal{M} = \mathcal{Q}_1 \otimes \mathcal{E},$$

for some closed subspace $\mathcal{E}$ of $H^2(\mathbb{D}^{n-1})$.

Proof. Let $\mathcal{M}$ be a $P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}$-reducing subspace of $\mathcal{Q}$. Then

$$(P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}) P_{\mathcal{M}} = P_{\mathcal{M}} (P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}), \tag{4.4}$$

or equivalently, $(P_{\mathcal{Q}_1} M_z|_{\mathcal{Q}_1} \otimes I_{H^2(\mathbb{D}^{n-1})}) P_{\mathcal{M}} = P_{\mathcal{M}} (P_{\mathcal{Q}_1} M_z|_{\mathcal{Q}_1} \otimes I_{H^2(\mathbb{D}^{n-1})})$. Now

$$I_{\mathcal{Q}} - (P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}})(P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}})^* = (P_{\mathcal{Q}_1} P_{\mathbb{C}}|_{\mathcal{Q}_1}) \otimes I_{H^2(\mathbb{D}^{n-1})}.$$

Further (4.4) yields $P_{\mathcal{M}}((P_{\mathcal{Q}_1} P_{\mathbb{C}}|_{\mathcal{Q}_1}) \otimes I_{H^2(\mathbb{D}^{n-1})}) = ((P_{\mathcal{Q}_1} P_{\mathbb{C}}|_{\mathcal{Q}_1}) \otimes I_{H^2(\mathbb{D}^{n-1})}) P_{\mathcal{M}}$. Let

$$\mathcal{L} := \mathcal{M} \cap \text{ran } ((P_{\mathcal{Q}_1} P_{\mathbb{C}}|_{\mathcal{Q}_1}) \otimes I_{H^2(\mathbb{D}^{n-1})}) = \mathcal{M} \cap (\mathcal{L}_1 \otimes H^2(\mathbb{D}^{n-1})),$$

where

$$\mathcal{L}_1 = \text{ran } (P_{\mathcal{Q}_1} P_{\mathbb{C}}|_{\mathcal{Q}_1}) \subseteq \mathcal{Q}_1.$$

Since $\mathcal{L} \subseteq \mathcal{L}_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$ and $\dim \mathcal{L}_1 = 1$ (otherwise, by Lemma 4.1 that $\mathcal{L}_1 = \{0\}$ is equivalent to $\mathcal{Q}_1 = \{0\}$), it follows that $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{E}$ for some closed subspace $\mathcal{E} \subseteq H^2(\mathbb{D}^{n-1})$. Hence $P_{\mathcal{M}}((P_{\mathcal{Q}_1} P_{\mathbb{C}}|_{\mathcal{Q}_1}) \otimes I_{H^2(\mathbb{D}^{n-1})}) = P_{\mathcal{L}} = P_{\mathcal{L}_1 \otimes \mathcal{E}}$. Claim:

$$\mathcal{M} = \bigvee_{l=0}^{\infty} P_{\mathcal{Q}} M^l_{z_1} \mathcal{L}.$$

Since $\mathcal{M}$ is $P_{\mathcal{Q}}M_{z_1}|_{\mathcal{Q}}$-reducing subspace and $\mathcal{M} \supseteq \mathcal{L}$, it follows that $\mathcal{M} \supseteq \vee_{l=0}^{\infty} P_{\mathcal{Q}}M_{z_1}^l \mathcal{L}$. To prove the reverse inclusion, we let $f \in \mathcal{M}$ and $f = \sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}$, where $a_{\boldsymbol{k}} \in \mathbb{C}$ for all $\boldsymbol{k} \in \mathbb{N}^n$. Then $f = P_{\mathcal{M}}P_{\mathcal{Q}}f = P_{\mathcal{M}}P_{\mathcal{Q}} \sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}$. Observe now that for all $\boldsymbol{k} \in \mathbb{N}^n$,

$$\begin{aligned} P_{\mathcal{M}}P_{\mathcal{Q}}\boldsymbol{z}^{\boldsymbol{k}} &= P_{\mathcal{M}}((P_{\mathcal{Q}_1} z_1^{k_1})(z_2^{k_2} \cdots z_n^{k_n})) \\ &= P_{\mathcal{M}}((P_{\mathcal{Q}_1} M_{z_1}^{k_1} P_{\mathcal{Q}_1} 1)(z_2^{k_2} \cdots z_n^{k_n})) \\ &= P_{\mathcal{Q}} M_{z_1}^{k_1} (P_{\mathcal{M}}(P_{\mathcal{Q}_1} 1 \otimes z_2^{k_2} \cdots z_n^{k_n})), \end{aligned}$$

by (4.4), where the second equality follows from $\langle z_1^{k_1}, f\rangle = \langle 1, (M_{z_1}^{k_1})^* f\rangle = \langle P_{\mathcal{Q}_1} M_{z_1}^{k_1} P_{\mathcal{Q}_1} 1, f\rangle$, for all $f \in \mathcal{Q}_1$. By the fact that $P_{\mathcal{Q}_1} 1 \in \mathcal{L}_1$ one gets $P_{\mathcal{M}}(P_{\mathcal{Q}_1} 1 \otimes z_2^{k_2} \cdots z_n^{k_n}) \in \mathcal{L}$ and infer $P_{\mathcal{M}}P_{\mathcal{Q}}\boldsymbol{z}^{\boldsymbol{k}} \in \vee_{l=0}^{\infty} P_{\mathcal{Q}_1} M_{z_1}^l \mathcal{L}$ for all $\boldsymbol{k} \in \mathbb{N}^n$. Therefore $f \in \vee_{l=0}^{\infty} P_{\mathcal{Q}} M_{z_1}^l \mathcal{L}$ and hence $\mathcal{M} = \vee_{l=0}^{\infty} P_{\mathcal{Q}} M_{z_1}^l \mathcal{L}$, and the claim follows. Finally, $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{E}$ yields

$$\mathcal{M} = \bigvee_{l=0}^{\infty} P_{\mathcal{Q}} M_{z_1}^l \mathcal{L} = (\bigvee_{l=0}^{\infty} P_{\mathcal{Q}_1} M_{z_1}^l \mathcal{L}_1) \otimes \mathcal{E},$$

and therefore by Lemma 4.1, $\mathcal{M} = \mathcal{Q}_1 \otimes \mathcal{E}$. The converse part is trivial. This finishes the proof. ∎

4.3. **Tensor product of Jordan blocks.** Let $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ be n quotient modules of $H^2(\mathbb{D})$. Then the module multiplication operators on $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$ are given by

$$\{I_{\mathcal{Q}_1} \otimes \cdots \otimes P_{\mathcal{Q}_i} M_z|_{\mathcal{Q}_i} \otimes \cdots \otimes I_{\mathcal{Q}_n}\}_{i=1}^n,$$

that is, $\mathcal{Q}$ is a doubly commuting quotient module. The following theorem provides a converse statement.

THEOREM 4.4. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{D}^n)$. Then $\mathcal{Q}$ is doubly commuting if and only if there exists quotient modules $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ of $H^2(\mathbb{D})$ such that*

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n.$$

Proof. Let $\mathcal{Q}$ be a doubly commuting quotient module of $H^2(\mathbb{D}^n)$. Set

$$\tilde{\mathcal{Q}}_1 = \overline{\text{span}}\{z_2^{l_2} \cdots z_n^{l_n} \mathcal{Q} : l_2, \ldots, l_n \in \mathbb{N}\},$$

a joint $(M_{z_2}, \ldots, M_{z_n})$-reducing subspace of $H^2(\mathbb{D}^n)$. By Proposition 4.2, it follows that

$$\tilde{\mathcal{Q}}_1 = \mathcal{Q}_1 \otimes H^2(\mathbb{D}^{n-1}),$$

for some closed subspace $\mathcal{Q}_1$ of $H^2(\mathbb{D})$. Since $\tilde{\mathcal{Q}}_1$ is $M_{z_1}^*$-invariant subspace, that $\mathcal{Q}_1$ is a M_z^*-invariant subspace of $H^2(\mathbb{D})$, that is, $\mathcal{Q}_1$ is a quotient module of $H^2(\mathbb{D})$.
Note that $\mathcal{Q} \subseteq \tilde{\mathcal{Q}}_1$. Claim: $\mathcal{Q}$ is a $M_{z_1}^*|_{\tilde{\mathcal{Q}}_1}$-reducing subspace of $\tilde{\mathcal{Q}}_1$, that is,

$$P_{\mathcal{Q}}(M_{z_1}^*|_{\tilde{\mathcal{Q}}_1}) = (M_{z_1}^*|_{\tilde{\mathcal{Q}}_1}) P_{\mathcal{Q}}.$$

In order to prove the claim, first observe that $C_{z_1}^* C_{z_i}^l = C_{z_i}^l C_{z_1}^*$ for all $l \geq 0$ and $2 \leq i \leq n$, and hence

$$C_{z_1}^* C_{z_2}^{l_2} \cdots C_{z_n}^{l_n} = C_{z_2}^{l_2} \cdots C_{z_n}^{l_n} C_{z_1}^*,$$

for all $l_2, \ldots, l_n \geq 0$, that is, $M_{z_1}^* P_{\mathcal{Q}} M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_{\mathcal{Q}} = P_{\mathcal{Q}} M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} M_{z_1}^* P_{\mathcal{Q}}$ or,

$$M_{z_1}^* P_{\mathcal{Q}} M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_{\mathcal{Q}} = P_{\mathcal{Q}} M_{z_1}^* M_{z_2}^{l_2} \cdots M_{z_n}^{l_n} P_{\mathcal{Q}}.$$

From this it follows that for all $f \in \mathcal{Q}$ and $l_2, \ldots, l_n \geq 0$,

$$(P_{\mathcal{Q}} M_{z_1}^*|_{\tilde{\mathcal{Q}}_1})(z_2^{l_2} \cdots z_n^{l_n} f) = P_{\mathcal{Q}} M_{z_1}^*(z_2^{l_2} \cdots z_n^{l_n} f) = (M_{z_1}^* P_{\mathcal{Q}})(z_2^{l_2} \cdots z_n^{l_n} f).$$

Also by $P_{\mathcal{Q}} \tilde{\mathcal{Q}}_1 \subseteq \tilde{\mathcal{Q}}_1$ one gets

$$P_{\mathcal{Q}} P_{\tilde{\mathcal{Q}}_1} = P_{\tilde{\mathcal{Q}}_1} P_{\mathcal{Q}} P_{\tilde{\mathcal{Q}}_1}.$$

This yields

$$\begin{aligned}(P_{\mathcal{Q}} M_{z_1}^*|_{\tilde{\mathcal{Q}}_1})(z_2^{l_2} \cdots z_n^{l_n} f) &= (M_{z_1}^* P_{\mathcal{Q}})(z_2^{l_2} \cdots z_n^{l_n} f)\\ &= M_{z_1}^* P_{\mathcal{Q}} P_{\tilde{\mathcal{Q}}_1}(z_2^{l_2} \cdots z_n^{l_n} f)\\ &= (M_{z_1}^*|_{\tilde{\mathcal{Q}}_1} P_{\mathcal{Q}})(z_2^{l_2} \cdots z_n^{l_n} f),\end{aligned}$$

for all $f \in \mathcal{Q}$ and $l_2, \ldots, l_n \geq 0$, and therefore

$$P_{\mathcal{Q}}(M_{z_1}^*|_{\tilde{\mathcal{Q}}_1}) = (M_{z_1}^*|_{\tilde{\mathcal{Q}}_1}) P_{\mathcal{Q}}.$$

Hence $\mathcal{Q}$ is a $M_{z_1}^*|_{\tilde{\mathcal{Q}}_1}$-reducing subspace of $\tilde{\mathcal{Q}}_1 = \mathcal{Q}_1 \otimes H^2(\mathbb{D}) \otimes \cdots \otimes H^2(\mathbb{D})$. Now by Proposition 4.3, there exists a closed subspace $\mathcal{E}_1$ of $H^2(\mathbb{D}^{n-1})$ such that

$$\mathcal{Q} = \mathcal{Q}_1 \otimes \mathcal{E}_1.$$

Moreover, since

$$\bigvee_{l=0}^{\infty} z_1^l \mathcal{Q} = \bigvee_{l=0}^{\infty} z_1^l (\mathcal{Q}_1 \otimes \mathcal{E}_1) = H^2(\mathbb{D}) \otimes \mathcal{E}_1,$$

and $\vee_{l=0}^{\infty} z_1^l \mathcal{Q}$ is a doubly commuting quotient module of $H^2(\mathbb{D}^n)$, it follows that $\mathcal{E}_1 \subseteq H^2(\mathbb{D}^{n-1})$, a doubly commuting quotient module of $H^2(\mathbb{D}^{n-1})$.
By the same argument as above, we conclude that $\mathcal{E}_1 = \mathcal{Q}_2 \otimes \mathcal{E}_2$, for some doubly commuting quotient module of $H^2(\mathbb{D}^{n-2})$. Continuing this process, we have $\mathcal{Q} = \mathcal{Q}_1 \otimes \cdots \otimes \mathcal{Q}_n$, where $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ are quotient modules of $H^2(\mathbb{D})$. This completes the proof. ∎

As a corollary, one can easily derive the following fact concerning Jordan blocks of $H^2(\mathbb{D}^n)$.

Corollary 4.5. *Let $\mathcal{Q}$ be a closed subspace of $H^2(\mathbb{D}^n)$. Then $\mathcal{Q}$ is doubly commuting quotient module if and only if there exists $\{\Theta_i\}_{i=1}^n \subseteq H^\infty(\mathbb{D})$ such that each Θ_i is either inner or the zero function for all $1 \leq i \leq n$ and*

$$\mathcal{Q} = \mathcal{Q}_{\Theta_1} \otimes \cdots \otimes \mathcal{Q}_{\Theta_n}.$$

4.4. **Beurling's representation.** The aim of this subsection is to relate the Hilbert tensor product structure of the doubly commuting quotient modules to the Beurling like representations of the corresponding co-doubly commuting submodules.

The following piece of notation will be used in the rest of the subsection.
Let $\Theta_i \in H^\infty(\mathbb{D})$ be a given function indexed by $i \in \{1, \ldots, n\}$. In what follows, $\tilde{\Theta}_i \in H^\infty(\mathbb{D}^n)$ will denote the extension function defined by

$$\tilde{\Theta}_i(\boldsymbol{z}) = \Theta_i(z_i),$$

for all $\boldsymbol{z} \in \mathbb{D}^n$.

The reader is referred to Lemma 2.5 in [Sa14b] for a proof of the following lemma.

LEMMA 4.6. *Let $\{P_i\}_{i=1}^n$ be a collection of commuting orthogonal projections on a Hilbert space $\mathcal{H}$. Then*

$$\mathcal{L} := \sum_{i=1}^{n} ranP_i,$$

is closed and the orthogonal projection of $\mathcal{H}$ onto $\mathcal{L}$ is given by

$$\begin{aligned} P_{\mathcal{L}} &= P_1(I-P_2)\cdots(I-P_n) + P_2(I-P_3)\cdots(I-P_n) + \cdots + P_{n-1}(I-P_n) + P_n \\ &= P_n(I-P_{n-1})\cdots(I-P_1) + P_{n-1}(I-P_{n-2})\cdots(I-P_1) + \cdots + P_2(I-P_1) + P_1. \end{aligned}$$

Moreover,

$$P_{\mathcal{L}} = I - \prod_{i=1}^{n}(I-P_i).$$

The following provides an explicit correspondence between the doubly commuting quotient modules and the co-doubly commuting submodules of $H^2(\mathbb{D}^n)$.

THEOREM 4.7. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{D}^n)$ and $\mathcal{Q} \neq H^2(\mathbb{D}^n)$. Then $\mathcal{Q}$ is doubly commuting if and only if there exists inner functions $\Theta_{i_j} \in H^\infty(\mathbb{D})$ for $1 \leq i_1 < \ldots < i_m \leq n$ for some integer $m \in \{1, \ldots, n\}$ such that*

$$\mathcal{Q} = H^2(\mathbb{D}^n)/[\tilde{\Theta}_{i_1} H^2(\mathbb{D}^n) + \cdots + \tilde{\Theta}_{i_m} H^2(\mathbb{D}^n)],$$

where $\tilde{\Theta}_{i_j}(\boldsymbol{z}) = \Theta_{i_j}(z_{i_j})$ for all $\boldsymbol{z} \in \mathbb{D}^n$.

Proof. The proof follows from Corollary 4.5 and Lemma 4.6. ■

The conclusion of this subsection concerns the orthogonal projection formulae for the co-doubly commuting submodules and the doubly commuting quotient modules of $H^2(\mathbb{D}^n)$. It can be treated as a co-doubly commuting submodules analogue of Beurling's theorem on submodules of $H^2(\mathbb{D})$.

COROLLARY 4.8. *Let $\mathcal{Q}$ be a doubly commuting submodule of $H^2(\mathbb{D}^n)$. Then there exists an integer $m \in \{1, \ldots, n\}$ and inner functions $\{\Theta_{i_j}\}_{j=1}^m \subseteq H^\infty(\mathbb{D})$ such that*

$$\mathcal{Q}^\perp = \sum_{1 \leq i_1 < \ldots < i_m \leq n} \tilde{\Theta}_{i_j} H^2(\mathbb{D}^n),$$

where $\tilde{\Theta}_i(\boldsymbol{z}) = \Theta_{i_j}(z_{i_j})$ for all $\boldsymbol{z} \in \mathbb{D}^n$. Moreover,

$$P_{\mathcal{Q}} = I_{H^2(\mathbb{D}^n)} - \prod_{j=1}^{m}(I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}),$$

and

$$P_{\mathcal{Q}^\perp} = \prod_{j=1}^{m}(I_{H^2(\mathbb{D}^n)} - M_{\tilde{\Theta}_{i_j}} M^*_{\tilde{\Theta}_{i_j}}).$$

This is an immediate consequence of Theorem 4.7. The reason to state Theorem 4.7 explicitly is its usefulness.

Further results and comments:

(1) An efficient solution to the algebraic problems, posed in the introduction of this section, would likely have practical applications.
(2) The study of the doubly commuting quotient modules of $H^2(\mathbb{D}^2)$ was initiated by Douglas and Yang in [DoY98] and [DoY00] (also see [BerCL78]). Later in [INS04] Izuchi, Nakazi and Seto obtained the tensor product classification of doubly commuting quotient modules of $H^2(\mathbb{D}^2)$.
(3) The results of this section can be found in the papers [Sa13a] and [Sa14b]. For the base case $n = 2$, they were obtained by Izuchi, Nakazi and Seto [IN04], [INS04].
(4) The tensor product representations of doubly commuting quotient modules of $H^2(\mathbb{D}^2)$ has deep and far reaching applications to the general study of submodules and quotient modules of the Hardy module $H^2(\mathbb{D}^2)$. See the work by K. J. Izuchi, K. H. Izuchi and Y. Izuchi [III11], [III11] and Yang [Y05a], [Y05b].
(5) The techniques embodied in this section can be used to give stronger results concerning doubly commuting quotient modules of a large class of reproducing kernel Hilbert modules over $\mathbb{D}^n$ including the weighted Bergman modules (see [CDSa14]).
(6) Other related work concerning submodules and quotient modules of $H^2(\mathbb{D}^n)$ appears in Berger, Coburn and Lebow [BerCL78], Yang [Y01], [Y05b], Guo and Yang [GuY04] and the book by Chen and Guo [ChGu03].
(7) In connection with Beurling representations for submodules of $H^2(\mathbb{D}^n)$, we refer the reader to Cotlar and Sadosky [CoSa98].

5. Similarity to free Hilbert modules

This section begins by describing the notion of "split short exact sequence"from commutative algebra. Let M_1 and M_2 be modules over a ring R. Then $M_1 \oplus M_2$, module direct sum of M_1 and M_2, yields the short exact sequence

$$0 \longrightarrow M_1 \xrightarrow{i} M_1 \oplus M_2 \xrightarrow{\pi} M_2 \longrightarrow 0,$$

where i is the embedding, $m_1 \mapsto (m_1, 0)$ and π is the projection, $(m_1, m_2) \mapsto m_2$ for all $m_1 \in M_1$ and $m_2 \in M_2$. A short exact sequence of modules

$$0 \longrightarrow M_1 \xrightarrow{\varphi_1} M \xrightarrow{\varphi_2} M_2 \longrightarrow 0, ,$$

is called *split exact sequence* if there exists a module isomorphism $\varphi : M \to M_1 \oplus M_2$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{\varphi_1} & M & \xrightarrow{\varphi_2} & M_2 & \longrightarrow & 0 \\
 & & I_{M_1}\Big\downarrow & & \varphi\Big\downarrow & & I_{M_2}\Big\downarrow & & \\
0 & \longrightarrow & M_1 & \xrightarrow{i} & M_1 \oplus M_2 & \xrightarrow{\pi} & M_2 & \longrightarrow & 0
\end{array}$$

commutes. It is well known that a short exact sequence of modules

$$0 \longrightarrow M_1 \xrightarrow{\varphi_1} M \xrightarrow{\varphi_2} M_2 \longrightarrow 0,$$

splits if and only if φ_2 has a right inverse, if and only if φ_1 has a left inverse.

Now let $\mathcal{R} \subseteq \mathcal{O}(\Omega, \mathbb{C})$ be a reproducing kernel Hilbert module. In the rest of this section focus will be on the quotient module $\mathcal{Q}_\Theta$ of $\mathcal{R} \otimes \mathcal{E}_*$ given by the exact sequence of Hilbert modules

$$\cdots \longrightarrow \mathcal{R} \otimes \mathcal{E} \xrightarrow{M_\Theta} \mathcal{R} \otimes \mathcal{E}_* \xrightarrow{\pi_\Theta} \mathcal{Q}_\Theta \longrightarrow 0,$$

where $\Theta \in \mathcal{M}_{\mathcal{R}}(\mathcal{E}, \mathcal{E}_*)$ is a multiplier. In other words,

$$\mathcal{Q}_\Theta = (\mathcal{R} \otimes \mathcal{E}_*)/\mathrm{ran} M_\Theta.$$

The exact sequence is called *split* if π_Θ has a module right inverse, that is, if there exists a module map $\sigma_\Theta : \mathcal{Q}_\Theta \to \mathcal{R} \otimes \mathcal{E}_*$ such that

$$\pi_\Theta \sigma_\Theta = I_{\mathcal{R} \otimes \mathcal{E}_*}.$$

5.1. **Complemented submodules.** This subsection provides a direct result concerning splitting of Hilbert modules, which involves a mixture of operator theory and algebra.

THEOREM 5.1. *Let $\mathcal{R} \subseteq \mathcal{O}(\Omega, \mathbb{C})$ be a reproducing kernel Hilbert module and $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E}, \mathcal{E}_*)}(\mathcal{R})$ be a multiplier for Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$ such that ran M_Θ is closed. Then ran M_Θ is complemented in $\mathcal{R} \otimes \mathcal{E}_*$ if and only if π_Θ is right invertible, that is, there exists a module map $\sigma_\Theta : \mathcal{Q}_\Theta \to \mathcal{R} \otimes \mathcal{E}_*$ such that $\pi_\Theta \sigma_\Theta = I_{\mathcal{R} \otimes \mathcal{E}_*}$.*

Proof. Let $\mathcal{S}$ be a submodule of $\mathcal{R} \otimes \mathcal{E}_*$ such that $\mathcal{R} \otimes \mathcal{E}_* = \mathrm{ran}\, M_\Theta \dot{+} \mathcal{S}$. Then

$$Y = \pi_\Theta|_{\mathcal{S}} : \mathcal{S} \to \mathcal{Q}_\Theta$$

is one-to-one. Also for $f \in \mathcal{Q}_\Theta$ and $f = f_1 + f_2$ with $f_1 \in \mathrm{ran} M_\Theta$ and $f_2 \in \mathcal{S}$ one gets

$$\pi_\Theta f_2 = \pi_\Theta (f - f_1) = f - \pi_\Theta f_1 = f.$$

Consequently, $Y = \pi_\Theta|_{\mathcal{S}}$ is onto. Hence

$$Y^{-1} : (\mathcal{R} \otimes \mathcal{E}_*) \,/\, \mathrm{ran}\, M_\Theta \to \mathcal{S},$$

is bounded by the closed graph theorem. Let

$$\sigma_\Theta = i\, Y^{-1},$$

where $i : \mathcal{S} \to \mathcal{R} \otimes \mathcal{E}_*$ is the inclusion map. Again for $f = f_1 + f_2 \in \mathcal{Q}_\Theta$ with $f_1 \in \mathrm{ran} M_\Theta$ and $f_2 \in \mathcal{S}$ one gets

$$\pi_\Theta \sigma_\Theta f = \pi_\Theta (i Y^{-1})(f_1 + f_2) = \pi_\Theta i f_2 = \pi_\Theta f_2 = \pi_\Theta (f - f_1) = \pi_\Theta f = f.$$

Therefore, σ_Θ is a right inverse for π_Θ. Also that σ_Θ is a module map follows from the fact that Y is a module map.
Conversely, let $\sigma_\Theta : \mathcal{Q}_\Theta \to \mathcal{R} \otimes \mathcal{E}_*$ be a module map which is a right inverse of π_Θ. Then $\sigma_\Theta \pi_\Theta$ is an idempotent on $\mathcal{R} \otimes \mathcal{E}_*$ such that $\mathcal{S} = \mathrm{ran}\, \sigma_\Theta \pi_\Theta$ is a complementary submodule for the closed submodule ran M_Θ in $\mathcal{R} \otimes \mathcal{E}_*$. ■

Examples in the case $n = 1$ show that the existence of a right inverse for π_Θ does not imply that ran M_Θ is closed. However, if $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E}, \mathcal{E}_*)}(\mathcal{R})$ and ran M_Θ is complemented in $\mathcal{R} \otimes \mathcal{E}_*$, then ran M_Θ is closed.

5.2. **Lifting and range-inclusion theorems.** This subsection concerned with the study of lifting and Drury-Arveson module (see [Ar98] or [Sa14a]). The *commutant lifting theorem* will be used to extend some algebraic results for the case of quotient modules of the Drury-Arveson module. The commutant lifting theorem for the Drury-Arveson module is due to Ball, Trent and Vinnikov, Theorem 5.1 in [BaTrVi01].

THEOREM 5.2. *Let $\mathcal{N}$ and $\mathcal{N}_*$ be quotient modules of $H^2_n \otimes \mathcal{E}$ and $H^2_n \otimes \mathcal{E}_*$ for some Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, respectively. If $X : \mathcal{N} \to \mathcal{N}_*$ is a bounded module map, that is,*

$$XP_{\mathcal{N}}(M_{z_i} \otimes I_{\mathcal{E}})|_{\mathcal{N}} = P_{\mathcal{N}_*}(M_{z_i} \otimes I_{\mathcal{E}_*})|_{\mathcal{N}_*}X,$$

for $i = 1, \ldots, n$, then there exists a multiplier $\Phi \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H^2_n)$ such that $\|X\| = \|M_\Phi\|$ and $P_{\mathcal{N}_*}M_\Phi = X$.

In the language of Hilbert modules, one has the following commutative diagram

$$\begin{array}{ccc} H^2_n \otimes \mathcal{E} & \xrightarrow{M_\Phi} & H^2_n \otimes \mathcal{E}_* \\ {\scriptstyle \pi_{\mathcal{N}}}\Big\downarrow & & {\scriptstyle \pi_{\mathcal{N}_*}}\Big\downarrow \\ \mathcal{N} & \xrightarrow{X} & \mathcal{N}_* \end{array}$$

where $\pi_{\mathcal{N}}$ and $\pi_{\mathcal{N}_*}$ are the quotient maps.

As one knows, by considering the $n = 1$ case, there is more than one multiplier $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H^2_n)$ for Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$ with the same range and thus yielding the same quotient. Things are even more complicated for $n > 1$. However, the following result using the commutant lifting theorem introduces some order.

THEOREM 5.3. *Let $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H^2_n)$ be a multiplier with closed range for Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$ and $\Phi \in \mathcal{M}_{\mathcal{B}(\mathcal{F},\mathcal{E}_*)}(H^2_n)$ for some Hilbert space $\mathcal{F}$. Then*

$$\text{ran } M_\Phi \subseteq \text{ran } M_\Theta,$$

if and only if

$$\Phi = \Theta\Psi,$$

for some multiplier $\Psi \in \mathcal{M}_{\mathcal{B}(\mathcal{F},\mathcal{E})}(H^2_n)$.

Proof. If $\Psi \in \mathcal{M}(\mathcal{F}, \mathcal{E})$ such that $\Phi = \Theta\Psi$, then $M_\Phi = M_\Theta M_\Psi$ and hence

$$\text{ran } M_\Phi = \text{ran } M_\Theta M_\Psi \subseteq \text{ran } M_\Theta.$$

Suppose ran $M_\Phi \subseteq$ ran M_Θ. Consider the module map $\hat{M}_\Theta : (H^2_n \otimes \mathcal{E}) \,/\, \ker M_\Theta \longrightarrow \text{ran } M_\Theta$ defined by

$$\hat{M}_\Theta \gamma_\Theta = M_\Theta,$$

where $\gamma_\Theta : H^2_n \otimes \mathcal{E} \longrightarrow (H^2_n \otimes \mathcal{E})/ \ker M_\Theta$ is the quotient module map. Since ran M_Θ is closed that $\hat{M}_\Theta$ is invertible. Then

$$\hat{X} := \hat{M}_\Theta^{-1} : \text{ ran } M_\Theta \to (H^2_n \otimes \mathcal{E})/ \ker M_\Theta$$

is bounded by the closed graph theorem and so is

$$\hat{X} M_\Phi : H^2_n \otimes \mathcal{F} \to (H^2_n \otimes \mathcal{E}) \,/ \ker M_\Theta.$$

By Theorem 5.2 there exists a multiplier $\Psi \in \mathcal{M}_{\mathcal{B}(\mathcal{F},\mathcal{E})}(H^2_n)$ so that

$$\gamma_\Theta M_\Psi = \hat{X} M_\Phi,$$

and hence

$$M_\Theta M_\Psi = (\hat{M}_\Theta \gamma_\Theta) M_\Psi = \hat{M}_\Theta (\hat{X} M_\Phi) = M_\Phi,$$

or $\Phi = \Theta\Psi$ which completes the proof. ∎

5.3. **Regular inverse and similarity problem.** The purpose of this subsection is to establish an equivalent condition which will allow one to tell when the range of a multiplier will be complemented.

A multiplier $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H^2_n)$ is said to have a regular inverse if there exists $\Psi \in \mathcal{M}_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(H^2_n)$ such that

$$\Theta(\boldsymbol{z})\Psi(\boldsymbol{z})\Theta(\boldsymbol{z}) = \Theta(\boldsymbol{z}). \qquad (\boldsymbol{z} \in \mathbb{B}^n)$$

THEOREM 5.4. *Let $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H^2_n)$. Then Θ admits a regular inverse if and only if $ran\, M_\Theta$ is complemented in $H^2_n \otimes \mathcal{E}_*$, or*

$$H^2_n \otimes \mathcal{E}_* = ran\, M_\Theta \dot{+} \mathcal{S},$$

for some submodule $\mathcal{S}$ of $H^2_n \otimes \mathcal{E}_*$.

Proof. If $H^2_n \otimes \mathcal{E}_* = \operatorname{ran} M_\Theta \dot{+} \mathcal{S}$ for some (closed) submodule $\mathcal{S}$, then ran M_Θ is closed. Consider the module map $\hat{M}_\Theta : (H^2_n \otimes \mathcal{E}) \,/\, \ker M_\Theta \longrightarrow (H^2_n \otimes \mathcal{E}_*) \,/\, \mathcal{S}$ defined by

$$\hat{M}_\Theta \gamma_\Theta = \pi_{\mathcal{S}} M_\Theta,$$

where $\gamma_\Theta : H^2_n \otimes \mathcal{E} \to (H^2_n \otimes \mathcal{E}) \,/\, \ker M_\Theta$ and $\pi_{\mathcal{S}} : H^2_n \otimes \mathcal{E}_* \to (H^2_n \otimes \mathcal{E}_*) \,/\, \mathcal{S}$ are quotient maps. This map is one-to-one and onto and thus has a bounded inverse $\hat{X} = \hat{M}_\Theta^{\,-1} : (H^2_n \otimes \mathcal{E}_*) \,/\, \mathcal{S} \to (H^2_n \otimes \mathcal{E}) \,/\, \ker M_\Theta$ by the closed graph theorem. Since $\hat{X}$ satisfies the hypotheses of the commutant lifting theorem, there exists $\Psi \in \mathcal{M}_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(H^2_n)$ such that $\gamma_\Theta M_\Psi = \hat{X} \pi_{\mathcal{S}}$. Further, $\hat{M}_\Theta \gamma_\Theta = \pi_{\mathcal{S}} M_\Theta$ yields

$$\pi_{\mathcal{S}} M_\Theta M_\Psi = \hat{M}_\Theta \gamma_\Theta M_\Psi = \hat{M}_\Theta \hat{X} \pi_{\mathcal{S}} = \pi_{\mathcal{S}},$$

and therefore,

$$\pi_{\mathcal{S}}(M_\Theta M_\Psi M_\Theta - M_\Theta) = 0.$$

Since $\pi_{\mathcal{S}}$ is one-to-one on $\operatorname{ran} M_\Theta$, it follows that $M_\Theta M_\Psi M_\Theta = M_\Theta$.
Now suppose there exists $\Psi \in \mathcal{M}_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(H^2_n)$ such that $M_\Theta M_\Psi M_\Theta = M_\Theta$. This implies that

$$(M_\Theta M_\Psi)^2 = M_\Theta M_\Psi,$$

and hence $M_\Theta M_\Psi$ is an idempotent. From the equality $M_\Theta M_\Psi M_\Theta = M_\Theta$ one obtains both that $\operatorname{ran} M_\Theta M_\Psi$ contains $\operatorname{ran} M_\Theta$ and that $\operatorname{ran} M_\Theta M_\Psi$ is contained in $\operatorname{ran} M_\Theta$. Therefore,

$$\operatorname{ran} M_\Theta M_\Psi = \operatorname{ran} M_\Theta,$$

and

$$\mathcal{S} = \operatorname{ran}(I - M_\Theta M_\Psi),$$

is a complementary submodule of ran M_Θ in $H^2_n \otimes \mathcal{E}_*$. ∎

COROLLARY 5.5. *Assume $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H^2_n)$ for Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$ such that ran M_Θ is closed and $\mathcal{H}_\Theta$ is defined by*

$$H^2_n \otimes \mathcal{E} \xrightarrow{M_\Theta} H^2_n \otimes \mathcal{E}_* \longrightarrow \mathcal{H}_\Theta \longrightarrow 0.$$

If $\mathcal{H}_\Theta$ is similar to $H^2_n \otimes \mathcal{F}$ for some Hilbert space $\mathcal{F}$, then the sequence splits.

Proof. First, assume that there exists an invertible module map $X : H^2_n \otimes \mathcal{F} \to \mathcal{H}_\Theta$, and let $\Phi \in \mathcal{M}_{\mathcal{B}(\mathcal{F},\mathcal{E}_*)}(H^2_n)$ be defined by the commutant lifting theorem, Theorem 5.2, so that

$$\pi_\Theta M_\Phi = X,$$

where $\pi_\Theta : H^2_n \otimes \mathcal{E}_* \to (H^2_n \otimes \mathcal{E}_*) \,/\, \text{ran } M_\Theta$ is the quotient map. Since X is invertible one gets

$$H^2_n \otimes \mathcal{E}_* = \text{ran } M_\Phi \dot{+} \text{ran} M_\Theta.$$

Thus ran M_Θ is complemented and hence it follows from Theorem 5.1 that the sequence splits. ■

Finally, the following weaker converse to Corollary 5.5 always holds.

COROLLARY 5.6. *Let $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H^2_n)$ for Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, and set $\mathcal{H}_\Theta = (H^2_n \otimes \mathcal{E}_*)/$ clos [ran M_Θ]. Then the following statements are equivalent:*
(i) Θ is left invertible, that is, there exists $\Psi \in \mathcal{M}_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(H^2_n)$ such that $\Psi\Theta = I_{\mathcal{E}}$.
(ii) ran M_Θ is closed, ker $M_\Theta = \{0\}$ and $\mathcal{H}_\Theta$ is similar to a complemented submodule $\mathcal{S}$ of $H^2_n \otimes \mathcal{E}_*$.

Proof. If (i) holds, then ran M_Θ is closed and ker $M_\Theta = \{0\}$. Further, $M_\Theta M_\Psi$ is an idempotent on $H^2_n \otimes \mathcal{E}_*$ such that ran $M_\Theta M_\Psi =$ ran M_Θ and $\mathcal{H}_\Theta$ is isomorphic to

$$\mathcal{S} = \text{ran } (I - M_\Theta M_\Psi) \subseteq H^2_n \otimes \mathcal{E}_*,$$

and

$$H^2_n \otimes \mathcal{E}_* = \text{ran } M_\Theta \dot{+} \mathcal{S},$$

so $\mathcal{S}$ is complemented.
Now assume that (ii) holds and there exists an isomorphism $X : \mathcal{H}_\Theta \to \mathcal{S} \subseteq H^2_n \otimes \mathcal{E}_*$, where $\mathcal{S}$ is a complemented submodule of $H^2_n \otimes \mathcal{E}_*$. Then

$$Y = X\pi_\Theta : H^2_n \otimes \mathcal{E}_* \to H^2_n \otimes \mathcal{E}_*,$$

is a module map and hence there exists a multiplier $\Xi \in \mathcal{M}_{\mathcal{B}(\mathcal{E}_*,\mathcal{E}_*)}(H^2_n)$ so that $Y = M_\Xi$. Since X is invertible, ran $M_\Xi = \mathcal{S}$, which is complemented by assumption, and hence by Theorem 5.4 there exists $\Psi \in \mathcal{M}_{\mathcal{B}(\mathcal{E}_*,\mathcal{E}_*)}(H^2_n)$ such that $M_\Xi = M_\Xi M_\Psi M_\Xi$ or $M_\Xi(I - M_\Psi M_\Xi) = 0$. Therefore,

$$\text{ran } (I - M_\Psi M_\Xi) \subseteq \text{ker } M_\Xi = \text{ker } Y = \text{ker } \pi_\Theta = \text{ran } M_\Theta.$$

Applying Theorem 5.3, we obtain $\Phi \in \mathcal{M}_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(H^2_n)$ so that

$$I - M_\Psi M_\Xi = M_\Theta M_\Phi.$$

Thus using $M_\Xi M_\Theta = 0$ we see that $M_\Theta M_\Phi M_\Theta = (I - M_\Psi M_\Xi)M_\Theta = M_\Theta$, or, $M_\Theta = M_\Theta M_\Phi M_\Theta$. Since ker $M_\Theta = \{0\}$, we have $M_\Phi M_\Theta = I_{H^2_n \otimes \mathcal{E}}$, which completes the proof. ■

Theorem 5.4 and Corollary 5.5 yields the main result of the present subsection.

Corollary 5.7. *Given $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H_n^2)$ for Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$ such that ran M_Θ is closed, consider the quotient Hilbert module $\mathcal{H}_\Theta$ given by*

$$H_n^2 \otimes \mathcal{E} \xrightarrow{M_\Theta} H_n^2 \otimes \mathcal{E}_* \longrightarrow \mathcal{H}_\Theta \longrightarrow 0.$$

If $\mathcal{H}_\Theta$ is similar to $H_n^2 \otimes \mathcal{F}$ for some Hilbert space $\mathcal{F}$, then Θ has a regular inverse.

Further results and comments:

(1) Most of the material in this section can be found in [DoFoSa12].
(2) It is well known that (see [NaFo70a]) a contractive Hilbert module $\mathcal{H}$ over $A(\mathbb{D})$ is similar to a unilateral shift if and only if its characteristic function $\Theta_{\mathcal{H}}$ has a left inverse. Various approaches to this result have been given but the present one uses the commutant lifting theorem and, implicitly, the Beurling-Lax-Halmos theorem. In particular, the proof does not involve, at least explicitly, the geometry of the dilation space for the contraction.
(3) For $n = 1$, Corollary 5.7 yields a more general result concerning similarity of contractive Hilbert modules over $A(\mathbb{D})$:

Theorem 5.8. *Let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces and $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ be a bounded analytic function such that ker $M_\Theta = \{0\}$ and ran M_Θ is closed. Then the quotient module $\mathcal{H}_\Theta$ given by*

$$0 \longrightarrow H^2_{\mathcal{E}}(\mathbb{D}) \xrightarrow{M_\Theta} H^2_{\mathcal{E}_*}(\mathbb{D}) \longrightarrow \mathcal{H}_\Theta \longrightarrow 0,$$

is similar to $H^2_{\mathcal{F}}(\mathbb{D})$ for some Hilbert space $\mathcal{F}$ if and only if $\Theta\Psi\Theta = \Theta$ for some $\Psi \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D})$.

(4) If $\mathcal{H}$ is a Hilbert module over $\mathbb{C}[\boldsymbol{z}]$, Corollary 5.7 remains true under the assumption that the analogue of the commutant lifting theorem holds for the class of Hilbert modules under consideration. In particular, Corollary 5.7 can be generalized to any other reproducing kernel Hilbert modules where the kernel is given by a complete Nevanlinna-Pick kernel (see [AgMc00]).
(5) For other results concerning similarity in both commutative and noncommutative setup see Popescu [Po11].

6. Generalized canonical models and similarity

This section describes conditions for certain quotient Hilbert modules to be similar to the reproducing kernel Hilbert modules from which they are constructed.

In particular, it is shown that the similarity criterion for a certain class of quotient Hilbert modules is independent of the choice of the basic Hilbert module "building blocks" as in the isomorphism case, so long as the multiplier algebras are the same.

6.1. **Corona pairs.** This subsection begins with the case in which the existence of a left inverse for the multiplier depends only on a positive answer to the corona problem for the domain.

THEOREM 6.1. *Let $\mathcal{R} \subseteq \mathcal{O}(\Omega, \mathbb{C})$ be a reproducing kernel Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Assume that $\theta_1, \theta_2, \psi_1, \psi_2$ are in $\mathcal{M}(\mathcal{R})$ and that $\theta_1\psi_1 + \theta_2\psi_2 = 1$. Then the quotient Hilbert module $\mathcal{R}_\Theta = (\mathcal{R} \otimes \mathbb{C}^2)/M_\Theta \mathcal{R}$ is similar to $\mathcal{R}$, where $M_\Theta f = \theta_1 f \otimes e_1 + \theta_2 f \otimes e_2 \in \mathcal{H} \otimes \mathbb{C}^2$ and $f \in \mathcal{R}$, with $\{e_1, e_2\}$ the standard orthonormal basis for $\mathbb{C}^2$.*

Proof. Let $R_\Psi : \mathcal{R} \oplus \mathcal{R} \to \mathcal{R}$ be the bounded module map defined by $R_\Psi(f \oplus g) = \psi_1 f + \psi_2 g$ for $f, g \in \mathcal{R}$. Note that

$$R_\Psi M_\Theta = I_{\mathcal{H}},$$

or that R_Ψ is a left inverse for M_Θ. Then for any $f \oplus g \in \mathcal{R} \oplus \mathcal{R}$, one gets

$$f \oplus g = (M_\Theta R_\Psi(f \oplus g)) + (f \oplus g - M_\Theta R_\Psi(f \oplus g)),$$

with $M_\Theta R_\Psi(f \oplus g) \in \operatorname{ran} M_\Theta$ and $f \oplus g - M_\Theta R_\Psi(f \oplus g) \in \ker R_\Psi$. This decomposition, along with

$$\operatorname{ran} M_\Theta \cap \ker R_\Psi = \{0\}$$

implies that

$$\mathcal{R} \oplus \mathcal{R} = \operatorname{ran} M_\Theta \dot{+} \ker R_\Psi.$$

Thus, there exists a module idempotent $Q \in \mathcal{B}(\mathcal{R} \oplus \mathcal{R})$ with matrix entries in $\mathcal{M}(\mathcal{R})$ such that $Q(\Theta f + g) = g$ for $f \in \mathcal{R}$ and $g \in \ker R_\Psi$. Moreover, $\operatorname{ran} M_\Theta = \ker Q$ and $\ker R_\Psi = \operatorname{ran} Q$. The composition $Q \circ \pi_\Theta^{-1} : \mathcal{R}_\Theta \to \mathcal{R}$ is well-defined and is the required invertible module map establishing the similarity of $\mathcal{R}_\Theta$ and $\mathcal{R}$. ■

COROLLARY 6.2. *Let $\theta_1, \theta_2 \in \mathcal{M}(H_n^2)$ satisfy $|\theta_1(\boldsymbol{z})|^2 + |\theta_2(\boldsymbol{z})|^2 \geq \epsilon$ for all $\boldsymbol{z} \in \mathbb{B}^n$ and some $\epsilon > 0$. Then the quotient Hilbert module $(H_n^2)_\Theta = (H_n^2 \otimes \mathbb{C}^2)/M_\Theta H_n^2$ is similar to H_n^2.*

Proof. The corollary follows from Theorem 6.1 using the corona theorem for $\mathcal{M}(H_n^2)$ (see [CSW11] or [OF00]). ■

6.2. **Left invertible multipliers.** The question of similarity of a quotient Hilbert module to the building block Hilbert module can be raised in the context of a split short exact sequence. More precisely, suppose $\mathcal{R}, \tilde{\mathcal{R}} \subseteq \mathcal{O}(\Omega, \mathbb{C})$ be reproducing kernel Hilbert modules and $\mathcal{M}(\mathcal{R}) = \mathcal{M}(\tilde{\mathcal{R}})$. Moreover, suppose $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E}, \mathcal{E}_*)}(\mathcal{R})$ and hence $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E}, \mathcal{E}_*)}(\tilde{\mathcal{R}})$. Consider the generalized canonical models

$$\mathcal{R} \otimes \mathcal{E} \xrightarrow{M_\Theta} \mathcal{R} \otimes \mathcal{E}_* \longrightarrow \mathcal{R}_\Theta \longrightarrow 0, \quad \text{and} \quad \tilde{\mathcal{R}} \otimes \mathcal{E} \xrightarrow{M_\Theta} \tilde{\mathcal{R}} \otimes \mathcal{E}_* \longrightarrow \tilde{\mathcal{R}}_\Theta \longrightarrow 0.$$

One can propose the following assertion: $\mathcal{R}_\Theta$ is similar to $\mathcal{R} \otimes \mathcal{F}$ for some Hilbert space $\mathcal{F}$ if and only if $\tilde{\mathcal{R}}_\Theta$ is similar to $\tilde{\mathcal{R}} \otimes \tilde{\mathcal{F}}$ for some Hilbert space $\tilde{\mathcal{F}}$.

Where the answer to the above question is not affirmative, however, the following hold:

THEOREM 6.3. *Let $\mathcal{H}, \tilde{\mathcal{H}} \in B_1^*(\Omega)$ for $\Omega \subseteq \mathbb{C}^n$, be such that $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{M}(\tilde{\mathcal{H}})$ and let $\Theta \in \mathcal{M}_{\mathcal{B}(\mathbb{C}^p, \mathbb{C}^q)}(\mathcal{H})$, for $1 \leq p < q$, be left invertible. Then the similarity of $\mathcal{H}_\Theta = (\mathcal{H} \otimes \mathbb{C}^q)/M_\Theta(\mathcal{H} \otimes \mathbb{C}^p)$ to $\mathcal{H} \otimes \mathbb{C}^{q-p}$ implies the similarity of $\tilde{\mathcal{H}}_\Theta = (\tilde{\mathcal{H}} \otimes \mathbb{C}^q)/M_\Theta(\tilde{\mathcal{H}} \otimes \mathbb{C}^p)$ to $\tilde{\mathcal{H}} \otimes \mathbb{C}^{q-p}$.*

Proof. Since $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{M}(\tilde{\mathcal{H}})$, $\Theta \in \mathcal{M}_{\mathcal{B}(\mathbb{C}^p,\mathbb{C}^q)}(\mathcal{H})$ and $\tilde{\mathcal{H}}_\Theta$ is well-defined. Moreover, by Theorem 2.1, we have $\mathcal{H}_\Theta, \tilde{\mathcal{H}}_\Theta \in B^*_{q-p}(\Omega)$. Let σ_Θ be a module cross-section for $\mathcal{H}_\Theta$; that is, $\sigma_\Theta : \mathcal{H}_\Theta \to \mathcal{H} \otimes \mathbb{C}^q$ such that $\pi_\Theta \sigma_\Theta = I_{\mathcal{H}_\Theta}$. Since $Q := \sigma_\Theta \pi_\Theta$ is a module idempotent on $\mathcal{H} \otimes \mathbb{C}^q$, it follows that

$$\text{ran } Q \dotplus \text{ran } M_\Theta = \mathcal{H} \otimes \mathbb{C}^q.$$

But there exists a $\Phi \in \mathcal{M}_{\mathcal{B}(\mathbb{C}^q)}(\mathcal{H})$ such that

$$M_\Phi = Q,$$

and $\Phi(z)$ is an idempotent on $\mathcal{B}(\mathbb{C}^q)$ for $z \in \Omega$. An easy argument using localization shows that

$$\text{ran } \Phi(z) \dotplus \text{ran } \Theta(z) = \mathbb{C}^q,$$

for $z \in \Omega$. But this fact is independent of $\mathcal{H}$.
Therefore

$$\tilde{\sigma}_\Theta := M_\Phi \tilde{\pi}_\Theta^{-1}$$

is a module map from $\tilde{\mathcal{H}}_\Theta$ to $\tilde{\mathcal{H}} \otimes \mathbb{C}^q$, where $\tilde{\pi}_\Theta$ is the quotient map of the short exact sequence for $\tilde{\mathcal{H}}_\Theta$. Moreover, the idempotent $\tilde{Q} = \tilde{\sigma}_\Theta \tilde{\pi}_\Theta$ is again represented by M_Φ.

Suppose that $\mathcal{H}_\Theta$ is similar to $\mathcal{H} \otimes \mathbb{C}^{q-p}$. Then there exists an invertible module map $X : \mathcal{H} \otimes \mathbb{C}^{q-p} \to \mathcal{H}_\Theta$. Compose the module maps σ_Θ and X to obtain $Y = \sigma_\Theta X : \mathcal{H} \otimes \mathbb{C}^{q-p} \to \mathcal{H} \otimes \mathbb{C}^q$ and let $\Gamma \in \mathcal{M}_{\mathcal{B}(\mathbb{C}^{q-p},\mathbb{C}^q)}(\mathcal{H})$ so that $Y = M_\Gamma$. Since $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{M}(\tilde{\mathcal{H}})$, one can use Γ to define

$$M_\Gamma : \tilde{\mathcal{H}} \otimes \mathbb{C}^{q-p} \to \tilde{\mathcal{H}} \otimes \mathbb{C}^q.$$

Composing M_Γ^{-1} and $\tilde{\sigma}_\Theta$, one gets an invertible module map $M_\Gamma^{-1} \tilde{\sigma}_\Theta$ from $\tilde{\mathcal{H}}_\Theta$ to $\tilde{\mathcal{H}} \otimes \mathbb{C}^{q-p}$, which shows that $\tilde{\mathcal{H}}_\Theta$ is similar to $\tilde{\mathcal{H}} \otimes \mathbb{C}^{q-p}$. ∎

Theorem 6.3 yields the following corollary.

Corollary 6.4. *Let $\mathcal{H} \in B_1^*(\mathbb{D})$ be a contractive Hilbert module over $A(\mathbb{D})$ and $\Theta \in H^\infty_{\mathcal{B}(\mathbb{C}^p,\mathbb{C}^q)}(\mathbb{D})$ is left-invertible. Then $\mathcal{H}_\Theta$ is similar to $\mathcal{H} \otimes \mathbb{C}^{q-p}$.*

Proof. Here one can use Theorem 6.3 with $\tilde{\mathcal{H}} = H^2(\mathbb{D})$ and $\mathcal{H}$ the given Hilbert module. Clearly $\mathcal{M}(\tilde{\mathcal{H}}) = H^\infty(\mathbb{D})$. The proof is completed by appealing to a result of Sz.-Nagy and Foias about a left invertible Θ (cf. [NaFo70a]). ∎

Further results and comments:

(1) Results of this section can be found in [DoKKSa14].
(2) The question of similarity is equivalent to a problem in complex geometry (cf. [Do88]). In general, for a split short exact sequence

$$0 \longrightarrow \mathcal{H} \otimes \mathbb{C}^p \xrightarrow{M_\Theta} \mathcal{H} \otimes \mathbb{C}^q \xrightarrow{\pi_\Theta} \mathcal{H}_\Theta \longrightarrow 0,$$

one can define the idempotent function $\Gamma : \Omega \to \mathcal{B}(\mathbb{C}^q)$, where ran Γ yields a hermitian holomorphic subbundle $\mathcal{F}$ of the trivial bundle $\Omega \times \mathbb{C}^q$. If Ω is contractible, then $\mathcal{F}$ is trivial. The question of similarity is equivalent to whether one can find a trivializing frame for which the corresponding Gramian G is uniformly bounded above and below when $\mathcal{M}(\mathcal{H}) = H^\infty(\Omega)$, or it and its inverse lie in the multiplier algebra when it is smaller. As mentioned above, this question is related to the corona problem and the commutant lifting theorem.

7. Free resolutions of Hilbert modules

Consideration of dilations such as those in Section 8 in [Sa14a], raises the question of what kind of resolutions exist for co-spherically contractive Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$. In particular, Theorem 8.3 in [Sa14a] yields a unique resolution of an arbitrary pure co-spherically contractive Hilbert module $\mathcal{H}$ over $\mathbb{C}[\boldsymbol{z}]$ in terms of Drury-Arveson modules and inner multipliers.

Let $\mathcal{H}$ be a co-spherically contractive Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. By Theorem 8.3 in [Sa14a], there exists a unique (assuming the minimality) free module $\mathcal{F}_0$ such that $\mathcal{F}_0$ is a dilation of $\mathcal{H}$. That is, there exists a module co-isometry

$$\pi_{\mathcal{H}} = X_0 : \mathcal{F}_0 \to \mathcal{H}.$$

The kernel of X_0 is a closed submodule of $\mathcal{F}_0$ and again by Theorem 8.3 in [Sa14a], there exists a free module $\mathcal{F}_1$ and a partially isometric module map $X_1 : \mathcal{F}_1 \to \mathcal{F}_0$ such that

$$\text{ran} X_1 = \ker\, X_0.$$

By repeating this process, one obtains a sequence of free Hilbert modules $\{\mathcal{F}_i\}$ and an exact sequence:

$$\cdots \xrightarrow{X_2} \mathcal{F}_1 \xrightarrow{X_1} \mathcal{F}_0 \xrightarrow{X_0} \mathcal{H} \longrightarrow 0., \tag{7.5}$$

A basic question is whether such a resolution can have finite length or, equivalently, whether one can take $\mathcal{E}_N = \{0\}$ for some finite N. That will be the case if and only if some X_k is an isometry or, equivalently, if $\ker X_k = \{0\}$.

7.1. **Isometric multipliers.** Let $V \in \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$ be an isometry. Then $\Phi_V \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H^2_n)$, defined by $\Phi_V = I_{H^2_n} \otimes V$ is an isometric multiplier. The purpose of this subsection is to prove that all isometric multipliers are of this form.

Theorem 7.1. *For $n > 1$, if $V : H^2_n \otimes \mathcal{E} \to H^2_n \otimes \mathcal{E}_*$ is an isometric module map for Hilbert spaces $\mathcal{E}$ and $\mathcal{E}_*$, then there exists an isometry $V_0 : \mathcal{E} \to \mathcal{E}_*$ such that*

$$V(\boldsymbol{z}^{\boldsymbol{k}} \otimes x) = \boldsymbol{z}^{\boldsymbol{k}} \otimes V_0 x, \quad \textit{for } \boldsymbol{k} \in \mathbb{N}^n, x \in \mathcal{E}_*.$$

Moreover, ran V is a reducing submodule of $H^2_n \otimes \mathcal{E}_*$ of the form $H^2_n \otimes (\text{ran}\, V_0)$.

Proof. For $x \in \mathcal{E}$, $\|x\| = 1$, one can compute

$$V(1 \otimes x) = f(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathbb{N}^n} a_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}, \qquad \text{for } \{a_k\} \subseteq \mathcal{E}.$$

Then

$$V(z_1 \otimes x) = V M_{z_1}(1 \otimes x) = M_{z_1} V(1 \otimes x) = M_{z_1} f = z_1 f,$$

and

$$\|z_1 f\|^2 = \|z_1 V(1 \otimes x)\|^2 = \|z_1 \otimes x\|^2 = 1 = \|f\|^2.$$

Therefore

$$\sum_{\boldsymbol{k}\in\mathbb{N}^n} \|a_{\boldsymbol{k}}\|^2_{\mathcal{E}_*}\|\boldsymbol{z}^{\boldsymbol{k}}\|^2 = \sum_{\boldsymbol{k}\in\mathbb{N}^n} \|a_{\boldsymbol{k}}\|^2_{\mathcal{E}_*}\|\boldsymbol{z}^{\boldsymbol{k}+e_1}\|^2, \text{ where } \boldsymbol{k}+e_1 = (k_1+1,\dots,k_n),$$

or

$$\sum_{\boldsymbol{k}\in\mathbb{N}^n} \|a_{\boldsymbol{k}}\|^2_{\mathcal{E}_*}\{\|\boldsymbol{z}^{\boldsymbol{k}+e_1}\|^2 - \|\boldsymbol{z}^{\boldsymbol{k}}\|^2\} = 0.$$

If $\boldsymbol{k} = (k_1,\dots k_n)$, then

$$\begin{aligned}\|\boldsymbol{z}^{\boldsymbol{k}+e_1}\|^2 &= \frac{(k_1+1)!\cdots k_n!}{(k_1+\cdots+k_n+1)!} = \frac{k_1!\cdots k_n!}{(k_1+\cdots+k_n)!}\frac{k_1+1}{k_1+\cdots+k_n+1} \\ &< \frac{k_1!\cdots k_n!}{(k_1+\cdots+k_n)!} = \|\boldsymbol{z}^{\boldsymbol{k}}\|^2,\end{aligned}$$

unless $k_2 = k_3 = \ldots = k_n = 0$. Since, $a_{\boldsymbol{k}} \neq 0$ implies $\|\boldsymbol{z}^{\boldsymbol{k}+e_1}\| = \|\boldsymbol{z}^{\boldsymbol{k}}\|$ we have $k_2 = \cdots = k_n = 0$. Repeating this argument using $i = 2,\dots,n$, it follows that $a_{\boldsymbol{k}} = 0$ unless $\boldsymbol{k} = (0,\dots,0)$ and therefore, $f(\boldsymbol{z}) = 1 \otimes y$ for some $y \in \mathcal{E}_*$. Set $V_0 x = y$ to complete the first part of the proof.

Finally, since ran $V = H^2_n \otimes (\text{ran} V_0)$, it follows that ran V is a reducing submodule, which completes the proof. ■

7.2. **Inner resolutions.** This subsection begins with a definition based on the dilation result in Section 8 in [Sa14a].

An *inner resolution* of length N, for $N = 1,2,3,\dots,\infty$, for a pure co-spherical contractive Hilbert module $\mathcal{H}$ is given by a collection of Hilbert spaces $\{\mathcal{E}_k\}_{k=0}^N$, inner multipliers $\varphi_k \in \mathcal{M}_{\mathcal{B}(\mathcal{E}_k,\mathcal{E}_{k-1})}(H^2_n)$ for $k = 1,\dots,N$ with $X_k = M_{\varphi_k}$ and a co-isometric module map $X_0 : H^2_n \otimes \mathcal{E}_0 \to \mathcal{H}$ so that

$$\text{ran}\, X_k = \ker X_{k-1},$$

for $k = 0,1,\dots,N$. To be more precise, for $N < \infty$ one has the finite resolution

$$0 \longrightarrow H^2_n \otimes \mathcal{E}_N \xrightarrow{X_N} H^2_n \otimes \mathcal{E}_{N-1} \longrightarrow \cdots \longrightarrow H^2_n \otimes \mathcal{E}_1 \xrightarrow{X_1} H^2_n \otimes \mathcal{E}_0 \xrightarrow{X_0} \mathcal{H} \longrightarrow 0,$$

and for $N = \infty$, the infinite resolution

$$\cdots \longrightarrow H^2_n \otimes \mathcal{E}_N \xrightarrow{X_N} H^2_n \otimes \mathcal{E}_{N-1} \longrightarrow \cdots \longrightarrow H^2_n \otimes \mathcal{E}_1 \xrightarrow{X_1} H^2_n \otimes \mathcal{E}_0 \xrightarrow{X_0} \mathcal{H} \longrightarrow 0.$$

The following result shows that an inner resolution does not stop when $n > 1$, unless $\mathcal{H}$ is a Drury-Arveson module and the resolution is a trivial one. In particular, the resolution in (7.5) never stops unless $\mathcal{H} = H^2_n \otimes \mathcal{E}$ for some Hilbert space $\mathcal{E}$.

Theorem 7.2. *If the pure, co-spherically contractive Hilbert module $\mathcal{M}$ possesses a finite inner resolution, then $\mathcal{H}$ is isometrically isomorphic to $H^2_n \otimes \mathcal{F}$ for some Hilbert space $\mathcal{F}$.*

Proof. By applying Theorem 7.1 to X_N, one can decompose

$$\mathcal{E}_{N-1} = \mathcal{E}^1_{N-1} \oplus \mathcal{E}^2_{N-1},$$

so that

$$\tilde{X}_{N-1} = X_{N-1}|_{H^2_n\otimes\mathcal{E}^2_{N-1}} \in \mathcal{L}(H^2_n \otimes \mathcal{E}^2_{N-1}, H^2_n \otimes \mathcal{E}_{N-2}),$$

is an isometry onto $\operatorname{ran} X_{N-1}$. Hence, one can apply the same theorem to $\tilde{X}_{N-1}$. Therefore, the desired conclusion follows by induction. ■

The following statement proceeds directly from the theorem.

COROLLARY 7.3. *If* $\Theta \in \mathcal{M}_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(H^2_n)$ *is an inner multiplier for the Hilbert spaces* $\mathcal{E}$ *and* $\mathcal{E}_*$ *with* $\ker M_\Theta = \{0\}$*, then the quotient module* $\mathcal{H}_\Theta = (H^2_n \otimes \mathcal{E}_*)/\operatorname{ran} M_\Theta$ *is isometrically isomorphic to* $H^2_n \otimes \mathcal{F}$ *for a Hilbert space* $\mathcal{F}$*. Moreover,* $\mathcal{F}$ *can be identified with* $(\operatorname{ran} V_0)^\perp$*, where* V_0 *is the isometry from* $\mathcal{E}$ *to* $\mathcal{E}_*$ *given in Theorem 7.1.*

Note that in the preceding corollary, one has $\dim \mathcal{E}_* = \dim \mathcal{E} + \dim \mathcal{F}$.

A resolution of $\mathcal{H}$ can always be made longer in a trivial way. Suppose we have the resolution

$$0 \longrightarrow H^2_n \otimes \mathcal{E}_N \xrightarrow{X_N} H^2_n \otimes \mathcal{E}_{N-1} \longrightarrow \cdots \longrightarrow H^2_n \otimes \mathcal{E}_0 \xrightarrow{X_0} \mathcal{H} \longrightarrow 0.$$

If $\mathcal{E}_{N+1}$ is a nontrivial Hilbert space, then define X_{N+1} as the inclusion map of $H^2_n \otimes \mathcal{E}_{N+1} \subseteq H^2_n \otimes (\mathcal{E}_N \oplus \mathcal{E}_{N+1})$. Further, set $\tilde{X}_N$ equal to X_N on $H^2_n \otimes \mathcal{E}_N \subseteq H^2_n \otimes (\mathcal{E}_{N+1} \oplus \mathcal{E}_N)$ and equal to 0 on $H^2_n \otimes \mathcal{E}_{N+1} \subseteq H^2_n \otimes (\mathcal{E}_N \oplus \mathcal{E}_{N+1})$. Extending $\tilde{X}_N$ to all of $H^2_n \otimes \mathcal{E}_{N+1}$ linearly, one obtains a longer resolution essentially equivalent to the original one

$$0 \longrightarrow H^2_n \otimes \mathcal{E}_{N+1} \xrightarrow{X_{N+1}} H^2_n \otimes (\mathcal{E}_{N+1} \oplus \mathcal{E}_N) \xrightarrow{\tilde{X}_N} \cdots \longrightarrow \mathcal{H} \longrightarrow 0.$$

Moreover, the new resolution will be inner if the original one is.

The proof of the preceding theorem shows that any finite inner resolution by Drury-Arveson modules is equivalent to a series of such trivial extensions of the resolution

$$0 \longrightarrow H^2_n \otimes \mathcal{E} \xrightarrow{X} H^2_n \otimes \mathcal{E} \longrightarrow 0,$$

for some Hilbert space $\mathcal{E}$ and $X = I_{H^2_n \otimes \mathcal{E}}$. Such a resolution will be referred as *trivial inner resolution*. The proof of the following statement is now straightforward.

COROLLARY 7.4. *All finite inner resolutions for a pure co-spherically contractive Hilbert module* $\mathcal{H}$ *are trivial inner resolutions.*

7.3. **Localizations of free resolutions.** Let $\varphi \in Aut(\mathbb{B}^n)$ and $\varphi = (\varphi_1, \ldots, \varphi_n)$ where $\varphi_i : \mathbb{B}^n \to \mathbb{D}$ is the i-th coordinate function of φ and $1 \leq i \leq n$. Denote $(H^2_n)_\varphi$ by the Hilbert module

$$\mathbb{C}[\boldsymbol{z}] \times H^2_n \to H^2_n, \qquad (p, f) \mapsto p(\varphi_1(M_z), \ldots, \varphi_n(M_z))f. \qquad (p \in \mathbb{C}[\boldsymbol{z}], f \in H^2_n)$$

One can check that $(H^2_n)_\varphi$ is a co-spherically contractive Hilbert module over $\mathbb{C}[\boldsymbol{z}]$. Moreover, as in $n = 1$ case, $H^2_n \cong (H^2_n)_\varphi$ (see [Gr03]) for all $\varphi \in Aut(\mathbb{B}^n)$.

In [Gr03], D. Green proved the following surprising theorem.

THEOREM 7.5. *Let* $\mathcal{H}$ *be a co-spherically contractive Hilbert module over* $\mathbb{C}[\boldsymbol{z}]$ *and* $\varphi \in Aut(\mathbb{B}^n)$ *with* $\boldsymbol{w} = \varphi^{-1}(0)$ *and* $\boldsymbol{w} \in \mathbb{B}^n$*. Let (7.5) be the free resolution of* $\mathcal{H}$ *with* $\mathcal{F}_i = H^2_n(\mathcal{E}_i) \oplus \mathcal{S}_i$ *for some Hilbert space* $\mathcal{E}_i$ *and spherical Hilbert module* $\mathcal{S}_i$ *(*$i \geq 0$*). Then the homology of*

$$\cdots \xrightarrow{X_3(\boldsymbol{w})} \mathcal{E}_2 \xrightarrow{X_2(\boldsymbol{w})} \mathcal{E}_1 \xrightarrow{X_1(\boldsymbol{w})} \mathcal{E}_0,$$

the localization of the free resolution of $\mathcal{H}$ *at* $\boldsymbol{w} \in \mathbb{B}^n$*, is isomorphic to the homology of*

$$K((\mathcal{H})_\varphi) : 0 \longrightarrow \mathcal{E}^n_n((\mathcal{H})_\varphi) \overset{\partial_{n,(\mathcal{H})_\varphi}}{\longrightarrow} \mathcal{E}^n_{n-1}((\mathcal{H})_\varphi) \overset{\partial_{n-1,(\mathcal{H})_\varphi}}{\longrightarrow} \cdots \overset{\partial_{1,(\mathcal{H})_\varphi}}{\longrightarrow} \mathcal{E}^n_1((\mathcal{H})_\varphi) \longrightarrow 0,$$

the Koszul complex of $(\mathcal{H})_\varphi$*. Therefore, for all* $i \geq 1$ *we have*

$$ker\ \partial_{i,(\mathcal{H})_\varphi}/ran\ \partial_{i+1,(\mathcal{H})_\varphi} \cong ker\ X_i(\boldsymbol{w})/ran\ X_{i+1}(\boldsymbol{w}),$$

for each $\boldsymbol{w} \in \mathbb{B}^n$ *and* $\varphi \in Aut(\mathbb{B}^n)$ *such that* $\varphi(\boldsymbol{w}) = 0$*.*

The following result is an immediate consequence of Theorem 7.5.

Corollary 7.6. *Let*

$$\cdots \overset{X_3(\boldsymbol{w})}{\longrightarrow} \mathcal{E}_2 \overset{X_2(\boldsymbol{w})}{\longrightarrow} \mathcal{E}_1 \overset{X_1(\boldsymbol{w})}{\longrightarrow} \mathcal{E}_0,$$

be the localization at $\boldsymbol{w} \in \mathbb{B}^n$ *of the free resolution (7.5) of a co-spherically contractive Hilbert module* $\mathcal{H}$ *over* $\mathbb{C}[\boldsymbol{z}]$*. Then for all* $i \geq n+1$*,*

$$ker\ X_i(\boldsymbol{w}) = ran\ X_{i+1}(\boldsymbol{w}).$$

Further results and comments:

(1) What happens when one relaxes the conditions on the module maps $\{X_k\}$ so that $\operatorname{ran} X_k = \ker X_{k-1}$ for all k but do not require them to be partial isometries? In this case, non-trivial finite resolutions do exist, completely analogous to what happens for the case of the Hardy or Bergman modules over $\mathbb{C}[\boldsymbol{z}]$ for $m > 1$. Here is one simple example:
Consider the module $\mathbb{C}_{(0,0)}$ over $\mathbb{C}[z_1, z_2]$ and the resolution:

$$0 \longrightarrow H^2_2 \overset{X_2}{\longrightarrow} H^2_2 \oplus H^2_2 \overset{X_1}{\longrightarrow} H^2_2 \overset{X_0}{\longrightarrow} \mathbb{C}_{(0,0)} \longrightarrow 0,$$

where $X_0 f = f(0,0)$ for $f \in H^2_2$, $X_1(f_1 \oplus f_2) = M_{z_1} f_1 + M_{z_2} f_2$ for $f_1 \oplus f_2 \in H^2_2 \oplus H^2_2$, and $X_2 f = M_{z_2} f \oplus (-M_{z_1} f)$ for $f \in H^2_2$. One can show that this sequence, which is closely related to the Koszul complex, is exact and non-trivial; in particular, it does not split as trivial resolutions do.

(2) It is not known if there exists any relationship between the inner resolution for a pure co-spherically contractive Hilbert module and more general, *not necessarily inner*, resolutions by Drury-Arveson modules. In particular, is there any relation between the minimal length of a not necessarily inner resolution and the inner resolution. Theorem 5.3 and Corollary 7.6 provides some information on this matter.

(3) A parallel notion of resolution for Hilbert modules was studied by Arveson [Ar04], [Ar07], which is different from the one considered in this section. For Arveson, the key issue is the behavior of the resolution at $0 \in \mathbb{B}^n$ or the localization of the sequence of connecting maps at 0. His main goal, which he accomplishes and is quite non trivial, is to extend an analogue of the Hilbert's syzygy theorem. In particular, he exhibits a resolution of Hilbert modules in his class which ends in finitely many steps.

(4) The resolutions considered in ([DoMi03], [DoMi05]) and this section are related to dilation theory although the requirement that the connecting maps are partial isometries is sometimes relaxed.

(5) Theorem 7.1 is related to an earlier result of Guo, Hu and Xu [GuHuXu04].

(6) Theorem 7.5 and Corollary 7.6 are due to Green [Gr03]. Except that, most of the material is from [DoFoSa12]. However, Theorem 7.1 was first proved by Arias [Ari04].

8. Rigidity

Let $\mathcal{H}$ be a Hilbert module over $A(\Omega)$ (or, over $\mathbb{C}[\boldsymbol{z}]$). Denote by $\mathcal{R}(\mathcal{H})$ the set of all non-unitarily equivalent submodules of $\mathcal{H}$, that is, if $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{R}(\mathcal{H})$ and that $\mathcal{S}_1 \cong \mathcal{S}_2$ then $\mathcal{S}_1 = \mathcal{S}_2$.
Problem: Determine $\mathcal{R}(\mathcal{H})$.

By virtue of the characterization results by Beurling and Richter (see Section 6 in [Sa14a]), we have

$$\mathcal{R}(H^2(\mathbb{D})) = \{\{0\}, H^2(\mathbb{D})\}, \qquad \text{and} \qquad \mathcal{R}(L^2_a(\mathbb{D})) = \{\mathcal{S} \subseteq L^2_a(\mathbb{D}) : \mathcal{S} \text{ is a submodule}\}.$$

A Hilbert module $\mathcal{H}$ over $A(\Omega)$ is said to be *rigid* if

$$\mathcal{R}(\mathcal{H}) = \{\mathcal{S} \subseteq \mathcal{H} : \mathcal{S} \text{ is a submodule}\} = \text{Lat}(\mathcal{H}).$$

Therefore, the Bergman module $L^2_a(\mathbb{D})$ is rigid. For the Hardy space $H^2(\mathbb{D}^n)$ with $n > 1$

$$\{\{0\}, H^2(\mathbb{D}^n)\} \subset \mathcal{R}(H^2(\mathbb{D}^n)) \subset \text{Lat}(H^2(\mathbb{D}^n))$$

The purpose of this section is to discuss some rigidity results for reproducing kernel Hilbert modules over $\mathbb{B}^n$ and $\mathbb{D}^n$. For the rest of the section, unless otherwise stated, it is assumed that $n > 1$.

8.1. **Rigidity of H^2_n.** In [GuHuXu04], Guo, Hu and Yu proved that two nested unitarily equivalent submodules of H^2_n must be equal.

Theorem 8.1. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two submodules of H^2_n and $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Then $\mathcal{S}_1 \cong \mathcal{S}_2$ if and only if $\mathcal{S}_1 = \mathcal{S}_2$.*

It is not known whether there exists proper submodules $\mathcal{S}_1$ and $\mathcal{S}_2$ of H^2_n such that $\mathcal{S}_1 \cong \mathcal{S}_2$ but $\mathcal{S}_1 \neq \mathcal{S}_2$. Recall that a submodule $\mathcal{S}$ of H^2_n is said to be proper if $\mathcal{S} \neq H^2_n$, or equivalently, $1 \notin \mathcal{S}$.

The following result provides a rather weaker version of Theorem 8.1.

Theorem 8.2. *If $\mathcal{S}$ is a submodule of H^2_n which is isometrically isomorphic to H^2_n, then $\mathcal{S} = H^2_n$.*

Proof. The result follows directly from Theorem 7.1. ■

8.2. **Rigidity of $L^2_a(\mu)$.** The purpose of this subsection is to prove that for a class of measures μ on the closure of Ω, two submodules of $L^2_a(\mu)$ are isometrically isomorphic if and only if they are equal. The subsection will be concluded by considering when two submodules of a subnormal Hilbert module $\mathcal{M}$ over $A(\Omega)$ can be isometrically isomorphic.

Let μ be the measure on $\overline{\mathbb{D}}$ obtained from the sum of Lebesgue measure on $\partial\mathbb{D}$ and the unit mass at 0, then $L^2_a(\mu)$ is not a Šilov module (see [DoPa89]). However, it is easy to see that the cyclic submodules generated by z and z^2, respectively, are isometrically isomorphic but distinct. A quick examination suggests the problem is that μ assigns positive measure to the intersection of a zero variety and $\mathbb{D}$. It turns out that if one excludes that possibility and $L^2(\nu)$ is not a Šilov module, then distinct submodules can not be isometrically isomorphic. The proof takes several steps.

LEMMA 8.3. *Let ν be a probability measure on clos Ω and f and g vectors in $L^2_a(\nu)$ so that the cyclic submodules of $L^2_a(\nu)$, $[f]$ and $[g]$, generated by f and g, respectively, are isometrically isomorphic with f mapping to g. Then $|f| = |g|$ a.e. ν.*

Proof. If the correspondence $Vf = g$ extends to an isometric module map, then

$$\langle z^{\boldsymbol{k}} f, z^{\boldsymbol{l}} f\rangle_{L^2_a(\nu)} = \langle z^{\boldsymbol{k}} g, z^{\boldsymbol{l}} g\rangle_{L^2_a(\nu)},$$

for monomials $z^{\boldsymbol{k}}$ and $z^{\boldsymbol{l}}$ in $\mathbb{C}[z]$. This implies that

$$\int\limits_{\text{clos }\Omega} z^{\boldsymbol{k}}\bar{z}^{\boldsymbol{l}}|f|^2 d\nu = \int\limits_{\text{clos }\Omega} z^{\boldsymbol{k}}\bar{z}^{\boldsymbol{l}}|g|^2 d\nu. \qquad (\boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}^n)$$

Since the linear span of the set $\{z^{\boldsymbol{k}}\bar{z}^{\boldsymbol{l}} : \boldsymbol{k}, \boldsymbol{l} \in \mathbb{N}^n\}$ forms a self-adjoint algebra which separates the points of clos Ω, it follows that the two measures $|f|^2\ d\nu$ and $|g|^2\ d\nu$ are equal or that $|f| = |g|$ a.e. ν. ■

The following theorem concerns measures for which point evaluation on Ω is bounded.

THEOREM 8.4. *Let ν be a probability measure on* clos Ω *such that point evaluation is bounded on $L^2_a(\Omega)$ with closed support properly containing $\partial\Omega$ but such that $\nu(X) = 0$ for X the intersection of* clos Ω *with a zero variety. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are isometrically isomorphic submodules of $L^2_a(\nu)$, then $\mathcal{S}_1 = \mathcal{S}_2$.*

Proof. Let V be an isometric module map from $\mathcal{S}_1$ onto $\mathcal{S}_2$. For $0 \neq f$ in $\mathcal{S}_1$, let $g = Vf$. Then by the previous lemma, it follows that $|f| = |g|$ a.e. ν. Since $\partial\Omega$ is contained in the closed support of ν, it follows that

$$|f(\boldsymbol{w})| = |g(\boldsymbol{w})|. \qquad (\boldsymbol{w} \in \partial\Omega)$$

Since point evaluation is bounded on $L^2_a(\Omega)$, f and g are holomorphic on Ω. If

$$X = \{\boldsymbol{w} \in \Omega : f(\boldsymbol{w}) = 0\},$$

then

$$\nu(X) = 0,$$

which implies that $\nu(\Omega\backslash X) > 0$. Now

$$\sup_{\boldsymbol{w}\in\Omega\backslash X} |h(\boldsymbol{w})| \leq 1,$$

where

$$h(\boldsymbol{w}) := \frac{g(\boldsymbol{w})}{f(\boldsymbol{w})}, \qquad (\boldsymbol{w} \in \Omega\backslash X).$$

Since there is $\boldsymbol{w}_0$ in the support of ν in $\Omega\backslash X$ such that $|h(\boldsymbol{w}_0)| = 1$, one gets $|h(\boldsymbol{w})| \equiv 1$ on $\Omega\backslash X$. Thus there is a constant $e^{i\theta}$ such that $f = e^{i\theta} g$ on Ω.
Since this holds for every f in $\mathcal{S}_1$, by considering f_1, f_2 and $f_1 + f_2$, it follows that $Vf = e^{i\theta} f$ for all f in $\mathcal{S}_1$ and hence $\mathcal{S}_1 = \mathcal{S}_2$. ■

This result contains the results of Richter [Ri88], Putinar [Pu94], and Guo–Hu–Xu [GuHuXu04] since area measure on $\mathbb{D}$ or volume measure on Ω satisfies the hypotheses of the theorem. However, so do the measures for the weighted Bergman spaces on $\mathbb{D}$ or weighted volume measure on any domain Ω.

COROLLARY 8.5. *If Ω is a bounded domain in $\mathbb{C}^n$ and $\mathcal{S}_1$ and $\mathcal{S}_2$ are isometrically isomorphic submodules of $L^2_a(\Omega)$, then $\mathcal{S}_1 = \mathcal{S}_2$.*

8.3. **Rigidity of $H^2(\mathbb{D}^n)$.** The purpose of this subsection is to discuss the rigidity issue for a simple class of submodules of $H^2(\mathbb{D}^n)$, namely, the co-doubly commuting submodules. There is an extensive literature on rigidity phenomenon for submodules of the Hardy module over $\mathbb{D}^n$. The reader is referred to the book by Chen and Guo [ChGu03], Chapter 3.

The following rigidity result is due to Agrawal, Clark and Douglas (Corollary 4 in [ACDo86]. See also [I87]).

THEOREM 8.6. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two submodules of $H^2(\mathbb{D}^n)$, both of which contain functions independent of z_i for $i = 1, \ldots, n$. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are unitarily equivalent if and only if they are equal.*

This yields the following results concerning rigidity of co-doubly commuting submodules of $H^2(\mathbb{D}^n)$ (see Section 4).

COROLLARY 8.7. *Let $\mathcal{S}_\Theta = \sum_{i=1}^n \tilde{\Theta}_i H^2(\mathbb{D})^n$ and $\mathcal{S}_\Phi = \sum_{i=1}^n \tilde{\Phi}_i H^2(\mathbb{D})^n$ be a pair of submodules of $H^2(\mathbb{D})^n$, where $\tilde{\Theta}_i(\boldsymbol{z}) = \Theta_i(z_i)$ and $\tilde{\Phi}_i(\boldsymbol{z}) = \Phi_i(z_i)$ for inner functions $\Theta_i, \Phi_i \in H^\infty(\mathbb{D})$ and $\boldsymbol{z} \in \mathbb{D}^n$ and $i = 1, \ldots, n$. Then $\mathcal{S}_\Theta$ and $\mathcal{S}_\Phi$ are unitarily equivalent if and only if $\mathcal{S}_\Theta = \mathcal{S}_\Phi$.*

Proof. Clearly $\tilde{\Theta}_i \in \mathcal{S}_\Theta$ and $\tilde{\Phi}_i \in \mathcal{S}_\Phi$ are independent of $\{z_1, \cdots, z_{i-1}, z_{i+1}, \ldots, z_n\}$ for all $i = 1, \ldots, n$. Therefore, the submodules $\mathcal{S}_\Theta$ and $\mathcal{S}_\Phi$ contains functions independent of z_i for all $i = 1, \ldots, n$. Consequently, if $\mathcal{S}_\Phi$ and $\mathcal{S}_\Phi$ are unitarily equivalent then $\mathcal{S}_\Theta = \mathcal{S}_\Phi$. ∎

COROLLARY 8.8. *Let $\mathcal{S}_\Theta = \sum_{i=1}^n \tilde{\Theta}_i H^2(\mathbb{D})^n$ be a submodules of $H^2(\mathbb{D})^n$, where $\tilde{\Theta}_i(\boldsymbol{z}) = \Theta_i(z_i)$ for inner functions $\Theta_i \in H^\infty(\mathbb{D})$ for all $i = 1, \ldots, n$ and $\boldsymbol{z} \in \mathbb{D}^n$. Then $\mathcal{S}_\Theta$ and $H^2(\mathbb{D}^n)$ are not unitarily equivalent.*

Proof. The result follows from the previous theorem along with the observation that $\mathcal{S}_\Theta^\perp \neq \{0\}$. ∎

Further results and comments:

(1) In [DoPaSY95], Douglas, Paulsen, Sah and Yan used algebraic localization techniques to obtain general rigidity results. In particular, under mild restrictions, they showed that the submodules obtained from the closure of ideals are equivalent if and only if the ideals coincide. See also [DoYa90], [ACDo86] for related results.
(2) Theorem 8.4 is from [DoSa08]. In connection with this section see Richter [Ri88], Putinar [Pu94], and Guo, Hu and Xu [GuHuXu04]. Corollaries 8.7 and 8.8 are from [Sa13a].
(3) In [Gu00], Guo used the notion of characteristic spaces [Gu99] and obtained a complete classification submodules of $H^2(\mathbb{D}^n)$ and $H^2(\mathbb{B}^n)$ generated by polynomials.

(4) In [I87], Izuchi proved the following results: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be submodules of $H^2(\mathbb{D}^n)$.
(1) If $\dim(\mathcal{S}_1 \ominus \mathcal{S}_2)$, $\dim(\mathcal{S}_2 \ominus \mathcal{S}_1) < \infty$, then $\mathcal{S}_1 \cong \mathcal{S}_2$ if and only if $\mathcal{S}_1 = \mathcal{S}_2$.
(2) Let φ_2 be an outer function and $\mathcal{S}_2 = [\varphi_2]$, the principle submodule generated by φ_2. If $\mathcal{S}_1 \cong \mathcal{S}_2$ then, $\mathcal{S}_1 = \Theta \mathcal{S}_2$ for some inner function $\Theta \in H^\infty(\mathbb{D}^n)$.

(5) For complete reference concerning rigidity for analytic Hilbert modules, the reader is referred to the book by Chen and Guo [ChGu03].

9. Essentially normal Hilbert modules

The purpose of this section is to introduce the notion of essentially normal Hilbert module, emphasizing a few highlights of the recent developments in the study of Hilbert modules.

9.1. **Introduction to essential normality.** A Hilbert module $\mathcal{H}$ over A, where $A = A(\Omega)$ or $\mathbb{C}[\boldsymbol{z}]$, is said to be *essentially reductive* or *essentially normal* if the cross-commutators

$$[M_i^*, M_j] = M_i^* M_j - M_j M_i^*,$$

are in the ideal of compact operators in $\mathcal{H}$ for all $1 \leq i, j \leq n$.

There are many natural examples of essentially normal Hilbert modules. In particular, H^2_n, $L^2_a(\mathbb{B}^n)$ and $H^2(\mathbb{B}^n)$ are essentially normal. However, $H^2(\mathbb{D}^n)$ and $L^2_a(\mathbb{D}^n)$ are not essentially normal whenever $n > 1$.

In [Do06], Douglas proved the following results: Let $\mathcal{H}$ be an essentially normal Hilbert module over A and $\mathcal{S}$ be a submodule of $\mathcal{H}$. Then $\mathcal{S}$ is essentially normal if and only if the quotient module $\mathcal{Q} := \mathcal{H}/\mathcal{S}$ is essentially normal.

Another variant of this result concerns a relationship between essentially normal Hilbert modules and resolutions of Hilbert modules (see Theorem 2.2 in [Do06]):

Theorem 9.1. *Let $\mathcal{H}$ be a Hilbert module over A with a resolution of Hilbert modules*

$$0 \longrightarrow \mathcal{F}_1 \xrightarrow{X} \mathcal{F}_2 \xrightarrow{\pi} \mathcal{H} \longrightarrow 0,$$

for some essentially normal Hilbert modules $\mathcal{F}_1$ and $\mathcal{F}_2$. Then $\mathcal{H}$ is essentially normal.

The preceding results raise questions about essentially normal submodules.
Problem: Let $\mathcal{S}$ be a submodule of $\mathcal{H}$, where $\mathcal{H} = H^2_n$ or $H^2(\mathbb{B}^n)$ or $L^2_a(\mathbb{B}^n)$ and $n > 1$. Does it follow that $\mathcal{S}$ is essentially normal?

This is one of the most active research areas in multivariable operator theory. For instance, if $\mathcal{S}$ is a submodule of $L^2_a(\mathbb{B}^n)$ and generated by a polynomial (by Douglas and Wang [DoW11]) or a submodule of H^2_n and generated by a homogeneous polynomial (by Guo and Wang [GuW08]), then $\mathcal{S}$ is p-essentially normal for all $p > n$ (see also [FXi09], [Es11] and [DaRS14]).

9.2. **Reductive modules.** This subsection continues the study of unitarily equivalent submodules of Hilbert modules (see Section 6 in [Sa14a]). In this context the following problem is of interest.: Let $\mathcal{R}$ be an essentially normal quasi-free Hilbert module over $A(\Omega)$ for which there exists a pure unitarily equivalent submodule. Does it follow that $\mathcal{R}$ is subnormal?

Now let $\mathcal{R}$ be a quasi-free Hilbert module over $A(\Omega)$. Then the Hilbert space tensor product $\mathcal{R} \otimes H^2(\mathbb{D})$ is a quasi-free Hilbert module over $A(\Omega \times \mathbb{D})$ which clearly contains the pure isometrically isomorphic submodule $\mathcal{R} \otimes H^2_0(\mathbb{D})$. Hence, one can say little without some additional hypothesis for Ω or $\mathcal{R}$ or both. Under the assumption of essential normality on $\mathcal{R}$ the following holds:

THEOREM 9.2. *Let $\mathcal{R}$ be an essentially normal Hilbert module over $A(\Omega)$ and U be an isometric module map U on $\mathcal{R}$ such that*

$$\bigcap_{k=0}^{\infty} U^k\mathcal{R} = \{0\}.$$

Then $\mathcal{R}$ is subnormal, that is, there exists a normal (reductive) Hilbert module $\mathcal{N}$ over $A(\Omega)$ with $\mathcal{R}$ as a submodule.

Proof. As in the proof of Proposition 6.1 in [Sa14a], there exists an isometric isomorphism Ψ from $\mathcal{R}$ onto $H^2_{\mathcal{W}}(\mathbb{D})$ with

$$\mathcal{W} = \mathcal{R} \ominus U\mathcal{R},$$

and $\varphi_1, \ldots, \varphi_n$ in $H^\infty_{\mathcal{L}(\mathcal{W})}(\mathbb{D})$ such that Ψ is a $\mathbb{C}[\boldsymbol{z}]$-module map relative to the module structure on $H^2_{\mathcal{W}}(\mathbb{D})$ defined so that

$$z_j \mapsto T_{\varphi_j}. \qquad (j = 1, \ldots, n)$$

It remains only to prove that the n-tuple $\{\varphi_1(e^{it}), \ldots, \varphi_n(e^{it})\}$ consists of commuting normal operators for e^{it}-a.e. on $\mathbb{T}$. Then $\mathcal{N}$ is $L^2_{\mathcal{W}}(\mathbb{T})$ with the module multiplication defined by $z_i \mapsto L_{\varphi_i}$, where L_{φ_i} denotes pointwise multiplication on $L^2_{\mathcal{W}}(\mathbb{T})$. Since the $\{\varphi_j(e^{it})\}_{j=1}^n$ are normal and commute, $L^2_{\mathcal{W}}(\mathbb{T})$ is a reductive Hilbert module.

The fact that $\mathcal{R}$ is essentially reductive implies that each T_{φ_i} is essentially normal and hence that the cross-commutators $[T^*_{\varphi_i}, T_{\varphi_j}]$ are compact for $1 \leq i, j \leq n$. To finish the proof it suffices to show that $[T^*_{\varphi_i}, T_{\varphi_j}]$ compact implies that $[L^*_{\varphi_i}, L_{\varphi_j}] = 0$ on $L^2_{\mathcal{W}}(\mathbb{T})$.

Fix f in $H^2_{\mathcal{W}}(\mathbb{D})$ and let N be a positive integer. Next observe that

$$\lim_{N\to\infty} \|(I-P)L_z^N L^*_{\varphi_i} L_{\varphi_j} f\| = 0, \tag{9.6}$$

and

$$\lim_{N\to\infty} \|(I-P)L_z^N L^*_{\varphi_i} f\| = 0, \tag{9.7}$$

where P is the projection of $L^2_{\mathcal{W}}(\mathbb{T})$ onto $H^2_{\mathcal{W}}(\mathbb{D})$. Consequently

$$\begin{aligned}\|[T^*_{\varphi_i}, T_{\varphi_j}]M_z^N f\| &= \|PL^*_{\varphi_i}PL_{\varphi_j}PL_z^N f - PL_{\varphi_j}PL^*_{\varphi_i}PL_z^N f\| \\ &= \|[L_z^N L^*\varphi_i L_{\varphi_j} f - (I-P)L_z^N L^*_{\varphi_i} L_{\varphi_j} f] \\ &\quad - [L_{\varphi_j} L_z^N L^*_{\varphi_i} f - L_{\varphi_j}(I-P)L_z^N L^*_{\varphi_i} f]\|.\end{aligned}$$

By (9.6) and (9.7) one gets

$$\begin{aligned}\lim_{N\to\infty} \|[T^*_{\varphi_i}, T_{\varphi_j}]L_z^N f\| &= \lim_{N\to\infty} \|(L_z^N L^*_{\varphi_i} L_{\varphi_j} - L_{\varphi_j} L_z^N L'_{\varphi_i}) f\| \\ &= \lim_{N\to\infty} \|L_z^N [L^*_{\varphi_i}, L_{\varphi_j}] f\| = \|[L^*_{\varphi_i}, L_{\varphi_j}] f\|.\end{aligned}$$

Since $[T^*_{\varphi_i}, T_{\varphi_j}]$ is compact and the sequence $\{e^{iNt} f\}$ converges weakly to 0, it follows that

$$\lim_{N\to\infty} \|[T^*_{\varphi_i}, T_{\varphi_j}]e^{iNt} f\| = 0.$$

Therefore,

$$\|[L^*_{\varphi_i}, L_{\varphi_j}]f\| = 0.$$

Finally, the set of vectors $\{e^{-iNt}f\}\colon\ N \geq 0, f \in H^2_{\mathcal{W}}(\mathbb{D})\}$ is norm dense in $L^2_{\mathcal{W}}(\mathbb{T})$ and

$$\|[L^*_{\varphi_i}, L_{\varphi_j}]e^{-iNt}f\| = \|[L^*_{\varphi_i}, L_{\varphi_j}]f\| = 0.$$

Therefore, $[L^*_{\varphi_i}, L_{\varphi_j}] = 0$, which completes the proof. ∎

The following result is complementary to Theorem 6.1, [Sa14a].

THEOREM 9.3. *Let $\mathcal{M}$ be an essentially reductive, finite rank, quasi-free Hilbert module over $A(\mathbb{D})$. Let U be a module isometry such that*

$$\cap_{k=0}^{\infty} U^k \mathcal{M} = \{0\}.$$

Then $\mathcal{M}$ is unitarily equivalent to $H^2_{\mathcal{F}}(\mathbb{D})$ for some Hilbert space $\mathcal{F}$ with

$$dim\ \mathcal{F} = rank\ \mathcal{M}.$$

Proof. As before (cf. Theorem 9.2) there is an isometrical isomorphism, $\Psi\colon\ H^2_{\mathcal{F}}(\mathbb{D}) \to \mathcal{M}$ such that $U = \Psi T_z \Psi^*$ and there exists φ in $H^\infty_{\mathcal{L}(\mathcal{F})}(\mathbb{D})$ such that $M_z = \Psi T_\varphi \Psi^*$. Further, since M_z is essentially normal and $M_z - \omega$ is Fredholm for ω in $\mathbb{D}$, it follows that M_z is an essential unitary. Finally, this implies

$$T^*_\varphi T_\varphi - I = T_{\varphi^*\varphi - I},$$

is compact and hence $\varphi^*(e^{it})\varphi(e^{it}) = I$ a.e. or φ is an inner function which completes the proof. ∎

9.3. **Essentially doubly commutativity.** Recall that the Hardy module $H^2(\mathbb{D}^n)$ with $n > 1$ is doubly commuting but not essentially normal. Therefore, a natural approach to measure a submodule of the Hardy module $H^2(\mathbb{D}^n)$ from being small is to consider the cross commutators $[R^*_{z_i}, R_{z_j}]$ for all $1 \leq i < j \leq n$.

It is difficult in general to characterize the class of essentially doubly commuting submodules of $H^2(\mathbb{D}^n)$. It is even more complicated to compute the cross-commutators of submodules of $H^2(\mathbb{D}^n)$. However, that is not the case for co-doubly commuting submodules [Sa13a]:

THEOREM 9.4. *Let $\mathcal{S} = \sum_{i=1}^n \tilde{\Theta}_i H^2(\mathbb{D}^n)$ be a co-doubly commuting submodule of $H^2(\mathbb{D}^n)$, where $\tilde{\Theta}_i(\boldsymbol{z}) = \Theta_i(z_i)$ for all $\boldsymbol{z} \in \mathbb{D}^n$ and each $\Theta_i \in H^\infty(\mathbb{D})$ is either an inner function or the zero function and $1 \leq i \leq n$. Then for all $1 \leq i < j \leq n$,*

$$[R^*_{z_i}, R_{z_j}] = I_{\mathcal{Q}_{\Theta_1}} \otimes \cdots \otimes \underbrace{P_{\mathcal{Q}_{\Theta_i}} M^*_z|_{\Theta_i H^2(\mathbb{D})}}_{i^{\text{th}}} \otimes \cdots \otimes \underbrace{P_{\Theta_j H^2(\mathbb{D})} M_z|_{\mathcal{Q}_{\Theta_j}}}_{j^{\text{th}}} \otimes \cdots \otimes I_{\mathcal{Q}_{\Theta_n}},$$

and

$$\|[R^*_{z_i}, R_{z_j}]\| = (1 - |\Theta_i(0)|^2)^{\frac{1}{2}}(1 - |\Theta_j(0)|^2)^{\frac{1}{2}}.$$

Proof. Let $\mathcal{S} = \sum_{i=1}^n \tilde{\Theta}_i H^2(\mathbb{D}^n)$, for some one variable inner functions $\Theta_i \in H^\infty(\mathbb{D})$. Let $\tilde{P}_i$ be the orthogonal projection in $\mathcal{L}(\mathcal{S})$ defined by

$$\tilde{P}_i = M_{\tilde{\Theta}_i} M^*_{\tilde{\Theta}_i},$$

for all $i = 1, \ldots, n$. By virtue of Corollary 4.8 and Lemma 4.6,

$$\begin{aligned} P_{\mathcal{S}} &= I_{H^2(\mathbb{D}^n)} - \prod_{i=1}^{n} (I_{H^2(\mathbb{D}^n)} - \tilde{P}_i) \\ &= \tilde{P}_1(I - \tilde{P}_2) \cdots (I - \tilde{P}_n) + \tilde{P}_2(I - \tilde{P}_3) \cdots (I - \tilde{P}_n) + \cdots + \tilde{P}_{n-1}(I - \tilde{P}_n) + \tilde{P}_n \\ &= \tilde{P}_n(I - \tilde{P}_{n-1}) \cdots (I - \tilde{P}_1) + \tilde{P}_{n-1}(I - \tilde{P}_{n-2}) \cdots (I - \tilde{P}_1) + \cdots + \tilde{P}_2(I - \tilde{P}_1) + \tilde{P}_1, \end{aligned}$$

and

$$P_{\mathcal{Q}} = \prod_{i=1}^{n} (I_{H^2(\mathbb{D}^n)} - \tilde{P}_i).$$

On the other hand, for all $1 \leq i < j \leq n$, one gets

$$[R_{z_i}^*, R_{z_j}] = P_{\mathcal{S}} M_{z_i}^* M_{z_j}|_{\mathcal{S}} - P_{\mathcal{S}} M_{z_j} P_{\mathcal{S}} M_{z_i}^*|_{\mathcal{S}},$$

and that

$$\begin{aligned} P_{\mathcal{S}} M_{z_i}^* M_{z_j} P_{\mathcal{S}} - P_{\mathcal{S}} M_{z_j} P_{\mathcal{S}} M_{z_i}^* P_{\mathcal{S}} &= P_{\mathcal{S}} M_{z_i}^* M_{z_j} P_{\mathcal{S}} - P_{\mathcal{S}} M_{z_j} (I - P_{\mathcal{Q}}) M_{z_i}^* P_{\mathcal{S}} \\ &= P_{\mathcal{S}} M_{z_j} P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{S}}. \end{aligned}$$

Furthermore, for all $1 \leq i < j \leq n$,

$$\begin{aligned} & P_{\mathcal{S}} M_{z_j} P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{S}} \\ &= [\tilde{P}_n(I - \tilde{P}_{n-1}) \cdots (I - \tilde{P}_1) + \tilde{P}_{n-1}(I - \tilde{P}_{n-2}) \cdots (I - \tilde{P}_1) + \cdots + \tilde{P}_2(I - \tilde{P}_1) + \tilde{P}_1] \\ & \quad M_{z_j} [\prod_{l=1}^{n} (I_{H^2(\mathbb{D}^n)} - \tilde{P}_l)] M_{z_i}^* \\ & \quad [\tilde{P}_1(I - \tilde{P}_2) \cdots (I - \tilde{P}_n) + \tilde{P}_2(I - \tilde{P}_3) \cdots (I - \tilde{P}_n) + \cdots + \tilde{P}_{n-1}(I - \tilde{P}_n) + \tilde{P}_n] \\ &= [\tilde{P}_n(I - \tilde{P}_{n-1}) \cdots (I - \tilde{P}_1) + \tilde{P}_{n-1}(I - \tilde{P}_{n-2}) \cdots (I - \tilde{P}_1) + \cdots + \tilde{P}_2(I - \tilde{P}_1) + \tilde{P}_1] \\ & \quad [\prod_{l \neq j} (I_{H^2(\mathbb{D}^n)} - \tilde{P}_l)] M_{z_j} M_{z_i}^* [\prod_{l \neq i} (I_{H^2(\mathbb{D}^n)} - \tilde{P}_l)] \\ & \quad [\tilde{P}_1(I - \tilde{P}_2) \cdots (I - \tilde{P}_n) + \tilde{P}_2(I - \tilde{P}_3) \cdots (I - \tilde{P}_n) + \cdots + \tilde{P}_{n-1}(I - \tilde{P}_n) + \tilde{P}_n] \\ &= [\tilde{P}_j(I - \tilde{P}_{j-1}) \cdots (I - \tilde{P}_1)] M_{z_i}^* M_{z_j} [\tilde{P}_i(I - \tilde{P}_{i+1}) \cdots (I - \tilde{P}_n)] \\ &= [(I - \tilde{P}_1) \cdots (I - \tilde{P}_{j-1}) \tilde{P}_j] M_{z_i}^* M_{z_j} [\tilde{P}_i(I - \tilde{P}_{i+1}) \cdots (I - \tilde{P}_n)]. \end{aligned}$$

These equalities shows that

$$\begin{aligned} [R_{z_i}^*, R_{z_j}] &= [(I - \tilde{P}_1) \cdots (I - \tilde{P}_i) \cdots (I - \tilde{P}_{j-1}) \tilde{P}_j] M_{z_i}^* M_{z_j} [\tilde{P}_i(I - \tilde{P}_{i+1}) \cdots (I - \tilde{P}_j) \cdots (I - \tilde{P}_n)] \\ &= (I - \tilde{P}_1)(I - \tilde{P}_2) \cdots (I - \tilde{P}_{i-1}) \; ((I - \tilde{P}_i) M_{z_i}^* \tilde{P}_i) \; (I - \tilde{P}_{i+1}) \cdots \\ & \quad \cdots (I - \tilde{P}_{j-1}) \; (\tilde{P}_j M_{z_j} (I - \tilde{P}_j)) \; (I - \tilde{P}_{j+1}) \cdots (I - \tilde{P}_n). \end{aligned}$$

Moreover,

$$[R_{z_i}^*, R_{z_j}] = [(I - \tilde{P}_1) \cdots (I - \tilde{P}_{j-1}) \tilde{P}_j] M_{z_i}^* M_{z_j} [(I - \tilde{P}_1) \cdots (I - \tilde{P}_{i-1}) \tilde{P}_i (I - \tilde{P}_{i+1}) \cdots (I - \tilde{P}_n)],$$

and

$$[R_{z_i}^*, R_{z_j}] = [(I - \tilde{P}_1) \cdots (I - \tilde{P}_{j-1}) \tilde{P}_j (I - \tilde{P}_{j+1}) \cdots (I - \tilde{P}_n)] M_{z_i}^* M_{z_j} [\tilde{P}_i (I - \tilde{P}_{i+1}) \cdots (I - \tilde{P}_n)].$$

Now we can conclude that

$$[R_{z_i}^*, R_{z_j}] = I_{\mathcal{Q}_{\Theta_1}} \otimes \cdots \otimes \underbrace{P_{\mathcal{Q}_{\Theta_i}} M_z^*|_{\Theta_i H^2(\mathbb{D})}}_{i^{\text{th}}} \otimes \cdots \otimes \underbrace{P_{\Theta_j H^2(\mathbb{D})} M_z|_{\mathcal{Q}_{\Theta_j}}}_{j^{\text{th}}} \otimes \cdots \otimes I_{\mathcal{Q}_{\Theta_n}}.$$

Further, note that

$$\begin{aligned}\|[R_{z_i}^*, R_{z_j}]\| &= \|I_{\mathcal{Q}_{\Theta_1}} \otimes \cdots \otimes P_{\mathcal{Q}_{\Theta_i}} M_z^*|_{\Theta_i H^2(\mathbb{D})} \otimes \cdots \otimes P_{\Theta_j H^2(\mathbb{D})} M_z|_{\mathcal{Q}_{\Theta_j}} \otimes \cdots \otimes I_{\mathcal{Q}_{\Theta_n}}\| \\ &= \|P_{\mathcal{Q}_{\Theta_i}} M_z^*|_{\Theta_i H^2(\mathbb{D})}\| \|P_{\Theta_j H^2(\mathbb{D})} M_z|_{\mathcal{Q}_{\Theta_j}}\|,\end{aligned}$$

and consequently by Proposition 2.3 in [Sa13a] it follows that

$$\|[R_{z_i}^*, R_{z_j}]\| = (1 - |\Theta_i(0)|^2)^{\frac{1}{2}} (1 - |\Theta_j(0)|^2)^{\frac{1}{2}}.$$

This completes the proof. ■

The following corollary reveals the significance of the identity operators in the cross commutators of the co-doubly commuting submodules of $H^2(\mathbb{D}^n)$ for $n > 2$.

Corollary 9.5. *Let $\mathcal{S} = \sum_{i=1}^n \tilde{\Theta}_i H^2(\mathbb{D}^n)$ be a submodule of $H^2(\mathbb{D}^n)$ for some one variable inner functions $\{\tilde{\Theta}_i\}_{i=1}^n \subseteq H^\infty(\mathbb{D}^n)$. Then*
(1) for $n = 2$: the rank of the cross commutator of $\mathcal{S}$ is at most one and the Hilbert-Schmidth norm of the cross commutator is given by

$$\|[R_{z_1}^*, R_{z_2}]\|_{HS} = (1 - |\Theta_1(0)|^2)^{\frac{1}{2}} (1 - |\Theta_2(0)|^2)^{\frac{1}{2}}.$$

In particular, $\mathcal{S}$ is essentially doubly commuting.
(2) for $n > 2$: $\mathcal{S}$ is essentially doubly commuting (or of Hilbert-Schmidth cross-commutators) if and only if that $\mathcal{S}$ is of finite co-dimension, that is,

$$dim\ [H^2(\mathbb{D}^n)/\mathcal{S}] < \infty.$$

Moreover, in this case, for all $1 \leq i < j \leq n$

$$\|[R_{z_i}^*, R_{z_j}]\|_{HS} = (1 - |\Theta_i(0)|^2)^{\frac{1}{2}} (1 - |\Theta_j(0)|^2)^{\frac{1}{2}}.$$

The following statements also proceeds directly from the theorem.

Corollary 9.6. *Let $n > 2$ and $\mathcal{S} = \sum_{i=1}^k \tilde{\Theta}_i H^2(\mathbb{D}^n)$ be a co-doubly commuting proper submodule of $H^2(\mathbb{D}^n)$ for some inner functions $\{\Theta_i\}_{i=1}^k$ and $k < n$. Then $\mathcal{S}$ is not essentially doubly commuting.*

Corollary 9.7. *Let $\mathcal{S}$ be a co-doubly commuting submodule of $H^2(\mathbb{D}^n)$ and $\mathcal{Q} := H^2(\mathbb{D}^n)/\mathcal{S}$ and $n > 2$. Then the following are equivalent:*
(i) $\mathcal{S}$ is essentially doubly commuting.
(ii) $\mathcal{S}$ is of finite co-dimension.
(iii) $\mathcal{Q}$ is essentially normal.

The following one is a "rigidity" type result.

Corollary 9.8. *Let $n \geq 2$ and $\mathcal{S} = \sum_{i=1}^n \tilde{\Theta}_i H^2(\mathbb{D}^n)$ be an essentially normal co-doubly commuting submodule of $H^2(\mathbb{D}^n)$ for some one variable inner functions $\{\Theta_i\}_{i=1}^n$. If $\mathcal{S}$ is of infinite co-dimension, then $n = 2$.*

Proof. The result follows from Theorem 9.4 and the fact that a finite co-dimensional submodule of an essentially doubly commuting Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ is essentially doubly commuting. ∎

It is now clear that the general picture of essentially doubly commuting submodules of $H^2(\mathbb{D})^n$ is much more complicated.

Further results and comments:

(1) It is an extremely interesting question as to whether essential reductivity is related to a lack of corners or not being a product.

(2) In [AhCl70], Ahern and Clark proved that there exists a bijective correspondence between submodules of $H^2(\mathbb{D}^n)$ of finite codimension, and the ideals in $\mathbb{C}[\boldsymbol{z}]$ of finite codimension whose zero sets are contained in $\mathbb{D}^n$. In [GuZhe01], Guo and Zheng characterized the finite co-dimensional submodules of the Bergman module and the Hardy module over $\mathbb{B}^n$ or $\mathbb{D}^n$ (also see Corollary 2.5.4 in [ChGu03]).

THEOREM 9.9. *Let $\Omega = \mathbb{B}^n$ or $\mathbb{D}^n$ and $\mathcal{S}$ be a submodule of $L^2_a(\Omega)$ or $H^2(\Omega)$. Then $\mathcal{S}$ is of finite co-dimension if and only if $\mathcal{S}^\perp$ consists of rational functions.*

(3) Second subsection is from [DoSa08] and the final subsection is from [Sa13a]. Part (1) of the Corollary 9.5 was obtained by R. Yang (Corollary 1.1, [Y05a]).

(4) In [BerSh74], Berger and Shaw proved a surprising result concerning essentially normal Hilbert modules. Suppose $\mathcal{H}$ be a *hyponormal* Hilbert module over $\mathbb{C}[z]$, that is, $[M^*, M] \geq 0$. Moreover, assume that $\mathcal{H}$ is *rationally finitely generated*, that is, there exists $m \in \mathbb{N}$ and $\{f_1, \ldots, f_m\} \subseteq \mathcal{H}$ such that

$$\{\sum_{i=1}^{m} r_i(M) f_i : r_i \in Rat(\sigma(M))\}$$

is dense in $\mathcal{H}$. Then

$$trace[M^*, M] \leq \frac{m}{\pi} \text{Area}(\sigma(M)).$$

In particular, every rationally finitely generated hyponormal Hilbert module is essentially normal. It is not known whether the Berger-Shaw theorem holds for "hyponormal" Hilbert modules over $\mathbb{C}[\boldsymbol{z}]$. However, in [DoY92], Douglas and Yan proposed a version of Berger-Shaw theorem in several variables under the assumption that the spectrum of the Hilbert module is contained in an algebraic curve (see also [Zh01]). The reader is also referred to the work of Chavan [Ch07] for a different approach to the Berger-Shaw theorem in the context of 2-hyperexpansive operators.

(5) In connection with trace formulae, integral operators, fundamental trace forms and pseudo-differential operators see also Pincus [Pi68], Helton and Howe [HeHo75] and Carey and Pincus [CaPi79], [CaPi77]. See also the recent article by Howe [Ho12].

(6) Let $\mathcal{S}$ be a homogeneous submodule of $H^2(\mathbb{D}^2)$. In [CuMY91], Curto, Muhly and Yan proved that $\mathcal{S}$ is always essentially doubly commuting.

(7) The reader is referred to the work by Ahern and Clark [AhCl70] for more details on finite co-dimensional submodules of the Hardy modules over $\mathbb{D}^n$ (see also [ChGu03]).

(8) In [AlDu03], Alpay and Dubi characterized finite co-dimensional subspaces of $H^2_n \otimes \mathbb{C}^m$ for $m \in \mathbb{N}$ (see also [AlDu05]).

References

[Ag82] J. Agler, *The Arveson extension theorem and coanalytic models*, Integral Equations Operator Theory 5 (1982), 608-631.

[Ag85] J. Agler, *Hypercontractions and subnormality*, J. Operator Theory 13 (1985), 203-217.

[AgMc00] J. Agler and J. McCarthy, *Complete Nevanlinna-Pick kernels*, J. Funct. Anal 175 (2000), 111-124.

[ACDo86] O. Agrawal, D. Clark and R. Douglas, *Invariant subspaces in the polydisk*, Pacific J. Math. 121 (1986), 1–11.

[AhCl70] P. Ahern and D. Clark, *Invariant subspaces and analytic continuations in several variables*, J. Math. Mech. 19 (1969/1970), 963–969.

[AlDu03] D. Alpay and C. Dubi, *Backward shift operator and finite dimensional de Branges Rovnyak spaces in the ball*, Linear Algebra Appl. 371 (2003), 277-285.

[AlDu05] D. Alpay and C. Dubi, *On commuting operators solving Gleason's problem*, Proc. Amer. Math. Soc. 133 (2005), no. 11, 3285-3293.

[AmEnMu02] C. Ambrozie, M. Englis, V. Muller, *Operator tuples and analytic models over general domains in $\mathbb{C}^n$*, J. Oper. Theory 47 (2002), 287-302.

[An63] T. Ando, *On a pair of commuting contractions*, Acta Sci. Math. (Szeged), 24 (1963), 88–90.

[AEn03] J. Arazy and M. Englis, *Analytic models for commuting operator tuples on bounded symmetric domains*, Trans. Amer. Math. Soc. 355 (2003), no. 2, 837-864.

[Ari04] A. Arias, *Projective modules on Fock spaces*, J. Operator Theory 52 (2004), no. 1, 139-172.

[Ar98] W. Arveson, *Subalgebras of C^*-algebras III: Multivariable operator theory*, Acta Math. 181 (1998), 159-228.

[Ar04] W. Arveson, *The free cover of a row contraction*, Doc. Math. 9 (2004), 137–161.

[Ar07] W. Arveson, *Quotients of standard Hilbert modules*, Trans. Amer. Math. Soc. 359 (2007), no. 12, 6027-6055.

[At87] A. Athavale, *Holomorphic kernels and commuting operators*, Trans. Amer. Math. Soc. 304 (1987), no. 1, 101-110.

[At90] A. Athavale, *On the intertwining of joint isometries*, J. Operator Theory 23 (1990), no. 2, 339-350.

[At92] A. Athavale, *Model theory on the unit ball in $\mathbb{C}^m$*, J. Operator Theory 27 (1992), no. 2, 347-358.

[BaTrVi01] J. Ball, T. Trent and V. Vinnikov, *Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces*, Operator theory and analysis (Amsterdam, 1997), 89–138, Oper. Theory Adv. Appl., 122, Birkhäuser, Basel, 2001.

[BerCL78] C. Berger, L. Coburn, A. Lebow, *Representation and index theory for C^*-algebras generated by commuting isometries*, J. Funct. Anal. 27 (1978), 51-99.

[BerSh74] C. Berger and B. Shaw, *Selfcommutators of multicyclic hyponormal operators are always trace class*, Bull. Amer. Math. Soc. 79 (1973), 1193-1199, (1974).

[CaPi77] R. Carey and J. Pincus, *Mosaics, principal functions, and mean motion in von Neumann algebras*, Acta Math. 138 (1977), no. 3-4, 153-218.

[CaPi79] R. Carey and J. Pincus, *Mean motion, principal functions, and the zeros of Dirichlet series*, Integral Equations Operator Theory 2 (1979), no. 4, 484-502.

[CDSa14] A. Chattopadhyay, B. K. Das and J. Sarkar, *Tensor product of quotient Hilbert modules*, preprint. arXiv:1310.5122.

[Ch07] S. Chavan, *On operators Cauchy dual to 2-hyperexpansive operators*, Proc. Edinb. Math. Soc. (2) 50 (2007), no. 3, 637-652.

[ChGu03] X. Chen and K. Guo, *Analytic Hilbert Modules*, Chapman & Hall/CRC Research Notes in Mathematics, 433. Chapman & Hall/CRC, Boca Raton, FL, 2003.

[CSW11] S. Costea, E. Sawyer and B. Wick, *The Corona Theorem for the Drury-Arveson Hardy space and other holomorphic Besov-Sobolev spaces on the unit ball in* $\mathbb{C}^n$, Anal. PDE 4 (2011), no. 4, 499–550.

[CoSa98] M. Cotlar and C. Sadosky, *A polydisk version of Beurling's characterization for invariant subspaces of finite multi-codimension*, Operator theory for complex and hypercomplex analysis (Mexico City, 1994), 51-56, Contemp. Math., 212, Amer. Math. Soc., Providence, RI, 1998.

[CoDo78] M. Cowen and R. Douglas, *Complex geometry and operator theory*, Acta Math. 141 (1978), 187–261.

[CuMY91] R. Curto, P. Muhly and K. Yan, *The* C^**-algebra of an homogeneous ideal in two variables is type I*, Current topics in operator algebras (Nara, 1990), 130-136, World Sci. Publ., River Edge, NJ, 1991.

[CuSal84] R. Curto and N. Salinas, *Generalized Bergman kernels and the Cowen-Douglas theory*, Amer J. Math. 106 (1984), 447-488.

[DaRS14] K. Davidson, C. Ramsey and O. Shalit, *Operator algebras for analytic varieties*, to appear in Transactions of the American Mathematical Society.

[Do88] R. Douglas, *On Silov resolution of Hilbert modules*, Special classes of linear operators and other topics (Bucharest, 1986), 5160, Oper. Theory Adv. Appl., 28, Birkhuser, Basel, 1988.

[Do06] R. Douglas, *Essentially reductive Hilbert modules*, J. Operator Theory 55 (2006), 117-133.

[DoFo93] R. Douglas and C. Foias, *Uniqueness of multi-variate canonical models*, Acta Sci. Math. (Szeged) 57 (1993), 79–81.

[DoFoSa12] R. Douglas, C. Foias and J. Sarkar, *Resolutions of Hilbert modules and similarity*, J. Geom. Anal. 22 (2012), 471-490.

[DoKKSa12] R. Douglas, Y. Kim, H. Kwon and J. Sarkar, *Curvature invariant and generalized canonical operator models - I*, Operator Theory Advances and Applications, 221 (2012), 293–304.

[DoKKSa14] R. Douglas, Y. Kim, H. Kwon and J. Sarkar, *Curvature invariant and generalized canonical operator models - II*, J. of Funct. Anal. 266 (2014), 2486–2502.

[DoMi03] R. Douglas and G. Misra, *Quasi-free resolutions of Hilbert modules*, Integral Equations Operator Theory 47 (2003), 435–456.

[DoMi05] R. Douglas and G. Misra, *On quasi-free Hilbert modules*, New York J. Math. 11 (2005), 547–561.

[DoMiSa12] R. Douglas, G. Misra and J. Sarkar, *Contractive Hilbert modules and their dilations*, Israel Journal of Math. 187 (2012), 141–165.

[DoPa89] R. Douglas and V. Paulsen, *Hilbert Modules over Function Algebras*, Research Notes in Mathematics Series, 47, Longman, Harlow, 1989.

[DoPaSY95] R. Douglas, V. Paulsen, C.-H. Sah and K. Yan, . *Algebraic reduction and rigidity for Hilbert modules*, Amer. J. Math. 117 (1995), 75-92.

[DoSa08] R. Douglas and J. Sarkar, *On unitarily equivalent submodules*, Indiana Univ. Math. J. 57 (2008), 2729–2743.

[DoW11] R. Douglas and K. Wang, *Essential normality of the cyclic submodule generated by any polynomial*, J Funct Anal, 261 (2011), 3155-3180.

[DoY92] R. Douglas and K. Yan, *A multi-variable Berger-Shaw theorem*, J. Operator Theory 27 (1992), 205-217.

[DoYa90] R. Douglas and K. Yan, *Rigidity of Hardy submodules*, Integral Equations Operator Theory 13 (1990), 350-363.

[DoY98] R. Douglas and R. Yang, *Quotient Hardy modules*, Houston J. Math. 24 (1998), 507-517.

[DoY00] R. Douglas and R. Yang, *Operator theory in the Hardy space over the bidisk. I*, Integral Equations Operator Theory, 38(2000), 207-221.

[Es11] J. Eschmeier, *Essential normality of homogeneous submodules*, Integral Equations Operator Theory 69 (2011), 171-182.

[EsS14] J. Eschmeier and J. Schmitt, *Cowen-Douglas operators and dominating sets*, J. of Operator Theory, 72 (2014) 277–290.

[FXi09] Q. Fang and J. Xia, *Schatten class membership of Hankel operators on the unit sphere*, J. Funct. Anal. 257 (2009), 3082-3134.

[Gr03] D. Greene, *Free resolutions in multivariable operator theory*, J. Funct. Anal. 200 (2003), 429-450.

[GriHar94] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994.

[Gu99] K. Guo, *Characteristic spaces and rigidity for analytic Hilbert modules*, J. Funct. Anal. 163 (1999), 133-151.

[Gu00] K. Guo, *Equivalence of Hardy submodules generated by polynomials*, J. Funct. Anal. 178 (2000), 343-371.

[GuHuXu04] K. Guo, J. Hu and X. Xu, *Toeplitz algebras, subnormal tuples and rigidity on reproducing* $\mathbb{C}[z_1,\ldots,z_d]$*-modules*, J. Funct. Anal. 210 (2004), 214–247.

[GuW08] K. Guo and K. Wang, *Essentially normal Hilbert modules and* K *-homology*, Math Ann. 340 (2008), 907-934.

[GuY04] K. Guo and R. Yang, *The core function of submodules over the bidisk*, Indiana Univ. Math. J. 53 (2004), 205-222.

[GuZhe01] K. Guo and D. Zheng, *Invariant subspaces, quasi-invariant subspaces*, and Hankel operators. J. Funct. Anal. 187 (2001), 308-342.

[HeHo75] J. W. Helton and R. Howe, *Traces of commutators of integral operators*, Acta Math. 135 (1975), 271-305.

[Ho12] R. Howe, *Traces of commutators of integral operatorsthe aftermath*, Mathematical methods in systems, optimization, and control, 221-231, Oper. Theory Adv. Appl., 222, Birkhauser/Springer Basel AG, Basel, 2012.

[I87] K. Izuchi, *Unitary equivalence of invariant subspaces in the polydisk*, Pacific J. Math. 130 (1987), 351-358.

[III11] K. J. Izuchi, K. H. Izuchi and Y. Izuchi, *Blaschke products and the rank of backward shift invariant subspaces over the bidisk*, J. Funct. Anal. 261 (2011), 1457-1468.

[III11] K. J. Izuchi, K. H. Izuchi and Y. Izuchi, *Ranks of invariant subspaces of the Hardy space over the bidisk*, J. Reine Angew. Math. 659 (2011), 101-139.

[IN04] K. J. Izuchi and T. Nakazi, *Backward shift invariant subspaces in the bidisc*, Hokkaido Math. J. 3 (2004), 247-254.

[INS04] K. Izuchi, T. Nakazi and M. Seto, *Backward shift invariant subspaces in the bidisc II*, J. Oper. Theory 51 (2004), 361-376.

[KT09] H. Kwon and S. Treil, *Similarity of operators and geometry of eigenvector bundles*, Publ. Mat. 53 (2009), 417-438.

[NaFo70a] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, Amsterdam, 1970.

[NaFo70b] B. Sz.-Nagy and C. Foias, *Modèle de Jordan pour une classe d́opérateurs de ĺespace de Hilbert*, Acta Sci. Math. (Szeged) 31 (1970) 91-115.

[OF00] J. Ortega and J. Fabrega, *Pointwise multipliers and decomposition theorems in analytic Besov spaces*, Math. Z. 235 (2000), 53–81.

[Pi68] J. Pincus, *Commutators and systems of singular integral equations. I*, Acta Math. 121 (1968) 219-249.

[Po11] G. Popescu, *Joint similarity to operators in noncommutative varieties*, Proc. Lond. Math. Soc. (3) 103 (2011), 331-370.

[Pu94] M. Putinar, *On the rigidity of Bergman submodules*, Amer. J. Math. 116 (1994), 1421-1432.

[Ri88] S. Richter, *Unitary equivalence of invariant subspaces of Bergman and Dirichlet spaces*, Pacific J. Math. 133 (1988), 151-156.

[Sa13a] J. Sarkar, *Submodules of the Hardy module over polydisc*, arXiv:1304.1564, to appear in Israel Journal of Mathematics.

[Sa13b] J. Sarkar, *Similarity of Quotient Hilbert modules in the Cowen-Douglas Class*, preprint. arXiv:1212.2707.

[Sa14a] J. Sarkar, *An Introduction to Hilbert Module Approach to Multivariable Operator Theory*, to appear in Handbook of Operator Theory, Springer.

[Sa14b] J. Sarkar, *Jordan Blocks of* $H^2(\mathbb{D}^n)$, Journal of Operator Theory, 72 (2014), 101–115.

[U90] M. Uchiyama, *Curvatures and similarity of operators with holomorphic eigenvectors*, Trans. Amer. Math. Soc. 319 (1990), 405–415.

[We80] R. O. Wells, *Differential analysis on complex manifolds, Graduate Texts in Mathematics*, Springer-Verlag, New York-Berlin, 1980.

[Y01] R. Yang, *Operator theory in the Hardy space over the bidisk. (III)*, J. Fund. Anal. 186 (2001) 521-545.

[Y05a] R. Yang, *Hilbert-Schmidt submodules and issues of unitary equivalence*, J. Operator Theory 53 (2005) 169-184.

[Y05b] R. Yang, *The core operator and congruent submodules*, J. Fund. Anal. 228 (2005) 459-489.

[Zh00] K. Zhu, *Operators in Cowen-Douglas classes*, Illinois J. Math.44 (2000), 767–783.

[Zh01] K. Zhu, *A trace formula for multiplication operators on invariant subspaces of the Bergman space*, Integral Equations Operator Theory 40 (2001), 244-255.

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