Commutant lifting and Nevanlinna-Pick interpolation in several variables

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Abstract. This paper concerns a commutant lifting theorem and a Nevanlinna-Pick type interpolation result in the setting of multipliers from vector-valued Drury-Arveson space to a large class of vector-valued reproducing kernel Hilbert spaces over the unit ball in $\mathbb{C}^n$. The special case of reproducing kernel Hilbert spaces includes all natural examples of Hilbert spaces like Hardy space, Bergman space and weighted Bergman spaces over the unit ball.

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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$. The classical Nevanlinna–Pick interpolation theorem [15, 17] states: Given distinct n points $\{z_i\}_{i=1}^n \subseteq \mathbb{D}$ (interpolation nodes) and n points $\{w_i\}_{i=1}^n \subseteq \mathbb{D}$ (target data), there exists a $\varphi \in H^\infty(\mathbb{D})$ such that $\|\varphi\|_\infty \leq 1$ and such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \ldots, n$, if and only if the *Pick matrix*

$$\left[\frac{1-w_i\bar{w}_j}{1-z_i\bar{z}_j}\right]_{i,j=1}^n,$$

is positive semi-definite. Here we denote by $H^\infty(\mathbb{D})$ the Banach algebra of all bounded analytic functions on $\mathbb{D}$ equipped with the norm $\|\varphi\| = sup\{|\varphi(z)| : z \in \mathbb{D}\}$, $\varphi \in H^\infty(\mathbb{D})$. In his seminal paper [18], Sarason proved the commutant lifting theorem for compressions of the shift operator to shift co-invariant subspaces of the Hardy space which gives a simpler and elegant proof of the Nevanlinna–Pick interpolation theorem.

Sarason's approach to the commutant lifting theorem, along with its direct application to Nevanlinna–Pick interpolation theorem, is deeply connected with a number of classical problems in function theory and operator theory and have been studied extensively in the past few decades (cf. [11]). There also has been a great deal of interest in analyzing the possibilities of commutant lifting theorem and interpolation (and other related problems) in the setting of general reproducing kernel Hilbert spaces over domains in $\mathbb{C}^n$, $n \geq 1$ (for instance, see [5], [10], [3], [7] and [9]).

In this paper we make a contribution to a commutant lifting theorem and a version of Nevanlinna-Pick interpolation in several variables. To be more precise, let $m \geq 1$ and let $\mathcal{H}_m$ denotes the reproducing kernel Hilbert space corresponding to the kernel k_m on $\mathbb{B}^n$, where

$$k_m(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-m} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n),$$

and $\mathbb{B}^n = \{\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^2 < 1\}$. Recall that $\mathcal{H}_m$ is the Drury-Arveson space (popularly denoted by H^2_n), the Hardy space, the Bergman space and the weighted Bergman space over $\mathbb{B}^n$ for $m = 1$, $m = n$, $m = n + 1$ and $m > n + 1$, respectively.

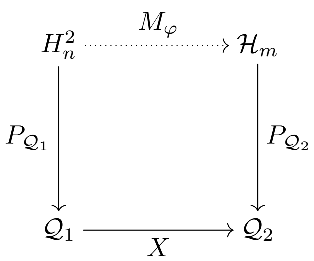
Our main results, restricted to $\mathcal{H}_m$, $m > 1$, can now be formulated as follows:
Commutant lifting theorem (Theorem 3.4*):* Suppose $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are joint $(M_{z_1}, \ldots, M_{z_n})$ co-invariant subspaces of $H^2_n(= \mathcal{H}_1)$ and $\mathcal{H}_m$, respectively. Let $X \in \mathcal{B}(\mathcal{Q}_1, \mathcal{Q}_2)$ and $\|X\| \leq 1$. If

$$X(P_{\mathcal{Q}_1} M_{z_i}|_{\mathcal{Q}_1}) = (P_{\mathcal{Q}_2} M_{z_i}|_{\mathcal{Q}_2})X,$$

for all $i = 1, \ldots, n$, then there exists a holomorphic function $\varphi : \mathbb{B}^n \to \mathbb{C}$ such that the multiplication operator $M_\varphi \in \mathcal{B}(H^2_n, \mathcal{H}_m)$, $\|M_\varphi\| \leq 1$ (that is, φ is a *contractive multiplier*), and

$$X = P_{\mathcal{Q}_2} M_\varphi|_{\mathcal{Q}_1}.$$

Thus, we have the following commutative diagram:



Given a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$ we denote by $P_{\mathcal{S}}$ the orthogonal projection of $\mathcal{S}$ on $\mathcal{H}$.

Nevanlinna–Pick interpolation theorem (Theorem 5.1*):* Given distinct n points

$$\{\boldsymbol{z}_i\}_{i=1}^n \subseteq \mathbb{B}^n,$$

and n points

$$\{w_i\}_{i=1}^n \subseteq \mathbb{D},$$

there exists a contractive multiplier φ such that

$$\varphi(\boldsymbol{z}_i) = w_i,$$

for all $i = 1, \ldots, n$ if and only if the matrix

$$\left[\frac{1}{(1-\langle \boldsymbol{z}_i, \boldsymbol{z}_j\rangle)^m} - \frac{w_i \bar{w}_j}{1-\langle \boldsymbol{z}_i, \boldsymbol{z}_j\rangle}\right]_{i,j=1}^n,$$

is positive semi-definite. Here $\langle \boldsymbol{z}, \boldsymbol{w}\rangle = \sum_{i=1}^n z_i \bar{w}_i$ for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{C}^n$.

We make strong use of the commutant lifting theorem in the setting of Drury-Arveson space (see Theorem 2.2) and a refined factorization result (see Theorem 4.2) concerning multipliers between Drury-Arveson space and a large class of analytic reproducing kernel Hilbert space over $\mathbb{B}^n$.

We point out that the above interpolation theorem, in the setting of normalized complete Pick kernel, is due to Aleman, Hartz, McCarthy and Richter [2]. Their proof relies on Leech's theorem (or Toeplitz corona theorem). From this point of view, in this paper we prove that the interpolation theorem is a consequence of the commutant lifting theorem. Furthermore, our interpolation result holds for operator-valued multipliers (see Theorem 5.1).

Note that there are also free noncommutative versions of interpolation theory (cf. [6]). We thanks the referee for pointing out that some of our results may extend to this setting. This will be the subject of future work.

The remainder of the paper is organized as follows. Section 2 discusses some useful and known facts about reproducing kernel Hilbert spaces. Section 3 presents the commutant lifting theorem. Section 4 is devoted to factorizations of multipliers. The factorization results obtained here may be of independent interest. Section 5 provides the interpolation theorem.

2. Preliminaries

The Drury-Arveson space over the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ will be denoted by H^2_n. Recall that H^2_n is a reproducing kernel Hilbert space corresponding to the kernel function

$$k_1(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^n z_i \bar{w}_i)^{-1} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

Let $k : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ be a kernel such that k is analytic in the first variables $\{z_1, \ldots, z_n\}$. We say that k is *regular* if there exists a kernel $\tilde{k} : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$, analytic in $\{z_1, \ldots, z_n\}$, such that

$$k(\boldsymbol{z}, \boldsymbol{w}) = k_1(\boldsymbol{z}, \boldsymbol{w})\tilde{k}(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

If k is a regular kernel, then $\mathcal{H}_k$, the reproducing kernel Hilbert space corresponding to the kernel k, will be referred as a *regular reproducing kernel Hilbert space.*

In the case of a regular reproducing kernel Hilbert space $\mathcal{H}_k$, it follows [13] that M_{z_i}, the multiplication operator by the coordinate function z_i, is bounded. Note that

$$(M_{z_i} f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}),$$

for all $f \in \mathcal{H}_k$, $\boldsymbol{w} \in \mathbb{B}^n$ and $i = 1, \ldots, n$. Moreover, it also follows that the commuting tuple $(M_{z_1}, \ldots, M_{z_n})$ on $\mathcal{H}_k$ is a *row contraction*, that is

$$\sum_{i=1}^{n} M_{z_i} M_{z_i}^* \leq I_{\mathcal{H}_k}.$$

If $\mathcal{E}$ is a Hilbert space, then we also say that $\mathcal{H}_k \otimes \mathcal{E}$ is a regular reproducing kernel Hilbert space. Note that the kernel function of $\mathcal{H}_k \otimes \mathcal{E}$ is given by

$$\mathbb{B}^n \times \mathbb{B}^n \ni (\boldsymbol{z}, \boldsymbol{w}) \mapsto k(\boldsymbol{z}, \boldsymbol{w}) I_{\mathcal{E}}.$$

The $\mathcal{E}$-valued Drury-Arveson space, denoted by $H_n^2(\mathcal{E})$, is the reproducing kernel Hilbert space corresponding to the $\mathcal{B}(\mathcal{E})$-valued kernel function

$$\mathbb{B}^n \times \mathbb{B}^n \ni (\boldsymbol{z}, \boldsymbol{w}) \mapsto k_1(\boldsymbol{z}, \boldsymbol{w}) I_{\mathcal{E}}.$$

To simplify the notation, we often identify $H_n^2(\mathcal{E})$ with $H_n^2 \otimes \mathcal{E}$ via the unitary map defined by $z^{\boldsymbol{k}} \eta \mapsto z^{\boldsymbol{k}} \otimes \eta$ for all $\boldsymbol{k} \in \mathbb{Z}_+^n$ and $\eta \in \mathcal{E}$. This also enable us to identify $(M_{z_1}, \ldots, M_{z_n})$ on $H_n^2(\mathcal{E})$ with $(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_n} \otimes I_{\mathcal{E}})$ on $H_n^2 \otimes \mathcal{E}$.

Typical examples of regular reproducing kernel Hilbert spaces arise from weighted Bergman spaces over $\mathbb{B}^n$. More specifically, let $\lambda > 1$, and let

$$k_\lambda(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-\lambda} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n). \tag{2.1}$$

Then $\mathcal{H}_{k_\lambda}$ is a regular reproducing kernel Hilbert space. Note that $\mathcal{H}_{k_\lambda}$ is the Hardy space, Bergman space and weighted Bergman space for $\lambda = n$, $n+1$ and $n + 1 + \alpha$ for any $\alpha > 0$, respectively.

Suppose $\mathcal{H}$ and $\mathcal{E}_*$ are Hilbert spaces and $(T_1, \ldots, T_n)$ is a commuting tuple of bounded linear operators on $\mathcal{H}$. We say that $(T_1, \ldots, T_n)$ on $\mathcal{H}$ *dilates* to $(M_{z_1} \otimes I_{\mathcal{E}_*}, \ldots, M_{z_n} \otimes I_{\mathcal{E}_*})$ on $H_n^2 \otimes \mathcal{E}_*$ if there exists an isometry $\Pi : \mathcal{H} \to H_n^2 \otimes \mathcal{E}_*$ such that

$$\Pi T_i^* = (M_{z_i} \otimes I_{\mathcal{E}_*})^* \Pi,$$

for all $i = 1, \ldots, n$ (cf. [19]). We often say that $\Pi : \mathcal{H} \to H_n^2 \otimes \mathcal{E}_*$ is a *dilation* of $(T_1, \ldots, T_n)$.

If $\mathcal{H} = \mathcal{H}_k$ is a regular reproducing kernel Hilbert space, then by [Theorem 6.1, [13]], it follows that $(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_n} \otimes I_{\mathcal{E}})$ on $\mathcal{H}_k \otimes \mathcal{E}$ dilates to $(M_{z_1} \otimes I_{\mathcal{E}_*}, \ldots, M_{z_n} \otimes I_{\mathcal{E}_*})$ on $H_n^2 \otimes \mathcal{E}_*$ for some Hilbert space $\mathcal{E}_*$. More specifically:

Theorem 2.1. *Let $\mathcal{E}$ be a Hilbert space. If $\mathcal{H}_k$ is a regular reproducing kernel Hilbert space, then there exist a Hilbert space $\mathcal{E}_*$ and an isometry*

$$\Pi_k : \mathcal{H}_k \otimes \mathcal{E} \to H_n^2 \otimes \mathcal{E}_*,$$

such that

$$\Pi_k(M_{z_i} \otimes I_{\mathcal{E}})^* = (M_{z_i} \otimes I_{\mathcal{E}_*})^*\Pi_k,$$

for all $i = 1, \dots, n$.

Since $(M_{z_1} \otimes I_{\mathcal{E}}, \dots, M_{z_n} \otimes I_{\mathcal{E}})$ on $\mathcal{H}_k \otimes \mathcal{E}$ is a pure row contraction [13], the above result also directly follows from Muller-Vasilescu [14] and Arveson [4].

In what follows, given a Hilbert space $\mathcal{H}$ and a closed subspace $\mathcal{Q}$ of $\mathcal{H}$, we will denote by $i_{\mathcal{Q}}$ the inclusion map

$$i_{\mathcal{Q}} : \mathcal{Q} \hookrightarrow \mathcal{H}.$$

Note that $i_{\mathcal{Q}}$ is an isometry and

$$i_{\mathcal{Q}} i_{\mathcal{Q}}^* = P_{\mathcal{Q}}.$$

We now recall the commutant lifting theorem in the setting of the Drury-Arveson space (see [3] or Theorem 5.1, page 118, [7]). A closed subspace $\mathcal{Q}$ of a regular reproducing kernel Hilbert space $\mathcal{H}_k \otimes \mathcal{E}$ is said to be *shift co-invariant* if

$$(M_{z_i} \otimes I_{\mathcal{E}})^*\mathcal{Q} \subseteq \mathcal{Q} \qquad (i = 1, \dots, n).$$

Theorem 2.2. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces. Suppose $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are shift co-invariant subspaces of $H^2_n(\mathcal{E}_1)$ and $H^2_n(\mathcal{E}_2)$, respectively, $X \in \mathcal{B}(\mathcal{Q}_1, \mathcal{Q}_2)$ and let $\|X\| \leq 1$. If*

$$X(P_{\mathcal{Q}_1} M_{z_i}|_{\mathcal{Q}_1}) = (P_{\mathcal{Q}_2} M_{z_i}|_{\mathcal{Q}_2})X,$$

for all $i = 1, \dots, n$, then there exists a multiplier $\Phi \in \mathcal{M}(H^2_n(\mathcal{E}_1), H^2_n(\mathcal{E}_2))$ such that $\|M_\Phi\| \leq 1$ and $P_{\mathcal{Q}_2} M_\Phi|_{\mathcal{Q}_1} = X$.

Recall also that, given regular reproducing kernel Hilbert spaces $\mathcal{H}_{k_1} \otimes \mathcal{E}_1$ and $\mathcal{H}_{k_2} \otimes \mathcal{E}_2$, a function $\Phi : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ is called a *multiplier* from $\mathcal{H}_{k_1} \otimes \mathcal{E}_1$ to $\mathcal{H}_{k_2} \otimes \mathcal{E}_2$ if

$$\Phi(\mathcal{H}_{k_1} \otimes \mathcal{E}_1) \subseteq \mathcal{H}_{k_2} \otimes \mathcal{E}_2.$$

The *multiplier space* $\mathcal{M}(\mathcal{H}_{k_1} \otimes \mathcal{E}_1, \mathcal{H}_{k_2} \otimes \mathcal{E}_2)$ is the set of all multipliers from $\mathcal{H}_{k_1} \otimes \mathcal{E}_1$ to $\mathcal{H}_{k_2} \otimes \mathcal{E}_2$. In what follows, $\mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ will denote the closed ball of radius one:

$$\mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2) = \{\Phi \in \mathcal{M}(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2) : \|M_\Phi\| \leq 1\}.$$

We have the following useful characterization of multipliers (cf. Proposition 4.2, [19]): Let $\mathcal{H}_k$ be a regular reproducing kernel Hilbert space, and let $X \in \mathcal{B}(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$. Then

$$X(M_{z_i} \otimes I_{\mathcal{E}_1}) = (M_{z_i} \otimes I_{\mathcal{E}_2})X,$$

if and only if $X = M_\Phi$ for some $\Phi \in \mathcal{M}(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$.

3. Commutant lifting theorem

We begin with a general result concerning intertwiner of bounded linear operators.

Lemma 3.1. *Suppose* $\Pi : \mathcal{H} \to \mathcal{K}$ *and* $\hat{\Pi} : \hat{\mathcal{H}} \to \hat{\mathcal{K}}$ *are isometries,* $V \in \mathcal{B}(\mathcal{K})$, $\hat{V} \in \mathcal{B}(\hat{\mathcal{K}})$, $T = \Pi^* V \Pi$ *and* $\hat{T} = \hat{\Pi}^* \hat{V} \hat{\Pi}$. *Moreover, let* $X \in \mathcal{B}(\mathcal{H}, \hat{\mathcal{H}})$ *satisfies*

$$XT = \hat{T}X.$$

If we define

$$\mathcal{Q} = \Pi\mathcal{H} \quad \textit{and} \quad \hat{\mathcal{Q}} = \hat{\Pi}\hat{\mathcal{H}},$$

and

$$\tilde{X} = \hat{\Pi} X \Pi^*|_{\mathcal{Q}},$$

then $\tilde{X} \in \mathcal{B}(\mathcal{Q}, \hat{\mathcal{Q}})$ *and*

$$\tilde{X}(P_{\mathcal{Q}} V|_{\mathcal{Q}}) = (P_{\hat{\mathcal{Q}}} \hat{V}|_{\hat{\mathcal{Q}}})\tilde{X}.$$

Proof. Notice that $P_{\mathcal{Q}} = \Pi\Pi^*$ and $P_{\hat{\mathcal{Q}}} = \hat{\Pi}\hat{\Pi}^*$. Hence

$$\tilde{X} = (\hat{\Pi}\hat{\Pi}^*)\hat{\Pi} X \Pi^*|_{\mathcal{Q}} = P_{\hat{\mathcal{Q}}}(\hat{\Pi} X \Pi^*)|_{\mathcal{Q}},$$

and in particular

$$(\hat{\Pi} X \Pi^*)\mathcal{Q} \subseteq \hat{\mathcal{Q}},$$

which shows that $\tilde{X} \in \mathcal{B}(\mathcal{Q}, \hat{\mathcal{Q}})$. Moreover

$$\begin{aligned}\tilde{X}(P_{\mathcal{Q}} V|_{\mathcal{Q}}) &= \hat{\Pi} X \Pi^* P_{\mathcal{Q}} V|_{\mathcal{Q}} = \hat{\Pi} X \Pi^* V|_{\mathcal{Q}} = \hat{\Pi} X T \Pi^*|_{\mathcal{Q}} \\ &= \hat{\Pi}\hat{T} X \Pi^*|_{\mathcal{Q}} = \hat{\Pi}\hat{\Pi}^* \hat{V} \hat{\Pi}(\hat{\Pi}^*\hat{\Pi}) X \Pi^*|_{\mathcal{Q}} \\ &= P_{\hat{\mathcal{Q}}} \hat{V}|_{\hat{\mathcal{Q}}} \hat{\Pi} X \Pi^*|_{\mathcal{Q}} = (P_{\hat{\mathcal{Q}}} \hat{V}|_{\hat{\mathcal{Q}}})\tilde{X}.\end{aligned}$$

□

Now we are ready to prove a variation, in terms of dilations, of Theorem 2.2.

Theorem 3.2. *Let* $\mathcal{H}$ *and* $\hat{\mathcal{H}}$ *be Hilbert spaces. Suppose* $T = (T_1, \ldots, T_n)$ *and* $\hat{T} = (\hat{T}_1, \ldots, \hat{T}_n)$ *are commuting tuples on* $\mathcal{H}$ *and* $\hat{\mathcal{H}}$*, respectively,* $X \in \mathcal{B}(\mathcal{H}, \hat{\mathcal{H}})$, $\|X\| \leq 1$, *and*

$$XT_i = \hat{T}_i X,$$

for all $i = 1, \ldots, n$. *If* $\Pi : \mathcal{H} \to H^2_n \otimes \mathcal{E}$ *and* $\hat{\Pi} : \hat{\mathcal{H}} \to H^2_n \otimes \hat{\mathcal{E}}$ *are dilations of* T *and* $\hat{T}$*, respectively, then there exists a multiplier* $\Phi \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}, H^2_n \otimes \hat{\mathcal{E}})$ *such that*

$$X = \hat{\Pi}^* M_\Phi \Pi.$$

Proof. Let

$$\mathcal{Q} = \Pi\mathcal{H} \quad \text{and} \quad \hat{\mathcal{Q}} = \hat{\Pi}\hat{\mathcal{H}}.$$

If

$$\tilde{X} = \hat{\Pi} X \Pi^*|_{\mathcal{Q}},$$

then by Lemma 3.1, it follows that $\tilde{X} \in \mathcal{B}(\mathcal{Q}, \hat{\mathcal{Q}})$ and

$$\tilde{X}(P_{\mathcal{Q}}(M_{z_i} \otimes I_{\mathcal{E}})|_{\mathcal{Q}}) = (P_{\hat{\mathcal{Q}}}(M_{z_i} \otimes I_{\hat{\mathcal{E}}})|_{\hat{\mathcal{Q}}})\tilde{X},$$

for all $i = 1, \ldots, n$. It then follows from the commutant lifting theorem, Theorem 2.2, that

$$\tilde{X} = P_{\hat{\mathcal{Q}}} M_\Phi|_{\mathcal{Q}},$$

for some $\Phi \in \mathcal{M}(H^2_n \otimes \mathcal{E}, H^2_n \otimes \hat{\mathcal{E}})$ and $\|M_\Phi\| \leq 1$. Then

$$\hat{\Pi} X \Pi^*|_{\mathcal{Q}} = P_{\hat{\mathcal{Q}}} M_\Phi|_{\mathcal{Q}}.$$

It then follows from

$$\mathcal{Q} = \text{ran } \Pi = \text{ran } \Pi\Pi^*,$$

that

$$(\hat{\Pi} X \Pi^*)(\Pi\Pi^*) = P_{\hat{\mathcal{Q}}} M_\Phi (\Pi\Pi^*).$$

Thus

$$\hat{\Pi} X = P_{\hat{\mathcal{Q}}} M_\Phi \Pi = (\hat{\Pi}\hat{\Pi}^*) M_\Phi \Pi,$$

and hence $X = \hat{\Pi}^* M_\Phi \Pi$. □

Now let $\mathcal{Q}$ be a shift co-invariant subspace of $\mathcal{H}_k \otimes \mathcal{E}$. An isometry $\Pi : \mathcal{Q} \to H^2_n \otimes \mathcal{E}_*$ is said to be a *dilation of* $\mathcal{Q}$ if

$$\Pi (P_{\mathcal{Q}} (M_{z_i} \otimes I_{\mathcal{E}})|_{\mathcal{Q}})^* = (M_{z_i} \otimes I_{\mathcal{E}_*})^* \Pi,$$

for all $i = 1, \ldots, n$, that is $(P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} M_{z_n}|_{\mathcal{Q}})$ on $\mathcal{Q}$ dilates to $(M_{z_1} \otimes I_{\mathcal{E}_*}, \ldots, M_{z_n} \otimes I_{\mathcal{E}_*})$ on $H^2_n \otimes \mathcal{E}_*$ via the isometry Π.

Lemma 3.3. *Let $\mathcal{H}_k$ be a regular reproducing kernel Hilbert space, and let $\mathcal{E}$ and $\mathcal{E}_*$ be a Hilbert spaces. Suppose $\mathcal{Q}$ is a shift co-invariant subspace of $\mathcal{H}_k \otimes \mathcal{E}$. If $\Pi : \mathcal{H}_k \otimes \mathcal{E} \to H^2_n \otimes \mathcal{E}_*$ is a dilation of $\mathcal{H}_k \otimes \mathcal{E}$, then $\Pi_{\mathcal{Q}} : \mathcal{Q} \to H^2_n \otimes \mathcal{E}_*$, defined by*

$$\Pi_{\mathcal{Q}} = \Pi \circ i_{\mathcal{Q}},$$

is a dilation $\mathcal{Q}$.

Proof. We first observe that

$$\Pi^*_{\mathcal{Q}} \Pi_{\mathcal{Q}} = i^*_{\mathcal{Q}} \Pi^* \Pi i_{\mathcal{Q}} = I_{\mathcal{Q}}.$$

Now we compute

$$\begin{aligned}
\Pi_{\mathcal{Q}} (P_{\mathcal{Q}} (M_{z_i} \otimes I_{\mathcal{E}})|_{\mathcal{Q}})^* &= \Pi i_{\mathcal{Q}} P_{\mathcal{Q}} (M_{z_i} \otimes I_{\mathcal{E}})^*|_{\mathcal{Q}} = \Pi (M_{z_i} \otimes I_{\mathcal{E}})^*|_{\mathcal{Q}} \\
&= (M_{z_i} \otimes I_{\mathcal{E}_*})^* \Pi|_{\mathcal{Q}} = (M_{z_i} \otimes I_{\mathcal{E}_*})^* (\Pi i_{\mathcal{Q}}) i^*_{\mathcal{Q}}|_{\mathcal{Q}} \\
&= (M_{z_i} \otimes I_{\mathcal{E}_*})^* \Pi_{\mathcal{Q}} i^*_{\mathcal{Q}}|_{\mathcal{Q}}.
\end{aligned}$$

Now

$$i^*_{\mathcal{Q}}|_{\mathcal{Q}} = I_{\mathcal{Q}},$$

and so

$$\Pi_{\mathcal{Q}} (P_{\mathcal{Q}} (M_{z_i} \otimes I_{\mathcal{E}})|_{\mathcal{Q}})^* = (M_{z_i} \otimes I_{\mathcal{E}_*})^* \Pi_{\mathcal{Q}},$$

for all $i = 1, \ldots, n$. □

We are now ready to present and prove the commutant lifting theorem.

Theorem 3.4. *Let $\mathcal{H}_k$ be a regular reproducing kernel Hilbert space, $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces, and let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be shift co-invariant subspaces of $H^2_n \otimes \mathcal{E}_1$ and $\mathcal{H}_k \otimes \mathcal{E}_2$, respectively. Let $X \in \mathcal{B}(\mathcal{Q}_1, \mathcal{Q}_2)$, and suppose that $\|X\| \leq 1$ and*

$$X(P_{\mathcal{Q}_1}(M_{z_i} \otimes I_{\mathcal{E}_1})|_{\mathcal{Q}_1}) = (P_{\mathcal{Q}_2}(M_{z_i} \otimes I_{\mathcal{E}_2})|_{\mathcal{Q}_2})X,$$

for all $i = 1, \ldots, n$. Then there exists a multiplier $\Phi \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ such that

$$X = P_{\mathcal{Q}_2} M_\Phi|_{\mathcal{Q}_1}.$$

Proof. Observe that the inclusion map $i_{\mathcal{Q}_1} : \mathcal{Q}_1 \hookrightarrow H^2_n \otimes \mathcal{E}_1$ is a dilation of $\mathcal{Q}_1$. Let $\Pi_k : \mathcal{H}_k \otimes \mathcal{E}_2 \to H^2_n \otimes \hat{\mathcal{E}}$ be a dilation of $\mathcal{H}_k$ (see Theorem 2.1), that is, Π_k is an isometry and

$$\Pi_k(M_{z_i} \otimes I_{\mathcal{E}_2})^* = (M_{z_i} \otimes I_{\hat{\mathcal{E}}})^* \Pi_k, \tag{3.1}$$

for all $i = 1, \ldots, n$ and some Hilbert space $\hat{\mathcal{E}}$. Set

$$\Pi_{\mathcal{Q}_2} = \Pi_k i_{\mathcal{Q}_2}.$$

By Lemma 3.3, it follows that $\Pi_{\mathcal{Q}_2} : \mathcal{Q}_2 \to H^2_n \otimes \hat{\mathcal{E}}$ is a dilation of $\mathcal{Q}_2$. Then Theorem 3.2 yields

$$X = \Pi^*_{\mathcal{Q}_2} M_{\Phi_1} i_{\mathcal{Q}_1},$$

for some multiplier $\Phi_1 \in \mathcal{M}(H^2_n \otimes \mathcal{E}_1, H^2_n \otimes \hat{\mathcal{E}})$. Hence

$$X = i^*_{\mathcal{Q}_2}(\Pi^*_k M_{\Phi_1}) i_{\mathcal{Q}_1}.$$

Since

$$M_{\Phi_1}(M_{z_i} \otimes I_{\mathcal{E}_1}) = (M_{z_i} \otimes I_{\hat{\mathcal{E}}}) M_{\Phi_1},$$

we have, using also the adjoint of (3.1),

$$\Pi^*_k M_{\Phi_1}(M_{z_i} \otimes I_{\mathcal{E}_1}) = \Pi^*_k (M_{z_i} \otimes I_{\hat{\mathcal{E}}}) M_{\Phi_1} = (M_{z_i} \otimes I_{\mathcal{E}_2}) \Pi^*_k M_{\Phi_1},$$

for all $i = 1, \ldots, n$, that is, $\Pi^*_k M_{\Phi_1} : H^2_n \otimes \mathcal{E}_1 \to \mathcal{H}_k \otimes \mathcal{E}_2$ intertwines the shifts. Consequently

$$\Pi^*_k M_{\Phi_1} = M_\Phi,$$

for some multiplier $\Phi \in \mathcal{M}(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$. Hence

$$X = i^*_{\mathcal{Q}_2} M_\Phi i_{\mathcal{Q}_1},$$

and thus

$$i_{\mathcal{Q}_2} X = P_{\mathcal{Q}_2} M_\Phi i_{\mathcal{Q}_1}.$$

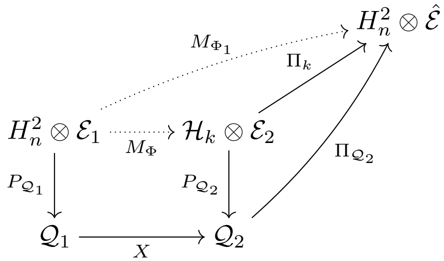
Hence, we have

$$X = P_{\mathcal{Q}_2} M_\Phi|_{\mathcal{Q}_1}.$$

Finally

$$\|M_\Phi\| \leq \|M_{\Phi_1}\| \leq 1. \qquad \square$$

A simpler way of presenting the above theorem, from Hilbert module point of view, is to say that the following diagram commutes:



4. Factorizations

Let k be a regular reproducing kernel on $\mathbb{B}^n$. Then there exists a positive definite kernel $\tilde{k} : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ such that

$$k(\boldsymbol{z}, \boldsymbol{w}) = k_1(\boldsymbol{z}, \boldsymbol{w})\tilde{k}(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

Let $\mathcal{H}_{\tilde{k}}$ be the reproducing kernel Hilbert space corresponding to the kernel $\tilde{k}$. Suppose $\boldsymbol{w} \in \mathbb{B}^n$ and $ev(\boldsymbol{w}) : \mathcal{H}_{\tilde{k}} \to \mathbb{C}$ is the evaluation map, that is

$$ev(\boldsymbol{w})(f) = f(\boldsymbol{w}) \qquad (f \in \mathcal{H}_{\tilde{k}}).$$

Then

$$\tilde{k}(\boldsymbol{z}, \boldsymbol{w}) = ev(\boldsymbol{z})ev(\boldsymbol{w})^* \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n),$$

and so

$$k(\boldsymbol{z}, \boldsymbol{w}) = k_1(\boldsymbol{z}, \boldsymbol{w})\Big(ev(\boldsymbol{z})ev(\boldsymbol{w})^*\Big) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n). \tag{4.1}$$

From Corollary 4.2 in [13] it follows that the map

$$(\pi F)(\boldsymbol{z}) := F(\boldsymbol{z}, \boldsymbol{z}),$$

for all $F \in H_n^2 \otimes \mathcal{H}_{\tilde{k}}$ and $\boldsymbol{z} \in \mathbb{B}^n$, defines a coisometry from $H_n^2 \otimes \mathcal{H}_{\tilde{k}} = \mathcal{H}_{k_1} \otimes \mathcal{H}_{\tilde{k}}$ to $\mathcal{H}_k = \mathcal{H}_{k_1\tilde{k}}$. If we view $H_n^2 \otimes \mathcal{H}_{\tilde{k}}$ as a reproducing kernel Hilbert space of functions with values in $\mathcal{H}_{\tilde{k}}$, then the map π is actually the multiplier M_{ev}; indeed, if we compute the action on reproducing kernels, we have

$$M_{ev}(f \otimes g)(\boldsymbol{w}) = f(\boldsymbol{w}) \otimes ev(\boldsymbol{w})(g) = f(\boldsymbol{w}) \otimes g(\boldsymbol{w}) = \pi(f \otimes g)(\boldsymbol{w}).$$

This formula may be extended by tensorizing with $I_{\mathcal{E}}$, where $\mathcal{E}$ is a Hilbert space. If we define $\Psi_k : \mathcal{H}_{\tilde{k}} \otimes \mathcal{E} \to \mathcal{E}$ by $\Psi_k := ev \otimes I_{\mathcal{E}}$, then Ψ_k is obviously also a coisometric multiplier. Taking into account (4.1), we obtain the following theorem (see also [12, Theorem 4.1] and [13, Theorem 6.2]):

Theorem 4.1. *Let $k : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ be a regular kernel, and let*

$$k(\boldsymbol{z}, \boldsymbol{w}) = k_1(\boldsymbol{z}, \boldsymbol{w})\tilde{k}(\boldsymbol{z}, \boldsymbol{w}) \qquad (z, \boldsymbol{w} \in \mathbb{B}^n),$$

for some kernel $\tilde{k}$ on $\mathbb{B}^n$. Suppose $\mathcal{H}_{\tilde{k}}$ is the reproducing kernel Hilbert space corresponding to the kernel $\tilde{k}$. If $\mathcal{E}$ is a Hilbert space, then there exists a co-isometric multiplier $\Psi_k \in \mathcal{M}(H^2_n \otimes (\mathcal{H}_{\tilde{k}} \otimes \mathcal{E}), \mathcal{H}_k \otimes \mathcal{E})$ such that

$$k(\boldsymbol{z},\boldsymbol{w})I_{\mathcal{E}} = \frac{\Psi_k(\boldsymbol{z})\Psi_k(\boldsymbol{w})^*}{1-\langle \boldsymbol{z},\boldsymbol{w}\rangle} \qquad (\boldsymbol{z},\boldsymbol{w} \in \mathbb{B}^n).$$

It is worth noting that except the explicit identification of the state space $\mathcal{H}_{\tilde{k}}$ and the fact that $\Psi_k \in \mathcal{M}(H^2_n \otimes (\mathcal{H}_{\tilde{k}} \otimes \mathcal{E}), \mathcal{H}_k \otimes \mathcal{E})$, Theorem 4.1 essentially follows from the Kolmogorov decomposition of a positive definite kernel.

It is instructive to consider, in particular, the familiar case: weighted Bergman spaces over $\mathbb{B}^n$. Let $m > 1$ be an integer and let

$$k_m(\boldsymbol{z},\boldsymbol{w}) = (1 - \sum_{i=1}^n z_i \bar{w}_i)^{-m} \qquad (\boldsymbol{z},\boldsymbol{w} \in \mathbb{B}^n).$$

Then

$$\tilde{k}_m(\boldsymbol{z},\boldsymbol{w}) = k_{m-1}(\boldsymbol{z},\boldsymbol{w}),$$

and hence $\Psi_{k_m}(\boldsymbol{w})^* : \mathcal{E} \to \mathcal{H}_{k_{m-1}} \otimes \mathcal{E}$ is given by

$$\Psi_{k_m}(\boldsymbol{w})^*\eta = k_{m-1}(\cdot,\boldsymbol{w}) \otimes \eta,$$

for all $\boldsymbol{z},\boldsymbol{w} \in \mathbb{B}^n$ and $\eta \in \mathcal{E}$. Note also that

$$\langle \Psi_{k_m}(\boldsymbol{w})(f \otimes \eta), \zeta\rangle = f(\boldsymbol{w})\langle \eta, \zeta\rangle,$$

for all $f \in \mathcal{H}_{k_{m-1}}$, $\eta, \zeta \in \mathcal{E}$ and $\boldsymbol{w} \in \mathbb{B}^n$.
For this particular case, the representation of Ψ_{k_m} has been computed explicitly in [8, Section 4] and [5].

Now suppose $\mathcal{E}_1$ and $\mathcal{E}_2$ are Hilbert spaces, and k is a regular kernel on $\mathbb{B}^n$. Let $\Theta : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ be an analytic function. From [16, Theorem 6.28] it follows that $\Theta \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ if and only if

$$k(\boldsymbol{z},\boldsymbol{w}) - k_1(\boldsymbol{z},\boldsymbol{w})\Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*$$

is a positive definite kernel. By virtue of Theorem 4.1, this is equivalent to positive definiteness of the kernel

$$(\boldsymbol{z},\boldsymbol{w}) \mapsto k_1(\boldsymbol{z},\boldsymbol{w})(\Psi_k(\boldsymbol{z})\Psi_k(\boldsymbol{w})^* - \Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*).$$

We may then apply [1, Theorem 8.57($(i) \Rightarrow (ii)$)] to obtain the following theorem.

Theorem 4.2. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces, and let $\Theta : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ be an analytic function. In the setting of Theorem* 4.1*, the following conditions are equivalent:*

(i) $\Theta \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$,

(ii) there exists $\tilde{\Theta} \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, H^2_n \otimes (\mathcal{H}_{\tilde{k}} \otimes \mathcal{E}_2))$ *such that*

$$M_\Theta = M_{\Psi_k} M_{\tilde{\Theta}}.$$

More specifically, the multiplier Ψ_k makes the following diagram commutative:

$$\begin{array}{ccc} & & H_n^2 \otimes (\mathcal{H}_{\tilde{k}} \otimes \mathcal{E}_2) \\ & \overset{M_{\tilde{\Theta}}}{\nearrow} & \Big\downarrow M_{\Psi_k} \\ H_n^2 \otimes \mathcal{E}_1 & \xrightarrow[M_\Theta]{} & \mathcal{H}_k \otimes \mathcal{E}_2 \end{array}$$

The above factorization theorem, in the scalar-valued multiplier case, is due to Aleman, Hartz, McCarthy and Richter (see Proposition 4.10 in [2]). The proof relies solely on Leech's theorem. One should also compare Theorems 4.1 and 4.2 with Lemma 4.1 and Theorem 4.2 in [8] and Theorem 2.1 in [5].

5. Nevanlinna-Pick interpolation

We now turn to the interpolation problem. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces. We denote by $\mathcal{B}_1(\mathcal{E}_1, \mathcal{E}_2)$ the open unit ball of $\mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$, that is

$$\mathcal{B}_1(\mathcal{E}_1, \mathcal{E}_2) = \{A \in \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2) : \|A\| < 1\}.$$

We aim to solve the following version of Pick-type interpolation problem: Suppose $\{\boldsymbol{z}_i\}_{i=1}^m \subseteq \mathbb{B}^n$, $\{W_i\}_{i=1}^m \subseteq \mathcal{B}_1(\mathcal{E}_1, \mathcal{E}_2)$ and $m \geq 1$. Find necessary and sufficient conditions (on $\{\boldsymbol{z}_i\}_{i=1}^m$ and $\{W_i\}_{i=1}^m$) for the existence of a multiplier $\Phi \in \mathcal{M}_1(H_n^2 \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ such that

$$\Phi(\boldsymbol{z}_i) = W_i, \tag{5.1}$$

for all $i = 1, \ldots, m$.

Given such data $\{\boldsymbol{z}_i\}_{i=1}^m \subseteq \mathbb{B}^n$, $\{W_i\}_{i=1}^m \subseteq \mathcal{B}_1(\mathcal{E}_1, \mathcal{E}_2)$, set

$$\mathcal{Q}_1 = \{\sum_{i=1}^m k_1(\cdot, \boldsymbol{z}_i)\zeta_i : \zeta_i \in \mathcal{E}_1\},$$

and

$$\mathcal{Q}_2 = \{\sum_{i=1}^m k(\cdot, \boldsymbol{z}_i)\eta_i : \eta_i \in \mathcal{E}_2\}.$$

Obviously $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are shift co-invariant subspaces of $H_n^2 \otimes \mathcal{E}_1$ and $\mathcal{H}_k \otimes \mathcal{E}_2$, respectively. Define $X : \mathcal{Q}_2 \to \mathcal{Q}_1$ by

$$Xk(\cdot, \boldsymbol{z}_i)\eta = k_1(\cdot, \boldsymbol{z}_i)(W_i^*\eta),$$

for all $i = 1, \ldots, m$ and $\eta \in \mathcal{E}_2$. Then

$$X(M_{z_i} \otimes I_{\mathcal{E}_2})^*|_{\mathcal{Q}_2} = (M_{z_i} \otimes I_{\mathcal{E}_1})^*|_{\mathcal{Q}_1} X,$$

for all $i = 1, \ldots, n$. Then, by Theorem 3.4, X is a contraction if and only if there exists $\Phi \in \mathcal{M}_1(H_n^2 \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ such that

$$P_{\mathcal{Q}_2} M_\Phi|_{\mathcal{Q}_1} = X^*.$$

In particular

$$\begin{aligned} k_1(\cdot, \boldsymbol{z}_i)(W_i^* \eta) &= X(k(\cdot, \boldsymbol{z}_i)\eta) \\ &= M_\Phi^*(k(\cdot, \boldsymbol{z}_i)\eta) \\ &= k_1(\cdot, \boldsymbol{z}_i)(\Phi(\boldsymbol{z}_i)^* \eta), \end{aligned}$$

for all $\eta \in \mathcal{E}_2$ and $i = 1, \ldots, m$, and so Φ satisfies (5.1). Conversely, if Φ satisfies (5.1), then it is easy to see that X defines a contractions from $\mathcal{Q}_2$ to $\mathcal{Q}_1$.

Now X is a contraction if and only if

$$\begin{aligned} &0 \leq \langle (I - X^* X) \sum_{i=1}^m k(\cdot, \boldsymbol{z}_i)\eta_i, \sum_{i=1}^m k(\cdot, \boldsymbol{z}_i)\eta_i \rangle \\ &\Rightarrow \sum_{1 \leq i,j \leq m} \langle k(\boldsymbol{z}_i, \boldsymbol{z}_j)\eta_j, \eta_i \rangle - \sum_{1 \leq i,j \leq m} \langle W_i k_1(\boldsymbol{z}_i, \boldsymbol{z}_j) W_j^* \eta_j, \eta_i \rangle \geq 0 \\ &\Rightarrow \sum_{1 \leq i,j \leq m} \langle \Big(k(\boldsymbol{z}_i, \boldsymbol{z}_j) I_{\mathcal{E}_2} - \frac{W_i W_j^*}{1 - \langle \boldsymbol{z}_i, \boldsymbol{z}_j \rangle} \Big) \eta_j, \eta_i \rangle \geq 0, \end{aligned}$$

for all $\eta_1, \ldots, \eta_m \in \mathcal{E}_2$, where the last equality follows from Theorem 4.1. On the other hand, Theorem 4.2 says that $\Phi \in \mathcal{M}_1(H_n^2 \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ if and only if there exists $\tilde{\Phi} \in \mathcal{M}_1(H_n^2 \otimes \mathcal{E}_1, H_n^2 \otimes (\mathcal{H}_{\tilde{k}} \otimes \mathcal{E}_2))$ such that

$$\Phi(\boldsymbol{z}) = \Psi_k(\boldsymbol{z})\tilde{\Phi}(\boldsymbol{z}),$$

for all $\boldsymbol{z} \in \mathbb{B}^n$. Summarizing, we have established the following interpolation theorem:

Theorem 5.1. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces, k be a regular kernel on $\mathbb{B}^n$, and let*

$$k(\boldsymbol{z}, \boldsymbol{w}) = k_1(\boldsymbol{z}, \boldsymbol{w})\tilde{k}(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n),$$

for some kernel $\tilde{k}$ on $\mathbb{B}^n$. Suppose $\{\boldsymbol{z}_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\{W_i\}_{i=1}^m \subseteq \mathcal{B}_1(\mathcal{E}_1, \mathcal{E}_2)$. Then the following conditions are equivalent:

(i) There exists a multiplier $\Phi \in \mathcal{M}_1(H_n^2 \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ such that $\Phi(\boldsymbol{z}_i) = W_i$ for all $i = 1, \ldots, m$.

(ii) $\displaystyle\sum_{1 \leq i,j \leq m} \langle \Big(k(\boldsymbol{z}_i, \boldsymbol{z}_j) I_{\mathcal{E}_2} - \frac{W_i W_j^*}{1 - \langle \boldsymbol{z}_i, \boldsymbol{z}_j \rangle} \Big) \eta_j, \eta_i \rangle$ *for all $\eta_1, \ldots, \eta_m \in \mathcal{E}_2$.*

(iii) There exists a multiplier $\tilde{\Phi} \in \mathcal{M}_1(H_n^2 \otimes \mathcal{E}_1, H_n^2 \otimes (\mathcal{H}_{\tilde{k}} \otimes \mathcal{E}_2))$ such that

$$\Psi_k(\boldsymbol{z}_i)\tilde{\Phi}(\boldsymbol{z}_i) = W_i \qquad (i = 1, \ldots, n).$$

As we pointed out before, in the case of scalar-valued multipliers (that is, $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{C}$), the equivalence of (i) and (ii) in Theorem 5.1 is due to Aleman, Hartz, McCarthy and Richter (see Proposition 4.4 in [2]). Moreover, if $n = 1$ and $\tilde{k}(z, w) = (1 - z\bar{w})^{-m}$, $m \in \mathbb{N}$ (that is, weighted Bergman space over $\mathbb{D}$ with an integer weight), then the equivalence of (i) and (ii) in Theorem 5.1 was proved by Ball and Bolotnikov [5].

Note that, the positivity condition in part (ii) of Theorem 5.1 does not hold in general:
Example: Consider the regular kernel k as the Bergman kernel on $\mathbb{D}$, that is

$$k(z,w) = \frac{1}{(1-z\bar{w})^2} \qquad (z,w \in \mathbb{D}).$$

Here

$$k(z,w) = \tilde{k}(z,w) = \Psi_k(z)\Psi_k^*(w) = \frac{1}{(1-z\bar{w})} \qquad (z,w \in \mathbb{D}).$$

Then, for a given pair of points $\{w_1, w_2\} \subseteq \mathbb{D}$, condition (ii) in Theorem 5.1 holds for some pair $\{z_1, z_2\} \subseteq \mathbb{D}$ if and only if

$$\begin{bmatrix} \frac{1}{1-|z_1|^2} - |w_1|^2 & \frac{1}{1-z_1\bar{z}_2} - w_1\bar{w}_2 \\ \frac{1}{1-z_2\bar{z}_1} - w_2\bar{w}_1 & \frac{1}{1-|z_2|^2} - |w_2|^2 \end{bmatrix} \diamond \begin{bmatrix} \frac{1}{1-|z_1|^2} & \frac{1}{1-z_1\bar{z}_2} \\ \frac{1}{1-z_2\bar{z}_1} & \frac{1}{1-|z_2|^2} \end{bmatrix} \geq 0,$$

where '$\diamond$' denotes the Schur product of matrices. However, if $z_1 = w_2 = 0$ and $z_2 \neq 0$, then it is easy to see that the positivity condition fails to hold for any $w_1 \in \mathbb{D}$ such that

$$\frac{1-|w_1|^2}{1-|z_2|^2} < 1.$$

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