TOPOLOGICAL VECTOR SPACES

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1. Topological Vector Spaces

Let X be a linear space over $\mathbb{R}$ or $\mathbb{C}$. We denote the scalar field by $\mathbb{K}$.

Definition 1.1. A topological vector space (tvs for short) is a linear space X (over $\mathbb{K}$) together with a topology $\mathcal{J}$ on X such that the maps $(x, y) \to x+y$ and $(\alpha, x) \to \alpha x$ are continuous from $X \times X \to X$ and $\mathbb{K} \times X \to X$ respectively, $\mathbb{K}$ having the usual Euclidean topology.

We will see examples as we go along.

Remark 1.2. Let X be a tvs. Then

(1) for fixed $x \in X$, the translation map $y \to x + y$ is a homeomorphism of X onto X and
(2) for fixed $\alpha \neq 0 \in \mathbb{K}$, the map $x \to \alpha x$ is a homeomorphism of X onto X.

Definition 1.3. A *base* for the topology at 0 is called a local base for the topology $\mathcal{J}$.

Theorem 1.4. *Let X be a tvs and let $\mathcal{F}$ be a local base at 0. Then*

(i) *$U, V \in \mathcal{F} \Rightarrow$ there exists $W \in \mathcal{F}$ such that $W \subseteq U \cap V$.*
(ii) *If $U \in \mathcal{F}$, there exists $V \in \mathcal{F}$ such that $V + V \subseteq U$.*
(iii) *If $U \in \mathcal{F}$, there exists $V \in \mathcal{F}$ such that $\alpha V \subseteq U$ for all $\alpha \in \mathbb{K}$ such that $|\alpha| \leq 1$.*
(iv) *Any $U \in \mathcal{J}$ is* absorbing, *i.e. if $x \in X$, there exists $\delta > 0$ such that $ax \in U$ for all a such that $|a| \leq \delta$.*

Conversely, let X be a linear space and let $\mathcal{F}$ be a non-empty family of subsets of X which satisfy (i)–(iv), define a topology $\mathcal{J}$ by :

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$A \subseteq X$ is open if, for each $x \in A$, there exists $U \in \mathcal{F}$ such that $x + U \subseteq A$. Then $\mathcal{J}$ is a topology which makes $(X, \mathcal{J})$ a tvs with $\mathcal{F}$ as a local base.

Proof. Exercise. $\square$

Some elementary facts concerning a tvs are summarized in

Lemma 1.5. *Let X be a tvs. Then*

- (i) *$\overline{x + A} = x + \overline{A}$.*
- (ii) *$\overline{A + B} \supseteq \overline{A} + \overline{B}$.*
- (iii) *If U is open and A is any subset then $A + U$ is open.*
- (iv) *C, D compact $\Rightarrow C + D$ compact.*
- (v) *If $A \subseteq X$, $\overline{A} = \cap\{(A + U) : U$ neighborhood of $0\}$.*
- (vi) *$\mathcal{J}$ is Hausdorff if and only if $\{0\}$ is closed if and only if $\{0\} = \cap\{U : U \in \mathcal{F}\}$, for any local base $\mathcal{F}$.*
- (vii) *If U is a neighborhood of 0, there exists a balanced neighborhood V of 0 such that $V \subseteq U$.*
- $(viii)$ *Closure of a convex set is convex; closure of a subspace is a subspace; closure of a balanced set is balanced.*
- (ix) *If C is compact, U neighborhood of C then there exists a neighborhood V of 0 such that $C + V \subseteq U$.*
- (x) *C compact, F closed $\Rightarrow C + F$ closed.*
- (xi) *If U is a balanced neighborhood of 0 then $int(U)$ is balanced.*
- (xii) *If U is any neighborhood of 0, U contains a closed balanced neighborhood of 0. In other words, the closed balanced neighborhoods form a local base (at 0).*
- $(xiii)$ *Every convex neighborhood of 0 contains a closed, balanced, convex neighborhood of 0.*

Proof. (x) Suppose $x \notin C + F$, i.e. $(x - F) \cap C = \emptyset$ or $C \subseteq (x - F)^c$. Now, $(x - F)^c$ is open and C compact, therefore by (ix), there exists a neighborhood V of 0 such that $C + V \subseteq (x - F)^c$, i.e. $(C + V) \cap (x - F) = \emptyset \Rightarrow x \notin C + F + V \Rightarrow x \notin \overline{C + F}$ by (v).

(xii) There exists a balanced neighborhood V of 0 such that $V + V \subseteq U$. Now $\overline{V}$ is also balanced and $\overline{V} \subseteq V + V \subseteq U$.

$(xiii)$ Let U be a convex neighborhood of 0 and define $V = \cap\{\alpha U : \alpha \in \mathbb{K}, |\alpha| = 1\}$. V is convex, balanced (easy to check!) and a neighborhood of 0 as

U contains a balanced neighborhood of 0 by (xii). Now, $\overline{\frac{1}{2}V} \subseteq \frac{1}{2}V + \frac{1}{2}V \subseteq V$, hence $\overline{\frac{1}{2}V}$ is a closed convex, balanced neighborhood of 0 contained in U. $\square$

The class of tvs mostly used in analysis is given by

Definition 1.6. A tvs X is called *locally convex* if there is a local base at 0 whose members are convex.

The topology of a lctvs is precisely that generated by a family of seminorms. Recall that

Definition 1.7. A function $p : X \to \mathbb{R}$ is sublinear if p is subadditive and positively homogeneous, i.e.,

(a) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$, and
(b) $p(\alpha x) = \alpha p(x)$ for all $x \in X$, and $\alpha \geq 0$

p is called a seminorm if for all $x, y \in X$ and $\alpha \in \mathbb{K}$,

(a) $p(x+y) \leq p(x) + p(y)$, and
(b) $p(\alpha x) = |\alpha| p(x)$.

If $p(x) = 0 \Rightarrow x = 0$, call p a norm.

We show now that a family of seminorms on a linear space generates a locally convex topology in the following sense :

Theorem 1.8. *Let $\{p_i : i \in I\}$ be a family of seminorms on the linear space X. Let $\mathcal{U}$ be the class of all finite intersections of sets of the form $\{x \in X : p_i(x) < \delta_i\}$ where $i \in I$, $\delta_i > 0$. Then $\mathcal{U}$ is a local base for a topology $\mathcal{J}$ that makes X a locally convex tvs. This topology is the weakest making all the p_i continuous, and for a net $\{x_\alpha\} \subseteq X$, $x_\alpha \to x$ in $\mathcal{J}$ if and only if $p_i(x_\alpha - x) \to 0$ for each $i \in I$.*

Proof. We check that all the conditions of Theorem 1.4 are satisfied.

(i). $\mathcal{U}$ is clearly closed under finite intersections.

(ii). Suppose $U = \{x : p_i(x) < \delta_i,\ i = 1, \dots, n\}$. Let $\delta = \min_i \delta_i/2$ and $V = \{x : p_i(x) < \delta,\ i = 1, \dots, n\}$. If $y, z \in V$ then $p_i(y+z) \leq p_i(y) + p_i(z) < \delta/2 + \delta/2 = \delta < \delta_i \Rightarrow V + V \subseteq U$.

(iii). Since $p_i(\alpha x) = |\alpha| p_i(x)$, each $U \in \mathcal{U}$ is balanced.

(iv). If $x \in X$, $p_i(\alpha x) = |\alpha| p_i(x) < \delta_i$ if α is sufficiently small.

$\mathcal{J}$ is a locally convex topology as each $U \in \mathcal{U}$ is convex.

Each $U = \{x : p_i(x) < \delta_i,\ i = 1, \ldots, n\}$ is open as if $x \in U$, $x + V \subseteq U$ where $V = \{z \in X : p_i(z) < \min_j[\delta_j - p_j(x)]\}$.

Now continuity of p_i is equivalent to continuity at 0 as $|p_i(x) - p_i(y)| \leq p_i(x - y)$ and the sets $U \in \mathcal{U}$ are open in any topology that makes each p_i continuous at 0. This proves that the given topology is the weakest, making the p_i's continuous.

Finally, $x_\alpha \to x \Leftrightarrow x_\alpha - x \to 0 \Leftrightarrow p_i(x_\alpha - x) \to 0$ for all i by the definition of $\mathcal{U}$. □

Remark 1.9. $\mathcal{J}$ is a Hausdorff topology if and only if the family $\{p_i : i \in I\}$ is separating, i.e., given $x \neq 0$, there exists p_i such that $p_i(x) \neq 0$.

Example 1.10. (a) Let X be the vector space of all $\mathbb{K}$-valued continuous functions on a topological space Ω. For each compact set $K \subseteq \Omega$, define $p_K(f) = \sup_{t \in K} |f(t)|$. The family $\{p_K : K \subseteq \Omega \text{ compact}\}$ gives a Hausdorff vector topology in which convergence means uniform convergence on all compact subsets of Ω. If K is restricted to finite subsets of Ω, we get the topology of pointwise convergence. In general, if the sets K are restricted to a class $\mathcal{C}$ of subsets of Ω, we obtain the topology of uniform convergence on sets in $\mathcal{C}$.

(b) Let $X = C^\infty[a, b]$, the vector space of all infinitely differentiable ($\mathbb{K}$-valued) functions on the closed bounded interval $[a, b]$. For each n, define $p_n(f) = \sup\{|f^{(n)}(t)| : t \in [a, b]\}$ where $f^{(n)}$ is the n-th derivative of f. In the topology defined by the p_n, convergence means uniform convergence of all derivatives.

We will now prove the converse of Theorem 1.8, that is, locally convex vector topologies are generated by families of seminorms. But we first examine convex sets in some detail.

Definition 1.11. A subset $K \subseteq X$ is said to be *radial at* x if and only if K contains a line segment through x in each direction, i.e. for every $y \in X$, there exists $\delta > 0$ such that $x + \lambda y \in K$ for all $\lambda \in [0, \delta]$ (sometimes x is called an *internal* point of K). If K is convex and radial at 0 (equivalently, K is absorbing), the *Minkowski functional of* K is defined as

$$p_K(x) = \inf\{r > 0 : x \in rK\}.$$

Intuitively, $p_K(x)$ is the factor by which x must be shrunk in order to reach the boundary of K.

Lemma 1.12. *Let K be convex and radial at 0.*

- (a) *p_K is sublinear.*
- (b) *$\{x \in X : p_K(x) < 1\} = \{x \in K : K$ is radial at $x\} \subseteq K \subseteq \{x : p_K(x) \leq 1\}$.*
- (c) *If K is balanced, p_K is a seminorm.*
- (d) *If X is a tvs and $0 \in K^\circ$, the interior of K, then p_K is continuous. $\overline{K} = \{p_K \leq 1\}$, $K^\circ = \{p_K < 1\}$, hence $\{p_K = 1\} = \partial K$, the boundary of K.*

Proof. (a). Let $\varepsilon > 0$ be given. There exists $r > 0$, $s > 0$ such that $r < p_K(x) + \varepsilon/2$, $s < p_K(y) + \varepsilon/2$ and $x/r, y/s \in K$. Now,

$$\frac{x+y}{r+s} = \frac{r}{r+s}\left(\frac{x}{r}\right) + \frac{s}{r+s}\left(\frac{y}{s}\right) \in K$$

by convexity, hence $p_K(x+y) \leq (r+s) < p_K(x) + p_K(y) + \varepsilon$.

(b). If $p_K(x) < 1$ then $x/r \in K$ for some $r < 1 \Rightarrow x = r(x/r) + (1-r)(0) \in K$. If $y \in X$ then $p_K(x + \lambda y) \leq p_K(x) + \lambda p_K(y) < 1$ if $\lambda > 0$ is sufficiently small, hence $\{p_K < 1\} \subseteq \{x \in K : K$ is radial at $x\}$. Conversely, if K is radial at x, then $x + \lambda x \in K$ for some $\lambda > 0$, hence $p_K(x + \lambda x) \leq 1$ by definition of $p_K \Rightarrow p_K(x) \leq \frac{1}{1+\lambda} < 1$. By definition of p_K, $K \subseteq \{p_K \leq 1\}$.

(c). If $x/r \in K$ and $a \neq 0 \in \mathbb{K}$, then $\frac{a}{|a|}\frac{x}{r} \in K$ (as K is balanced) $\Rightarrow p_K(ax) \leq |a|r$. Thus, $p_K(ax) \leq |a| p_K(x)$ (taking infimum over r).

Taking x/a instead of x, we get $p_K(x) \leq |a| p_K(x/a)$. Putting $b = 1/a$, we get $p_K(bx) \geq |b| p_K(x)$.

(d). $0 \in K^\circ \Rightarrow$ there exists neighborhood U of 0 with $U \subseteq K$. Let $\varepsilon > 0$ be given. If $y \in \varepsilon U$ then $p_K(y) = p_K(\varepsilon u) = \varepsilon p_K(u) \leq \varepsilon$ (since $x \in K \Rightarrow p_K(x) \leq 1$) $\Rightarrow p_K$ is continuous at $0 \Rightarrow p_K$ continuous everywhere.

p_K continuous $\Rightarrow \{p_K \leq 1\}$ is closed $\Rightarrow \overline{K} \subseteq \{p_K \leq 1\}$. Suppose $p_K(x) \leq 1$. If $0 < \lambda < 1$, then $p_K(\lambda x) = \lambda p_K(x) < 1 \Rightarrow \lambda x \in K$. If $\lambda \to 1$ then $\lambda x \to x$, hence $x \in \overline{K} \Rightarrow \overline{K} \supseteq \{p_K \leq 1\}$, hence $\overline{K} = \{p_K \leq 1\}$.

p_K continuous $\Rightarrow \{p_K < 1\}$ is open and hence $\subseteq K^\circ$. But if $p_K(x) = 1$ then $x_n = x/(1 - 1/n) \notin K$ as $p_K(x_n) = 1/(1-1/n) > 1$ and $K \subseteq \{p_K \leq 1\}$, but $x_n \to x$, so x is a limit of points not in K, hence $x \notin K^\circ$. $\square$

Theorem 1.13. *If X is a locally convex tvs, then its topology is generated by a family $\mathcal{P}$ of seminorms.*

Proof. If X has a local base consisting of convex sets, it has a local base $\mathcal{B}$ consisting of closed convex balanced neighborhoods of 0. For $U \in \mathcal{B}$, the Minkowski functional p_U is a seminorm. Since $U = \{p_u \leq 1\}$, the family $\{p_U : U \in \mathcal{B}\}$ generates the topolgy of X. □

Definition 1.14. A set E in a tvs Y is said to be *bounded* if, for every neighborhood U of 0 in Y, there exists $t \in \mathbb{R}^+$, such that $E \subseteq tU$.

Theorem 1.15. *Suppose X is locally convex, so its topology is generated by a family $\mathcal{P}$ of seminorms. Then $E \subseteq X$ is bounded if and only if each $p \in \mathcal{P}$ is bounded on E.*

Proof. Let $E \subseteq X$ be a bounded set. Let $p \in \mathcal{P}$. There exists $k > 0$ such that $E \subseteq k\{p \leq 1\} \Rightarrow p(E) \leq k$.

Conversely, suppose each $p \in \mathcal{P}$ is bounded on E. Let U be a neighborhood of 0. Then $U \supseteq \{p_1 \leq 1/n_1\} \cap \cdots \cap \{p_k \leq 1/n_k\}$ for some $p_1, \ldots, p_k \in \mathcal{P}$ and $n_1, \ldots, n_k \in \mathbb{N}$. There exists numbers $M_i > 0$ such that $p_i(E) < M_i$ $(i = 1, \ldots, k)$. Choose $M > M_i n_i$ $(1 \leq i \leq k)$. If $x \in E$ then $p_i(x/M) < M_i/M < 1/n_i \Rightarrow x/M \in U \Rightarrow x \in MU$. □

Example 1.16. Compact sets are bounded. In a Hausdorff tvs no subspace other than $\{0\}$ is bounded.

Remark 1.17. Suppose $\mathcal{P} = \{p_i : i = 1, 2, \ldots\}$ is a countable separating family of seminorms on a linear space X generating a vector topology $\mathcal{J}$. Then there exists a translation-invariant metric compatible with $\mathcal{J}$. Just define

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_i(x-y)}{1 + p_i(x-y)}.$$

The only trouble is that the balls $\{x : d(0, x) \leq r\}$ need not be convex (they are balanced though) as we see from the following example:

Example 1.18. Let $s = \{(x_n)_{n=1}^{\infty} : x_n \in \mathbb{K} \text{ for all } n \geq 1\}$, the space of all scalar sequences. The topology of pointwise convergence is described by the seminorms p_k, $(k \geq 1)$, $p_k((x_n)) = |x_k|$ and the metric is

$$d(\mathbf{x}, \mathbf{y}) = \sum \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \quad \mathbf{x} = (x_n), \quad \mathbf{y} = (y_n).$$

The ball $U = \{\mathbf{x} : d(0,\mathbf{x}) \leq \frac{1}{4}\}$ is not convex, since $(1,0,0,\ldots)$, $(0,1,0,0,\ldots) \in U$, but $\dfrac{3}{4}(1,0,0,\ldots)+\dfrac{1}{4}(0,1,0,\ldots) = (\dfrac{3}{4},\dfrac{1}{4},0,0,0,\ldots) \notin U$.

Can this be rescued? Indeed we have the following theorem whose proof is omitted (See Rudin, Functional Analysis, Theorem 1.24).

Theorem 1.19. *If $(X,\mathcal{J})$ is a (Hausdorff) tvs, with a countable local base, then there is a metric d on X such that*

- (a) *d is compatible with the topology $\mathcal{J}$,*
- (b) *the balls $\{x : d(0,x) \leq r\}$ are balanced*
- (c) *d is translation-invariant: $d(x+z,y+z) = d(x,y)$ for $x,y,z \in X$.*
  If, in addition, X is locally convex then d can be chosen so that
- (d) *all open balls $\{x : d(0,x) < r\}$ are convex.*

Remark 1.20. The notion of a Cauchy net for a tvs $(X,\mathcal{J})$ can be defined without reference to any metric. Fix a local base $\mathcal{F}$ at 0 for the topology $\mathcal{J}$. A net $\{x_\alpha\}$ is said to be ($\mathcal{J}$-) Cauchy if, for any $U \in \mathcal{F}$, there exists α_0 such that $\alpha \geq \alpha_0$ and $\beta \geq \alpha_0 \Rightarrow x_\alpha - x_\beta \in U$. It is clear that different local bases for $\mathcal{J}$ give rise to the same class of Cauchy nets. Now let $(X,\mathcal{J})$ be metrized by an invariant metric d. As d is invariant and the d-balls centered at 0 from a local base, we conclude that a sequence $\{x_n\} \subseteq X$ is a d-Cauchy sequence if and only if it is a $\mathcal{J}$-Cauchy sequence. Consequently any two invariant metrics on X that are compatible with the topology have the same Cauchy sequences and the same convergent sequences, viz. the $\mathcal{J}$-convergent ones.

By far the most widely discussed locally convex spaces are those for which the vector topologies are given by a single norm, the so-called normed spaces.

Theorem 1.21 (Kolmogoroff)**.** *A (Hausdorff) tvs is normable if and only if it has a bounded convex neighborhood of 0.*

Proof. Let X be normed by $\|\cdot\|$. Then the open unit ball $\{x : \|x\| < 1\}$ is a convex and bounded neighborhood of 0.

For the converse, let V be a bounded convex open neighborhood of 0. By Lemma 1.5 $(xiii)$, there exists neighborhood U of 0 such that $U \subseteq V$, U convex, open and balanced. Obviously, U is also bounded. For $x \in X$, define $\|x\| = p_U(x)$ where p_U is the Minkowski functional of U.

Claim: $\{\lambda U : \lambda > 0\}$ form a local base for the topology $\mathcal{J}$ of X.

Let $W \in \mathcal{J}$ be a neighborhood of 0. U bounded $\Rightarrow U \subseteq \lambda_0 W$ for some $\lambda_0 > 0 \Rightarrow \frac{1}{\lambda_0} U \subseteq W$.

Now, if $x \neq 0$ then $x \notin \lambda U$ for some $\lambda > 0 \Rightarrow \|x\| \geq \lambda \Rightarrow \|\cdot\|$ is actually a norm on X. As $U = \{p_U < 1\}$ is open, it's easy to see that $\{x : \|x\| < r\} = rU$ for all $r > 0$ and the norm topology coincides with the given one. $\square$

Exercise 1. Show that s, the space of all scalar sequences, is not normable where the topology on s is defined by the metric

$$d(0, \{x_n\}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n|}{1 + |x_n|}.$$

[**Hint:** Check that $U = \{\mathbf{x} : d(0, \mathbf{x}) < r\}$ is not bounded by verifying that $p_m(\mathbf{x}) = |x_m|$, $\mathbf{x} = (x_m)$, is not bounded on U.]

Exercise 2. Let X be the space of analytic functions on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ with the topology of uniform convergence on compact subsets. Show that X is metrizable but not normable.

[**Hint:** Let $p_n(f) = \sup\{|f(z)| : |z| \leq 1 - \frac{1}{n}\}$ and $U_n = \{f : p_n(f) \leq \frac{1}{n}\}$. Then U_n's form a local base for the topology, hence X is metrizable. To show that no U_n is bounded, fix N_0. Suppose there exists $\alpha_{n_1} < \infty$ such that $U_{N_0} \subseteq \alpha_{n_1} U_{n_1}$ for $n_1 > N_0$. Fix z_0 and z_{n_1} such that $1 - \frac{1}{N_0} < |z_0| < |z_{n_1}| \leq 1 - \frac{1}{n_1}$. The functions $f_k(z) = (\frac{z}{z_0})^k$ converge uniformly to 0 on $\{z : |z| \leq 1 - \frac{1}{N_0}\}$, hence $f_k/\alpha_{n_1} \in U_{n_1}$ for all $k \geq k_0$. But $\dfrac{f_k(z_{n_1})}{\alpha_{n_1}} = \dfrac{1}{\alpha_{n_1}} (\dfrac{z_{n_1}}{z_0})^k \to \infty$ as $k \to \infty$, contradiction].

2. Quotient Spaces

Let X be a tvs and $M \subseteq X$ a subspace. By definition, the quotient space X/M consists of cosets $x + M = [x]$ and the quotient map $\pi : X \to X/M$ is defined by $\pi(x) = x + M$. We define a quotient topology on X/M by stipulating that $U \subseteq X/M$ is open if and only if $\pi^{-1}(U)$ is open in X. With this definition, we have

Lemma 2.1. *X/M is a tvs, π is a continuous and open map, and X/M is Hausdorff if and only if M is closed.*

Proof. π is continuous by the definition of the quotient topology. Let V be open in X. Then $\pi(V)$ is open in $X/M \Leftrightarrow \pi^{-1}(\pi(V))$ is open in $X \Leftrightarrow V+M$ is open in X which is true, hence π is an open map.

Now if W is a neighborhood of 0 in X/M. Then there exists a neighborhood V of 0 in X such that $V+V \subseteq \pi^{-1}(W)$. Since π is an open map, $\pi(V)$ is a neighborhood of 0 in X/M and $\pi(V)+\pi(V) \subseteq W$. This proves the continuity of addition in X/M.

Check similarly that $\mathbb{K} \times X/M$ defined by $(\alpha, x+M) \to \alpha(x+M) = \alpha x+M$ is continuous. Hence the quotient topology is a vector topology.

Now X/M is Hausdorff if and only if $[0]$ is closed in X/M if and only if $\pi^{-1}[0] = M$ is closed in X. $\square$

Suppose that $(X, \mathcal{J})$ is a tvs which is metrized by an invariant metric d. Let $M \subseteq X$ be a subspace. If $x+M, y+M \in X/M$, define

$$\bar{d}([x],[y]) = d(x+M, y+M) = d(x-y, M)$$

which is the distance of $x-y$ to the subspace M. (The expression for $\bar{d}$ shows that it is well-defined). $\bar{d}$ is a psuedometric on X/M and it is a metric if M is closed. In the latter event, $\bar{d}$ is called the *quotient metric* on X/M.

If X is normed, this definition of $\bar{d}$ specializes to the *quotient norm* on X/M:

$$\|x+M\| = \bar{d}(0,[x]) = d(x, M) = \inf\{\|x+m\| : m \in M\}.$$

Theorem 2.2. *Let M be a closed subspace of a metrizable tvs X. With the quotient topology, X/M is a metrizable tvs. Indeed, if d is a translation-invariant metric which defines the topology on X, the quotient metric $\bar{d}$ induced by d induces the topology on X/M. If X is complete, so is X/M.*

Proof. Let $\pi : X \to X/M$ be the quotient map. It is easy to see that if $\mathcal{B}$ is a local base for X then $\{\pi(V) : V \in \mathcal{B}\}$ is a local base for X/M. Check that

$$\pi(\{x : d(0,x) < r\}) = \{[u] : \bar{d}(0,[u]) < r\}.$$

It follows that $\bar{d}$ is compatible with the quotient topology.

To show d complete $\Rightarrow \bar{d}$ complete, let $\{[u_n]\}$ be a $\bar{d}$-Cauchy sequence. We can choose a subsequence $\{[u_{n_i}]\}$ such that $\bar{d}([u_{n_i}],[u_{n_{i+1}}]) < 2^{-i}$.

Inductively choose $x_i \in X$ so that $d(x_i, x_{i+1}) < 2^{-i}$ and $\pi(x_i) = [u_{n_i}]$. Since d is complete, $x_i \to x \in X$ and as π is continuous, $\pi(x_i) = [u_{n_i}] \to \pi(x)$. Hence $\{[u_n]\}$, being Cauchy, must converge to $\pi(x)$. $\square$

Exercise 3. If M is a closed subspace of a metrizable tvs X and if both X/M and M are complete then X is complete.

3. Duals of tvs

Theorem 3.1. *Let X be a tvs and $f : X \to \mathbb{K}$ be a linear functional on X, $f \not\equiv 0$. Following are equivalent:*

- (i) *f is continuous*
- (ii) $\ker(f)$ *is closed*
- (iii) $\ker(f)$ *is not dense in X*
- (iv) *there exists a neighborhood of 0 on which f is bounded*
- (v) *the image under f of some non-empty open neighborhood of 0 is a proper subset of $\mathbb{K}$*

  If $\mathbb{K} = \mathbb{C}$, these are also equivalent to :
- (vi) *Ref is continuous.*

Proof. $(i) \Rightarrow (ii)$ and $(iv) \Rightarrow (v)$ trivial. $(ii) \Rightarrow (iii)$ clear as $f \not\equiv 0$.

$(iii) \Rightarrow (iv)$. Choose $x \in X$ and a balanced neighborhood U of 0 such that $(x + U) \cap \ker(f) = \emptyset \Rightarrow f(x) \notin -f(U)$ (which proves (v)). But $f(U)$ is balanced as U is balanced and hence $f(U)$ is bounded [as a proper balanced subset of $\mathbb{K}$ must be bounded] which proves (iv).

$(v) \Rightarrow (i)$. Assume that f maps a balanced neighborhood U of 0 onto a proper subset of $\mathbb{K}$. Hence $f(U)$ is bounded. We have a $k > 0$ such that $f(U) \subseteq \{z \in \mathbb{K} : |z| \leq k\}$. Let $\varepsilon > 0$. Then $f(\frac{\varepsilon}{k}U) \subseteq \{z \in \mathbb{K} : |z| \leq \varepsilon\} \Rightarrow f$ is continuous at 0.

$(vi) \Leftrightarrow (i)$ follows from the observation that $f(x) = Ref(x) - iRef(ix)$. $\square$

Remark 3.2. The last observation that $f(x) = Ref(x) - iRef(ix)$ is useful. We can consider a complex vector space X as a vector space over the $\mathbb{R}$ by restricting scalar multiplication to $\mathbb{R}$ and the space $X_{\mathbb{R}}$ thus obtained is called the *real restriction* of X. The above shows that $(X^*)_{\mathbb{R}}$ is (real) linearly isomorphic to $(X_{\mathbb{R}})^*$ under the map $f \to Ref$.

We want to know conditions which ensure that there exists non-zero continuous linear functionals on a tvs.

Theorem 3.3. *Let X be a tvs. The following are equivalent :*

- (i) *there exists non-zero continuous linear functional on X.*
- (ii) *there exists proper convex neighborhood of 0 in X.*
- (iii) *there exists a non-zero continuous seminorm on X.*

Proof. $(i) \Rightarrow (ii)$. If $f \neq 0$ and f is continuous then $U = \{x : |f(x)| < 1\}$ is a proper convex neighborhood of 0 in X.

$(ii) \Rightarrow (iii)$ If V is a convex neighborhood of 0 then V contains a convex balanced neighborhood U of 0. We know that p_U is continuous and that $U = \{p_u < 1\}$. As $U \neq X, p_u(x) \geq 1$ if $x \notin U$, hence $p_u \not\equiv 0$.

$(iii) \Rightarrow (i)$. This is the Hahn-Banach theorem which we will prove in Section 4. $\square$

Remark 3.4. This result makes it clear why lctvs are important viz. these have plenty *of non-zero* continuous linear functionals.

If X is a tvs then the continuous linear functionals on X form a linear space in the usual way, denoted by X^*, and is called the *dual* or *conjugate space* of X.

Exercise 4. Let X is a normed space and Y a Banach space then a linear operator $T : X \to Y$ is continuous if and only if $\sup_{\|x\|\leq 1} \|Tx\| < \infty$. If T is continuous, let $\|T\| = \sup_{\|x\|\leq 1} \|Tx\|$ and denote by $\mathcal{B}(X,Y)$ the space of all continuous linear operators from X to Y. Show that $\|\cdot\|$ is a norm on $B(X,Y)$ and that $B(X,Y)$ is complete in this norm, i.e. $B(X,Y)$ is a Banach space. Specializing to $Y = \mathbb{K}$, we get that X^* is always complete even though X may not be.

Example 3.5. For $1 \leq p < \infty$, $\ell^p = \{(x_n) : x_n \in \mathbb{K},\ \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ with norm $\|(x_n)\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$.

$\ell^\infty = \{(x_n) : x_n \in \mathbb{K},\ \sup_n |x_n| < \infty\}$ with norm $\|(x_n)\| = \sup_n |x_n|$.

$c = \{(x_n) \in \ell^\infty : \lim_n x_n \text{ exists}\}$.

$c_0 = \{(x_n) \in \ell^\infty : \lim_n x_n = 0\}$.

These are all Banach spaces and $(\ell^p)^* = \ell^q$ for $1 \leq p < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$, $c_0^* = \ell^1$, $c^* = \ell^1$. The continuous analogues of the ℓ^p-spaces are the $L^p(\mu)$

spaces where (Ω, Σ, μ) is a σ-finite measure space.

$$L^p(\mu) = \{f : \Omega \to \mathbb{K} : f \text{ measurable and } \int |f|^p d\mu < \infty\},$$

with norm $\|f\| = (\int |f|^p d\mu)^{1/p}, 1 \leq p < \infty$.

$$L^\infty(\mu) = \{f : \Omega \to \mathbb{K} : f \text{ measurable and essup}|f| < \infty\}$$

where essup $f = \inf\{M > 0 : |f| \leq M \ a.e. \ [\mu]\}$.

One knows that $(L^p)^* = L^q$ for $1 < p < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$. Also $(L^1)^* = L^\infty$ but $(L^\infty)^*$ is usually much larger than L^1. More specifically, $\Phi \in (L^p)^*$ if and only if there exists unique $g \in L^q$, unique upto sets of measure zero, such that $\Phi(f) = \int_\Omega f\bar{g} \ d\mu$, $f \in L^p$ and $\|\Phi\| = \|g\|_q$, so the correspondence between $(L^\rho)^*$ and L^q is given by $\Phi \leftrightarrow g$.

For a compact Hausdorff space X, $C(X)$ is a Banach space normed by $\|f\| = \sup_{x \in X} |f(x)|$ and $C(X)^* = M(X)$, the space of regular Borel measures on X normed by $\|\mu\| = \sup_{\|f\| \leq 1} |\mu(f)| = |\mu|(1)$, the total variation norm. More specifically, $\Phi \in C(X)^*$ if and only if there exists unique $\mu \in M(X)$ such that $\Phi(f) = \int f \ d\mu$, $(f \in C(X))$ and $\|\Phi\| = \|\mu\|$. If L is a positive linear functional, i.e. $L(f) \geq 0$ whenever $f \in C(X)$ and $f \geq 0$ then μ is a non-negative measure.

Exercise 5. When $0 < p < 1$, $L^p(d\mu)$ is defined in the usual way but because of the failure of Minkowski's inequality for such values of p, one does not get a norm. However, $d(f, g) = \int |f - g|^p d\mu$ does define an invariant metric on $L^p(d\mu)$. Show by using (ii) of Theorem 3.3, that $(L^p[0, 1])^* = \{0\}$, with μ as Lebesgue measure, in contrast to the case when $1 \leq p < \infty$.

Solution: Let $U_\varepsilon = \{f : |f|^p d\mu < \varepsilon\}$ is a basic neighborhood. Enough to show that $co(U_\varepsilon)$ is the whole space. Let $f \in L^p(d\mu)$ and let $\int |f|^p \ d\mu = M$. Since $x \rightsquigarrow \int_0^x |f(y)|^p d\mu(y)$ is a continuous function, there exists a point $x_0 \in [0, 1]$ such that $\int_0^{x_0} |f|^p \ d\mu = \frac{M}{2}$. Subdivide $[0, x_0]$ and $[x_0, 1]$ further in this way to obtain 2^n intervals I_n (n to be determined) in each of which $\int_{I_n} |f|^p \ d\mu = \frac{M}{2^n}$. Let $g_i = f\chi_{I_n}$. Then $f = \sum_{i=1}^{2^n} g_i$. Put $h_i = 2^n g_i$. Then

$$f = \sum_{i=1}^{2^n} \frac{1}{2^n} h_i \text{ and } \int |h_i|^p d\mu = \int_{I_i} 2^{np} |f_i|^p d\mu = \frac{M}{2^{n(1-p)}} < \varepsilon$$

if n is sufficiently large. Hence $h_i \in U_\varepsilon$ and $f \in co(U_\varepsilon)$.

4. The Hahn-Banach theorem

Theorem 4.1 (Hahn-Banach)**.** *Let X be a linear space (over $\mathbb{R}$) and p a sublinear map on X. Suppose $Y \subseteq X$ is a subspace and $f : Y \to \mathbb{R}$ a linear functional with $f \leq p$ on Y. Then there exists an extension of f to a linear functional $\widetilde{f}$ on X such that $\widetilde{f} \leq p$.*

We first prove the following

Lemma 4.2. *The theorem holds when Y has codimension one.*

Proof. Suppose $X = Y \oplus \mathbb{R}a$ where $a \in X \setminus Y$. For a fixed $k \in \mathbb{R}$, define $\widetilde{f}(y + \lambda a) = f(y) + \lambda k$, $y \in Y$. Notice that $\widetilde{f} \leq p \Longleftrightarrow$

$$\begin{aligned} f(y) + \lambda k &\leq p(y + \lambda a) \quad \forall \lambda \in \mathbb{R} \\ \Longleftrightarrow k &\leq p(u + a) - f(u) \quad \forall\, u \in Y \qquad (\lambda \geq 0) \\ \text{and } k &\geq f(v) - p(v - a) \quad \forall\, v \in Y \qquad (\lambda < 0). \end{aligned}$$

Thus $\widetilde{f} \leq p \Longleftrightarrow$

$$\sup\{f(v) - p(v - a) : v \in Y\} \leq k \leq \inf\{p(u + a) - f(u) : u \in Y\}$$

Now, if $u, v \in Y$,

$$f(u) + f(v) = f(u+v) \leq p(u+v) = p((u+a)+(v-a)) \leq p(u+a) + p(v-a),$$

hence $f(v) - p(v - a) \leq p(u + a) - f(u)$. Therefore

$$\sup\{f(v) - p(v - a) : v \in Y\} \leq \inf\{p(u + a) - f(u) : u \in Y\} \tag{1}$$

and it follows that k, as required, exists. □

Proof of Theorem 4.1. Let $\mathcal{G} = \{(V, g) : V \subseteq X$ a subspace, $Y \subseteq V$, $g \in V'$, $g|_Y = f$, $g \leq p$ on $V\}$ where V' is the *algebraic* dual of V.

Partially order $\mathcal{G}$ by: $(V_1, g_1) \prec (V_2, g_2)$ if and only if $V_1 \subseteq V_2$ and $g_2|_{V_1} = g_1$. It is clear that any chain in $\mathcal{G}$ has an upper bound, hence $\mathcal{G}$ has a maximal element $(Y', f') \succ (Y, f)$. If $Y' \neq X$, we could obtain an element larger than Y' (by one dimension) by Lemma 4.2. This completes the proof. □

Exercise 6. In the theorem, the extension $\widetilde{f}$ is unique if and only if (1) holds for all $a \in X \setminus Y$.

Exercise 7. If $Y = \{0\}$ then $\widetilde{f}$ is unique if and only if p is linear.

(Use (1) to get, in this case, $-p(-a) = p(a)$.)

Theorem 4.3 (Complex form of the Hahn-Banach theorem)**.** *Let X be a vector space over $\mathbb{C}$, p a* seminorm *on X, $Y \subseteq X$ a subspace and $f : Y \to \mathbb{C}$ a (complex) linear functional such that $|f(x)| \leq p(x)$. Then there exists an extension $\widetilde{f} : X \to \mathbb{C}$ also $\mathbb{C}$-linear and $|\widetilde{f}| \leq p$.*

Proof. Let $g = Ref$, then $g : Y \to \mathbb{R}$ is $\mathbb{R}$-linear and $g \leq p$. By Theorem 4.1, there exists $\mathbb{R}$-linear extension $\widetilde{g} : X \to \mathbb{R}$ with $\widetilde{g} \leq p$. Define $\widetilde{f} : X \to \mathbb{C}$ by $\widetilde{f}(x) = \widetilde{g}(x) - i\widetilde{g}(ix)$. It is clear that $\widetilde{f}$ is $\mathbb{C}$-linear and that $\widetilde{f}|_Y = f$. Now $\widetilde{g} \leq p \Rightarrow |\widetilde{g}| \leq p$. Also, $|\widetilde{f}| \leq p$, since if $\widetilde{f}(x) = re^{i\theta}$, $r \geq 0$, then $\widetilde{f}(e^{-i\theta}x) = r \in \mathbb{R}$ and hence,

$$r = |\widetilde{f}(x)| = \widetilde{f}(e^{-i\theta}x) = \widetilde{g}(e^{-i\theta}x) \leq p(e^{-i\theta}x) = p(x).$$

□

5. Consequences

Corollary 5.1. *Let X be a linear space and let $A \subseteq X$ be a convex balanced set which is radial at the origin. Let f be a linear functional on a subspace $M \subseteq X$ such that $|f(y)| \leq 1$ for all $y \in M \cap A$. Then there exists a linear functional g on X such that*

$$g|_M = f, \qquad |g(x)| \leq 1 \ \forall \ x \in A.$$

Proof. Let $y \in M$, choose $r > 0$ such that $y/r \in A$ (radiality of A) $\Rightarrow$ $|f(y/r)| \leq 1 \Rightarrow |f(y)| \leq r \Rightarrow |f(y)| \leq p(y)$ where $p = p_A$ is the Minkowski functional of A. p as we know is a seminorm. By HB Theorem, there exists a linear functional g on X extending f and $|g(x)| \leq p(x)$ for all $x \in X$. If $x \in A$, $p(x) \leq 1$ and it follows that $|g| \leq 1$ on A. □

Corollary 5.2. *If $(X, \|\cdot\|)$ is a normed linear space and $x_0 \in X$, there exists a linear functional g on X such that $\|g\| = 1$ and $g(x_0) = \|x_0\|$.*

Proof. Let $y = x_0/\|x_0\|$ and define f on $\mathbb{K}y$ by $f(\alpha y) = \alpha$. Now, $|f(\alpha y)| = |\alpha| \leq \|\alpha y\|$ for all $\alpha \in \mathbb{K}$, so by HB Theorem, there exists g on X such that $g|_{\mathbb{K}y} = f$ ($\Rightarrow g(x_0) = \|x_0\|$) and $|g(x)| \leq \|x\|$ for all x ($\Rightarrow \|g\| = 1$). □

Remark 5.3. It follows that $\|x\| = \sup\{|g(x)| : g \in X^*, \|g\| \leq 1\}$.

Theorem 5.4. *Let M be a subspace of a lctvs X. Then any continuous linear functional on M can be extended to a continuous linear functional on X (when X is a normed linear space this can be accomplished in a* norm-preserving *way).*

Proof. Let $f \in M^*$. Then there exists a neighborhood V of 0 in X such that $|f| \leq 1$ on $V \cap M$. Choose a convex balanced neighborhood U of 0 in X such that $U \subseteq V$ and we have $|f| \leq 1$ on $U \cap M$. By Corollary 5.1 above, there exists linear functional g on X such that $g|_M = f$, $|g| \leq 1$ on $U \Rightarrow g \in X^*$. □

Corollary 5.5. *If M is a closed subspace of a lctvs X and if $x \notin M$, there exists $f \in X^*$ such that $f|_M = 0$ and $f(x) \neq 0$.*

Proof. X/M is a lctvs and $x + M \neq [0]$. Find f on X/M by HB Theorem, such that $f \in (X/M)^*$ and $f(x+M) \neq 0$. Define $g = f \circ \pi \in X^*$. Hence $g(x) = f(x+M) \neq 0$. □

Corollary 5.6. *If X is locally convex and $x, y \in X$, $x \neq y$ then there exists $f \in X^*$ such that $f(x) \neq f(y)$, i.e. X^* separates points of X.*

Theorem 5.7 (The Separation Theorems)**.** *Suppose A, B are disjoint, non-empty convex sets in a tvs X.*

- (a) *If A is open, there exists $f \in X^*$ and $\lambda \in \mathbb{R}$ such that Re $f(x) < \lambda \leq$ Re $f(y)$ for all $x \in A$, $y \in B$.*
- (b) *If A is compact, B closed and X is locally convex, then there exists $f \in X^*$, $\lambda_1, \lambda_2 \in \mathbb{R}$ such that Re $f(x) < \lambda_1 < \lambda_2 <$ Re $f(y)$ for all $x \in A$, $y \in B$.*

Remark 5.8. It is enough to prove these results for tvs over $\mathbb{R}$.

Proof. (a). Fix $a_0 \in A$, $b_0 \in B$ and let $x_0 = b_0 - a_0$. Put $C = A - B + x_0$. Then C is a convex neighborhood of 0. Let $p = p_C$ be the Minkowski functional of C. Then p is a sublinear functional. Now, $A \cap B = \emptyset \Rightarrow x_0 \notin C \Rightarrow p(x_0) \geq 1$. Define $f(tx_0) = t$ on the subspace $M = \mathbb{R}x_0$.

Now, if $t \geq 0$, $f(tx_0) = t \leq tp(x_0) = p(tx_0)$ and if $t < 0$, $f(tx_0) = t < 0 \leq p(tx_0) \Rightarrow f \leq p$ on M.

By HB Theorem, extend f to a linear functional Λ on X so that $\Lambda \leq p$. In particular, $\Lambda \leq 1$ on $C \Rightarrow \Lambda(-x) \geq -1$ on $-C \Rightarrow |\Lambda| \leq 1$ on $C \cap -C$ which is neighborhood of $0 \in X \Rightarrow \Lambda \in X^*$.

If $a \in A$, $b \in B$ then $\Lambda(a)-\Lambda(b)+1 = \Lambda(a-b+x_0) \leq p(a-b+x_0) < 1$ (since C is open and $a-b+x_0 \in C$) $\Rightarrow \Lambda(a) < \Lambda(b)$. It follows that $\Lambda(A)$ and $\Lambda(B)$ are disjoint convex sets in $\mathbb{R}$ with $\Lambda(A)$ to the left of $\Lambda(B)$. Also $\Lambda(A)$ is open since A is open and every non-constant continuous linear functional on X is an open map. We get the result by letting λ be the right end point of $\Lambda(A)$.

(b). A compact. B closed $\Rightarrow B - A$ closed $\Rightarrow$ there exists convex neighborhood U of 0 such that $U \cap (B - A) = \emptyset$ (as $A \cap B = \emptyset$, $0 \notin B - A$) $\Rightarrow (U + A) \cap B = \emptyset$. By (a), there exists $\Lambda \in X^*$ such that $\Lambda(A + U)$ and $\Lambda(B)$ are disjoint convex subsets of $\mathbb{R}$ with $\Lambda(A+U)$ open and to the left of $\Lambda(B)$. Since $\Lambda(A)$ is a compact subset of $\Lambda(A + U)$, we get the result. ☐

Exercise 8. A closed convex set in a lctvs is the intersection of all the closed half spaces containing it.

6. Complete metrizable tvs

Definition 6.1. A set A in a linear space X is said to be *radial* at the origin if, for each $x \in X$, there exists $r_x > 0$ such that $0 \leq r \leq r_x \Rightarrow rx \in A$. Some authors call 0 a *core point* or *internal point* of A.

The basic result here is the following

Proposition 6.2. *Let X be a complete metrizable tvs. Let A be a balanced, closed set radial at 0. Then $A+A$ is a neighborhood of 0. So, if A is convex then A is a neighborhood of 0.*

Proof. A radial at 0 and A balanced $\Rightarrow X = \bigcup_{n=1}^{\infty} nA$.

A closed $\Rightarrow nA$ closed. By Baire Category Theorem, some nA has non-empty interior $\Rightarrow A$ has non-empty interior $\Rightarrow A + A$ ($= A - A$ as A is balanced) is a neighborhood of 0.

A convex $\Rightarrow \frac{1}{2}A + \frac{1}{2}A \subseteq A$. By applying the above argument to $\frac{1}{2}A$, we see that A is a neighborhood of 0 if A is convex. ☐

Theorem 6.3 (Equicontinuity Principle)**.** *Let X be a complete metrizable tvs and Y a tvs. Suppose $\{T_i\}_{i\in I}$ is a family of continuous linear transformations from X into Y and suppose that, for each $x \in X$, the set $\{T_i x : i \in I\}$ is a bounded set in Y. Then $\lim_{x\to 0} T_i x = 0$ uniformly for $i \in I$.*

Proof. Let V be a neighborhood of 0 in Y. It suffices to show that $\bigcap_{i\in I} T_i^{-1}(V)$ is a neighborhood of 0 in X. Choose a closed balanced neighborhood U of 0 in Y such that $U + U \subseteq V$. Let $A = \bigcap_{i\in I} T_i^{-1}(U)$. Then A is balanced (as T_i is linear for each i), closed (as each T_i is continuous and U is closed) and radial at 0 because if $x \in X$, there exists $r_x > 0$ such that $r_x(T_i x) \in U$ (for all i) by the boundedness of $\{T_i x : i \in I\}$ in Y. So by last Proposition, $A + A$ is a neighborhood of 0 and clearly $A + A \subseteq \bigcap_{i\in I} T_i^{-1}(V)$. □

Remark 6.4. The above result is often called the *Principle of Uniform Boundedness* when X, Y are, respectively, Banach and normed linear spaces. Specifically, in this context, the theorem reads:

If $\{T_i\}_{i\in I}$ is a family of continuous linear transformations from X to Y such that for each $x \in X$, $\sup\{\|T_i x\|_Y : i \in I\} < \infty$ then there exists $M > 0$ such that $\|T_i x\|_Y \le M\|x\|_X$ (for all i and $x \in X$), hence $\sup_i \|T_i\| < \infty$.

Theorem 6.5 (Open Mapping Theorem)**.** *Let X, Y be two complete metrizable TVS and let T be a linear continuous transformation from X onto Y. Then T is an open mapping.*

Proof. It suffices to prove that if V is a neighborhood of 0 in X then $T(V)$ is a neighborhood of 0 in Y. Let d be a invariant metric compatible with the topology of X. Define $V_n = \{x : d(0, x) < r \cdot 2^{-n}\}$, $(n = 0, 1, 2, \ldots)$ where $r > 0$ is so small that $V_0 \subseteq V$. We will prove that some neighborhood W of 0 in Y satisfies $W \subseteq \overline{T(V_1)} \subseteq T(V)$. Since $V_1 \supseteq V_2 + V_2$ and T is continuous, we get

$$\overline{T(V_1)} \supseteq \overline{T(V_2) + T(V_2)} \supseteq \overline{T(V_2)} + \overline{T(V_2)}.$$

But $\overline{T(V_2)}$ is closed, balanced and radial at 0 (the latter because V_2 is radial at 0 and T is onto), we have by our first proposition that $\overline{T(V_2)} + \overline{T(V_2)}$ contains a neighborhood W of 0 in Y.

Now to prove that $\overline{T(V_1)} \subseteq T(V)$. Fix $y_1 \in \overline{T(V_1)}$. Assume $n \ge 1$ and y_n has been chosen in $\overline{T(V_n)}$. By what was just proved, $\overline{T(V_{n+1})}$ contains a neighborhood of 0 in Y. Hence $[y_n - \overline{T(V_{n+1})}] \cap T(V_n) \neq \emptyset \Rightarrow$ there exists

$x_n \in V_n$ such that $Tx_n \in y_n - \overline{T(V_{n+1})}$. Put $y_{n+1} = y_n - Tx_n \in \overline{T(V_{n+1})}$ and we can continue the construction. Since $d(0, x_n) < \frac{r}{2^n}$, it is easy to see that the sums $x_1 + \ldots + x_n$ form a Cauchy sequence and hence $\sum x_n = x$ exists with $d(0, x) < r \Rightarrow x \in V_0 \subseteq V$. Moreover,

$$\sum_{n=1}^{m} Tx_n = \sum_{n=1}^{m} (y_n - y_{n+1}) = y_1 - y_{m+1}$$

and as $y_{m+1} \to 0$ as $m \to \infty$ (this is because by the continuity of T, $\{\overline{T(V_n)}\}_{n=1}^{\infty}$ forms a local base at 0 in Y), we conclude that $y_1 = Tx \in T(V)$. This completes the proof. □

Corollary 6.6. *Any continuous one-to-one linear map of one complete metrizable tvs onto another is a homeomorphism.*

Exercise 9. *Let X be a linear space which is complete under two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ and suppose there exists $K > 0$ such that $\|x\|_1 \le K\|x\|_2$ for all $x \in X$. Then there exists $L > 0$ such that $L\|x\|_2 \le \|x\|_1 \le K\|x\|_2$ for all $x \in X$, i.e. $\|\cdot\|_1$ and $\|\cdot\|_1$ are equivalent.*

Theorem 6.7 (Closed Graph Theorem)**.** *Let X, Y be complete metrizable tvs and let T be a linear transformation from X to Y. Then T is continuous if and only if the graph $G_T = \{(x, Tx) : x \in X\} \subseteq X \times Y$ is closed in $X \times Y$.*

Proof. Clearly, T continuous $\Rightarrow G_T$ closed.

Suppose now that G_T is closed. Consider $X \times Y$ with the product metric:

$$d((x, y), (x', y')) = d_1(x, x') + d_2(y, y')$$

where d_1, d_2 are the metrics inducing the topologies of X, Y respectively. G_T is a closed subspace of $X \times Y$. Let $P : G_T \to X$ be the projective map, i.e. $P\{(x, Tx)\} = x$. P is continuous linear, 1-1 and onto X. Hence by the above Corollary, P is a homeomorphism, i.e. P^{-1} is continuous which says: $x_n \to x \Rightarrow Tx_n \to Tx$, i.e. T is continuous. □

Exercise 10. Suppose Y is complete metrizable tvs such that the continuous functionals on Y separate points of Y, i.e. given $y_1 \ne y_2$ there exists $f \in Y^*$ such that $f(y_1) \ne f(y_2)$. If X is a complete tvs then $T : X \to Y$, (T assume to be linear) is continuous if and only if $f \circ T$ is continuous for all $f \in Y^*$.

Exercise 11. Prove that $\nexists$ a sequence $\{\lambda_n\} \subseteq \mathbb{C}$ such that $\sum a_n$ converges absolutely if and only if $\{\lambda_n a_n\}$ is bounded.

[**Hint:** Assume (λ_n) exists with $\lambda_n \neq 0$ for all n. Define $T : \ell^\infty \to \ell^1$ by $T[(c_n)] = \{c_n/\lambda_n\}$. Then $\|T\| \leq \sum |\frac{1}{\lambda_n}|$ and $\sum |\frac{1}{\lambda_n}| < \infty$ by the hypothesis. Hence T is continuous, linear, one-one, onto $\Rightarrow$ ℓ^∞ and ℓ^1 are homeomorphic which is impossible as ℓ^∞ is non-separable and ℓ^1 is separable in their respective norm topologies.]

7. Weak and Weak* Topologies

Lemma 7.1. *Let X be a linear space. Let $\mathcal{F}$ be a family of linear functionals on X which is* total, *i.e. $f(x) = 0$ for all $f \in \mathcal{F} \Rightarrow x = 0$ (equivalently: $\mathcal{F}$ is separating). Let $\mathcal{J}$ be the weakest topology on X relative to which every $f \in \mathcal{F}$ is continuous. Then $(X, \mathcal{J})$ is a lctvs and every continuous linear functional on $(X, \mathcal{J})$ is a linear combination of functionals in $\mathcal{F}$.*

Proof. A topology on X will make each $f \in \mathcal{F}$ continuous if and only if every set $U_{f,x,\varepsilon} = \{y \in X : |f(y) - f(x)| < \varepsilon\}$ is open. The collection of all such sets is translation-invariant and $\mathcal{J}$ is a translation invariant topology. Thus it is sufficient to look at the topology at the origin where a local base looks like $\{x : |f_j(x)| < \varepsilon_j,\ j = 1, 2, \ldots, n\}$. Such sets are convex, balanced and absorbing. Therefore, by Theorem 1.4, $(X, \mathcal{J})$ is locally convex.

Suppose f is a linear functional on X which is $\mathcal{J}$-continuous. Then there exists a $\mathcal{J}$-neighborhood of 0 on which f is bounded, i.e. there exists $f_1, \ldots, f_n \in \mathcal{F}$, $\varepsilon > 0$ and $M > 0$ such that

$$|f_j(x)| < \varepsilon \quad \forall j = 1, \ldots, n \Rightarrow |f(x)| \leq M.$$

Therefore, there exists $c > 0$ such that $|f(x)| \leq c \max_{1 \leq j \leq n} |f_j(x)|$ for any $x \in X$. In particular, $\ker(f) \supseteq \bigcap_{j=1}^n \ker(f_j) \Rightarrow f = \sum_{k=1}^n \alpha_k f_k$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$. □

Exercise 12. Prove the last statement in the above proof.

Remark 7.2. If $\mathcal{F}$ is a linear subspace (of the algebraic dual of X), it follows from the lemma that the (continuous) dual of $(X, \mathcal{J})$ is $\mathcal{F}$ itself.

Definition 7.3. Suppose that $(X, \mathcal{J})$ a lctvs. We know that X^* separates the points of X. Taking $\mathcal{F} = X^*$, the lc topology $\mathcal{J}_w$ that we get in the above lemma, is called the *weak topology* of X. Note that $\mathcal{J}_w \subseteq \mathcal{J}$. Moreover, a typical neighborhood of 0 in $\mathcal{J}_w$ looks like $\{x \in X : |f_k(x)| < \varepsilon,\ k = 1, \ldots, n\}$, $f_k \in X^*$.

Exercise 13. Let X be a lctvs and $E \subseteq X$ is a convex set. Then the closure of E in the weak and the original topology are the same.

Definition 7.4. Suppose that X is a lctvs For each $x \in X$, define the linear functional $\hat{x}$ on X^* by $\hat{x}(x^*) = x^*(x)$. Note that $\{\hat{x} : x \in X\}$ is a linear space and that it separates the points of X^*. We are now in the situation described by the lemma with X replaced by X^* and $\mathcal{F}$ replaced by $\{\hat{x} : x \in X\}$. The corresponding topology is called the *weak* topology* on X^*. A basic w*-neighborhood of 0 is then $\{x^* : |x^*(x_k)| < \varepsilon, x_k \in X, k = 1, \ldots, n\}$.

The most important result concerning w*-topology is the following

Theorem 7.5 (Banach-Alaoglu)**.** *Let X be a lctvs and suppose that U is a neighborhood of 0 in X. Then $U^\circ = \{f \in X^* : |f(x)| \leq 1$ for all $x \in U\}$ is compact in the w*-topology on X^*.*

Proof. First observe that if $U^{\circ\circ}$ is defined by $\{x \in X : |f(x)| \leq 1$ for all $f \in U^\circ\}$ then $U^{\circ\circ} = \overline{co}(U)$ and that $U^\circ = (U^{\circ\circ})^\circ$ (both are simple consequences of the 2nd form of the separation theorem). Since $U^{\circ\circ}$ is convex, balanced and closed, we may without loss of generality assume that the neighborhood U is closed, convex and balanced to start with.

Let $p \equiv p_U$ be the associated Minkowski functional of U. Now, $U = \{x : p(x) \leq 1\}$, hence $U^\circ = \{f \in X^* : |f(x)| \leq p(x)$ for all $x \in X\}$. p being a continuous seminorm, any linear functional g on X such that $|g| \leq p$ on X is automatically continuous on X.

For each $x \in X$, let $D_x = \{\alpha \in \mathbb{C} : |\alpha| \leq p(x)\}$. Look at $\Pi = \prod_{x \in X} D_x$. Π is compact (Tychonoff). Identifying f with $(f(x))_{x \in X}$, we see that $U^\circ \subseteq \Pi$. Now Π is the set of all functions $F : X \to \mathbb{C}$ such that $|F(x)| \leq p(x)$ for all x and $U^\circ = \{F \in \Pi : F$ linear $\}$. The topology on Π is the product topology, i.e. the weakest topology defined by the coordinate projections. One projection for each $x \in X : \hat{x}(F) = F(x)$. Therefore, the weak* topology on U° is its relative topology in Π. But U° is closed in Π:

$$U^\circ = \{F \in \Pi : F \text{ linear}\} = \bigcap_{\substack{\alpha \in \mathbb{C}, \beta \in \mathbb{C} \\ x, y \in X}} \ker[(\widehat{\alpha x + \beta y}) - \alpha \widehat{x} - \beta \widehat{y}].$$

$\square$

Corollary 7.6. *Let X be a normed linear space. Then the unit ball of X^* is weak* compact.*

Exercise 14. If X is a separable normed linear space. Show that the w*-topology of the unit ball of X^* is metrizable.

8. Extremal points of compact convex sets

Definition 8.1. Let C be a convex set in a linear space X.

(a) A point $x \in C$ is called an extreme point of C if x is not an interior point of any line segment in C. That is, if $y, z \in C$, $\lambda \in (0,1)$ and $x = \lambda y + (1-\lambda)z$, then $x = y = z$.

We will denote by $\text{ext}(C)$ the set of extreme points of C.

(b) A convex set $F \subseteq C$ is called a face of C if no point of F is an internal point of a line segment whose endpoints are in C but not in F. That is, $x, y \in C$, $\lambda \in (0,1)$ and $\lambda x + (1-\lambda)y \in F \Rightarrow x, y \in F$.

Thus, $x \in \text{ext}(C)$ if and only if $\{x\}$ is a singleton face of C.

If F has the above property, but is not convex, we call it an *extremal set.*

Example 8.2. Let K be a compact Hausdorff space and let P be the convex set of *regular* probability measures on K, so $P \subseteq X$ where $X = C(K)^*$.

(i) The discrete measures form a *face* of P (a measure μ in P is *discrete* if $\mu = \sum_{n=1}^{\infty} \alpha_n \delta_{x_n}$, $\sum_n \alpha_n = 1$, $\alpha_n \geq 0$, $x_n \in K$).

(ii) The continuous measures (i.e., measures without any point masses) from a face of P.

(iii) If $m \in P$, then $\{\mu \in P : \mu \ll m\}$ is a face of P.

(iv) If $m \in P$, then $\{\mu \in P : \mu \perp m\}$ is a face of P.

(v) The extreme points of P are the point masses $\{\delta_x : x \in K\}$ and conversely.

Proof. Any δ_x is clearly extreme. If $\mu \in P$ and if the (topological) support $S(\mu)$ of μ has two points x_1, x_2 find an open U such that $x_1 \in U$, $x_2 \notin U$. Now $0 < \mu[U \cap S(\mu)] < 1$. Let $\mu_1 = \dfrac{\mu|_{U \cap S(\mu)}}{\mu(U \cap S(\mu))}$, $\mu_2 = \dfrac{\mu|_{S(\mu) \setminus U}}{\mu(S(\mu) \setminus U)}$, then $\mu_1, \mu_2 \in P$, $\mu_1 \neq m_2$ and

$$\mu = \mu(U \cap S(\mu))\mu_1 + [1 - \mu(U \cap S(\mu))]\mu_2.$$

$\square$

(vi) Using (v), show that the extreme points of the unit ball of $C(K)^* = M(K)$ are exactly $\{\lambda\delta_x : \lambda \in \mathbb{C}, |\lambda| = 1, x \in K\} = B$.

(vii) Consider a subspace A of $C(K)$. Show that the extreme points of the unit ball of A^* are contained in B and find an example to show that all points in B may not be extreme.

(The proof of (vii) follows easily from the Krein-Milman theorem)

Theorem 8.3 (Krein-Milman Theorem)**.** *Suppose X is a topological vector space such that X^* separates points of X. If K is a compact convex set in X, then K is the closed convex hull of its extreme points, i.e.,*

$$K = \overline{co}(\text{ext}(K)).$$

Proof. Step I : If K is a compact convex set in X, then $\text{ext}(K) \neq \emptyset$.

Let $\mathcal{P}$ be the collection of all closed faces of K. Clearly, $K \in \mathcal{P}$.

Claim : If $S \in \mathcal{P}$, $f \in X^*$ and

$$S_f = \{x \in S : f(x) = \sup f(S)\},$$

then $S_f \in \mathcal{P}$.

To prove the claim, suppose $z = \lambda x + (1 - \lambda)y \in S_f$, $x, y \in K$, $\lambda \in (0, 1)$. Since $z \in S$ and $S \in \mathcal{P}$, we have $x, y \in S$. Hence $f(x) \leq \sup f(S)$, $f(y) \leq \sup f(S)$. Since $f(z) = \lambda f(x) + (1 - \lambda)f(y) = \sup f(S)$, we conclude $f(x) = f(y) = \sup f(S)$, *i.e.*, $x, y \in S_f$.

Order $\mathcal{P}$ by reverse inclusion, *i.e.*, say $K_1 \leq K_2$ if $K_2 \subseteq K_1$. Let $\mathcal{C}$ be chain in $\mathcal{P}$. Since $\mathcal{C}$ is a collection of compact sets having finite intersection property, the intersection M of all members of $\mathcal{C}$ is nonempty. It is easy to see that $M \in \mathcal{P}$. Thus, every chain in $\mathcal{P}$ has an upper bound. By Zorn's Lemma, $\mathcal{P}$ has maximal element, M_0. The maximality implies that no proper subset of M_0 belongs to $\mathcal{P}$. It now follows from the claim that every $f \in X^*$ is constant on M_0. Since X^* separates points of X, M_0 must be a singleton. Therefore M_0 is an extreme point of K.

Step II : If K is a compact convex set in X, then $K = \overline{co}(\text{ext}(K))$.

Let $H = \overline{co}(\text{ext}(K))$. If possible, let $x_0 \in K \setminus H$. By Separation Theorem, there is an $f \in X^*$ such that $f(x) < f(x_0)$ for every $x \in H$. If $K_f = \{x \in K : f(x) = \sup f(K)\}$, then, by the claim, $K_f \in \mathcal{P}$ and $K_f \cap H = \emptyset$. By Step I, $\text{ext}(K_f) \neq \emptyset$. But since K_f is a face of K, $\text{ext}(K_f) \subseteq \text{ext}(K)$. This contradiction completes the proof. $\square$

9. Integral Representations

Suppose K is a non-empty compact subset of a lctvs X and μ is a regular probability measure on K. A point $x \in X$ (if it exists!) is said to be *represented* by μ if $f(x) = \mu(f) = \int f d\mu$ for all $f \in X^*$. Also, one says that x is the *resultant* of μ and writes $x = r(\mu)$.

Proposition 9.1. *Suppose $C \subseteq X$ is compact and assume further that $K = \overline{co}(C)$ is compact. If $\mu \in P(C)$ then there exists a unique point $x \in X$ which is represented by μ.*

Proof. By K-M theorem, there exists a net μ_α of the form

$$\mu_\alpha = \sum_{i=1}^{n_\alpha} a_i^{(\alpha)} \delta_{y_i^{(\alpha)}} \qquad \left(a_i^\alpha \geq 0,\ \sum a_i^{(\alpha)} = 1,\ y_i^{(\alpha)} \in C \right).$$

With $\mu_\alpha \to \mu$ in the w*-topology (on $M(C)$). But $\sum_{i=1}^{n_\alpha} a_i^{(\alpha)} y_i^{(\alpha)} \in K$ and since K is compact, there exists a subset $\sum a_i^{(\beta)} y_i^{(\beta)}$, say, converging to a point $x \in K$. Consequently, $f(\sum a_i^{(\beta)} y_i^{(\beta)}) \to f(x)$ for all $f \in X^*$ and thus $\mu(f) = \lim_\beta \mu_\beta(f) = f(x)$ for all $f \in X^*$. The uniqueness of $r(\mu)$ is a direct consequence of the fact that X^* separates the points of X. $\square$

Proposition 9.2. *Suppose $C \subseteq X$ is compact and that $K = \overline{co}(C)$ is compact as before. Then $x \in K$ if and only if there exists $\mu \in M_1^+(C)$, the regular probability measures on C, which represents x.*

Proof. Suppose $\mu \in M_1^+(C)$. Then by the last result, $r(\mu) = x \in \overline{co}(C)$. Now, let $x \in \overline{co}(C)$. Then x is approximable by elements of the form $y_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha y_i^\alpha$ ($\alpha_i^\alpha \geq 0$, $\sum \lambda_i^\alpha = 1$, $y_i^\alpha \in C$). Consider the corresponding measures $\mu_\alpha = \sum \lambda_\alpha^i \delta_{y_i^\alpha} \in M_1^+(C)$, $r(\mu_\alpha) = y_\alpha$. There exists a subnet $\mu_\beta \to \mu \in M_1^+(C)$, hence for all $f \in X^*$, $\lim_\beta f(y_\beta) = \lim_\beta \mu_\beta(f) = \mu(f)$. But $y_\beta \to x$ (as $y_\alpha \to x$), and we have $\mu(f) = f(x)$ for all $f \in X^*$. $\square$

Remark 9.3. It now follows from the above result that the following statements are equivalent:

(a) If $K \subseteq X$ is a compact convex set, then $K = \overline{co}(\mathrm{ext}(K))$.
(b) Each $x \in K$ is the resultant of a $\mu \in M_1^+(K)$ with μ supported by $\overline{\mathrm{ext}(K)}$.

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