Shift Operators

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1 Introduction

In these lectures we will try to explore the paper written by A. Beurling in 1948 titled *"On 2-problems concerning linear operators on Hilbert spaces"*.

Let us first explain the various terms in the title.

1. Hilbert Spaces under consideration are the *Hardy Space* $H^2(\mathbb{D})$ or the *square summable sequence space* $l^2(\mathbb{N})$ where

$$H^2(\mathbb{D}) = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : f(z) = \sum_{n=0}^{\infty} a_n z^n, a_n \in \mathbb{C}, \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$$

and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk in the complex plane $\mathbb{C}$, and

$$l^2(\mathbb{N}) = \left\{ \{a_n\}_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$$

Now onwards we use the following notations

$$H^2(\mathbb{D}) := H^2$$

$$l^2(\mathbb{N}) := l^2$$

Exercise: l^2 is a Hilbert space with inner product defined as

$$\langle \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

and $\{e_n\}_{n=0}^{\infty}$ forms an orthonormal basis for l^2 where e_n is the sequence with 1 in the n^{th} position and zero elsewhere. We state without proof that $H^2(\mathbb{D})$ is a Hilbert Space with the inner product defined as

$$\left\langle \sum_{n=0}^{\infty} a_n z^n , \sum_{n=0}^{\infty} b_n z^n \right\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \tag{1.1}$$

For a proof please refer Duren [3] and Halmos [1].

That H^2 is a Hilbert space also follows readily from the following simple exercise.

Exercise: Show that the mapping

$$X : l^2 \to H^2$$

$$\{a_n\}_{n=0}^{\infty} \longmapsto \sum_{n=0}^{\infty} a_n z^n$$

is well defined and one to one and onto. Prove that H^2 is a Hilbert space with the inner product defined as in (1.1) (Note that under this isomorphism $e_n \longmapsto z^n$ and thus $\{1, z, z^2, ...\}$ forms an orthonormal basis of H^2).

2. The Linear Operator under consideration is the *shift*, S on l^2 or the shift M_z on H^2 (see Definition 2.1).

3. 2-Problems mentioned are

   - Take $f \in H^2/l^2$. When does $\overline{\bigvee} S^n f = H^2$ (or l^2)?
   - For $f \in H^2/l^2$. When does $C_f := \overline{\bigvee}(S^*)^n f$ is generated by the eigenvectors?

   Here $\overline{\bigvee}$ denotes the span closure and that S^* is the adjoint of S

2 Shift Operator on the Hardy Space H^2

2.1 Definition. Define shift S on l^2 by

$$S\left(\{a_n\}_{n=0}^{\infty}\right) = \{0, a_0, a_1, ...\},$$

for all $\{a_n\}_{n=0}^{\infty} \in l^2$.

2.2 Remarks.

1. S is a linear isometry on l^2, that is,

$$\|S(\{a_n\}_{n=0}^{\infty})\| = \|\{a_n\}_{n=0}^{\infty}\|.$$

2. For any $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \in l^2$

$$\begin{aligned} \langle S^*(\{a_n\}_{n=0}^{\infty}), \{b_n\}_{n=0}^{\infty}\rangle &= \langle \{a_n\}_{n=0}^{\infty}, S(\{b_n\}_{n=0}^{\infty})\rangle \\ &= \langle \{a_n\}_{n=0}^{\infty}, \{0, b_1, b_2, ...\}\rangle \\ &= \langle \{a_1, a_2, ...\}, \{b_0, b_1, ...\}\rangle \end{aligned}$$

$$\Rightarrow S^*(\{a_n\}_{n=0}^{\infty}) = \{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \dots\}.$$

3. $\ker S^* = \{\{a_0, 0, 0, ...\} : a_0 \in \mathbb{C}\} \cong \mathbb{C}$. In particular, $\dim[\ker S^*] = 1$

4. $S^*S = I$ but

$$SS^* = I - P_{\ker S^*} = I - P_{\mathbb{C}},$$

   where $P_{\ker S^*}$ is the orthogonal projection of H^2 onto $\ker S^*$. Therefore, that S is not *normal*.

2.3 Definition. Let H be a Hilbert space and $T \in L(H)$.

1. T is said to be $C_{.0}$ if $T^{*n} \longrightarrow 0$ in strong operator topology (that is, $\|T^{*n}h\| \to 0$ for all $h \in H$).

2. A closed subspace $M \subseteq H$ is said to be *invariant subspace* of $T \in L(H)$ (or, T-invariant) if $T(M) \subseteq M$.

3. A closed subspace $M \subseteq H$ is said to be *co-invariant subspace* of $T \in L(H)$ (or, T^*-invariant) if $T^*(M) \subseteq M$.

4. A closed subspace M is said to be *T-reducing* if $T(M), T^*(M) \subseteq M$.

5. $T \in L(H)$ is said to be *irreducible* if T has no reducing subspace except $\{0\}$ and H.

2.4 Definition. Define shift M_z on H^2 by

$$M_z(f) = zf, \qquad \forall f \in H^2.$$

Note that for all $f \in H^2$, zf is a function in H^2 defined by

$$(zf)(w) = wf(w),$$

for all $w \in \mathbb{D}$.

2.5 Remarks.

1. $M_z^* M_z = I_{H^2}$
2. $M_z M_z^* = I_{H^2} - P_{\mathbb{C}}$ where $P_{\mathbb{C}}$ is the orthogonal projection of H^2 onto the subspace of all constant functions, denoted by $\mathbb{C}$.
3. $M_z(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n z^{n+1}$
4. $M_z^*(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=1}^{\infty} a_n z^{n-1}$
5. $M_z^{*n} \longrightarrow 0$ in strong operator topology.
6. M_z on H^2 is irreducible.
7. $H^2 \ominus M_z H^2 = \mathbb{C}$.
8. That $H^2 \ominus M_z H^2 = \mathbb{C}$ satisfies the following relation:

$$\overline{\bigvee} \bigoplus_{n=0}^{\infty} z^n (H^2 \ominus M_z H^2) = \overline{\bigvee} \bigoplus_{n=0}^{\infty} z^n \mathbb{C} = H^2.$$

HW: Let $M \subseteq H^2$ be a M_z^*-invariant subspace. Then $M_z|_M \in L(H)$ is in $C_{\cdot 0}$. [What is the conclusion if that M is M_z-invariant?]

Example: Let $n \geq 1$ be a fixed integer and $M_n = \overline{\bigvee}\{z^n, z^{n+1}, ...\}$. Then M_n is invariant under M_z but not under M_z^*. Also, $Q_n = \bigvee\{1, z, z^2, ..., z^{n-1}\}$ is M_z^* invariant but not M_z invariant.

Questions:

(1) Is it true that $M_n = z^n H^2$?

(2) Is it true that $Q_n = (z^n H^2)^{\perp}$?

2.6 Definition. Magic/ kernel Vectors: For each $w \in \mathbb{D}$ define $k_w : \mathbb{D} \longrightarrow \mathbb{C}$ by

$$k_w(z) = \textstyle\sum_{n \geq 0} \overline{w}^n z^n \ .$$

2.7 Remarks.

1. $k_w \in H^2, \forall w \in \mathbb{D}$ and $\|k_w\| = (1 - |w|^2)^{1/2}$.
2. Let $f \in H^2$ with $f(z) = \sum_{n \geq 0} a_n z^n$ then

$$\left\langle \sum_{n=0}^{\infty} a_n z^n \ , \sum_{n=0}^{\infty} \overline{w}^n z^n \right\rangle = \sum_{n=0}^{\infty} a_n w^n.$$

   Consequently, for all $f \in H^2$ and $w \in \mathbb{D}$,

$$f(w) = \langle f, k_w \rangle.$$

   Hence, that k_w reproduce the value of $f \in H^2$ at each $w \in \mathbb{D}$.

3. $k_w \in \ker(M_z^* - \overline{w})$, $\forall w \in \mathbb{D}$ because,

$$\begin{aligned} M_z^*(k_w) &= M_z^*(1 + \overline{w}z + \overline{w}^2 z^2 + \cdots) \\ &= \overline{w} + \overline{w}^2 z + \overline{w}^3 z^2 + \cdots \\ &= \overline{w} k_w. \end{aligned}$$

4. Evaluation Functional: Define $ev_w : H^2 \longrightarrow \mathbb{C}$, for all $w \in \mathbb{D}$ by

$$ev_w(f) = f(w).$$

   Note that $|ev_w(f)| = |f(w)| = |\langle f, k_w \rangle| \leq \|f\| \|k_w\|$.
   What is $\|ev_w\|$?

5. The *Szego kernel* over $\mathbb{D}$ is the function, $k : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{C}$ defined by

$$k(\lambda, w) := (1 - \lambda \overline{w})^{-1},$$

   for all w, λ in $\mathbb{D}$.

   Note that

$$k(\lambda, w) = \langle k_w, k_\lambda \rangle, \forall \lambda, w \in \mathbb{D}$$

   and that k is holomorphic in the first variable and anti-holomorphic in the second variable.

6. Prove that $k(z, w) = ev_z \circ ev_w^*$.

7. Prove that $\{k_w : w \in \mathbb{D}\} \subseteq H^2$ is a total set, that is,

$$\overline{\bigvee} \{k_w : w \in \mathbb{D}\} = H^2.$$

To proceed further we recall the following important notion.

Let H be a Hilbert space and $T \in L(H)$ then the *spectrum* of $T, \sigma(T)$ is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

We recall that a bounded linear operator T on H is invertible if and only if that T is bounded below (that is, there exists $c > 0$ such that $\|Tf\| > c\|f\|$ for all $f (\neq 0) \in H$) and of dense range (that is $\overline{\text{ran}} T = H$).

The *approximate point spectrum* of $T, \sigma_a(T)$ is defined as

$$\sigma_a(T) = \{\lambda \in \mathbb{C} : \exists f_n \subseteq H \ni \|f_n\| = 1, \|(T - \lambda) f_n\| \longrightarrow 0\},$$

and the *point spectrum*, $\sigma_p(T)$ is defined as

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : Tf = \lambda f \text{ for some } f \neq 0\}.$$

Finally, the *compression spectrum*, $\Pi(T)$ is defined as

$$\Pi(T) = \{\lambda \in \mathbb{C} : \overline{\text{ran}}(T - \lambda I) \subsetneqq H\}.$$

2.8 Theorem. $\partial \sigma(T) \subseteq \sigma_a(T)$.

Proof. See Problem 78 in [1]. ■

2.9 Theorem.

(i) $\forall w \in \mathbb{D}, ker(M_z^* - \overline{w}) = span\{k_w\} = \mathbb{C} \cdot k_w$.
In particular, $dim[ker(M_z^* - \bar{w})] = 1$.

(ii) $\sigma_p(M_z^*) = \mathbb{D}$.

(iii) $\sigma(M_z^*) = \overline{\mathbb{D}} = \sigma_a(M_z^*)$.

Proof. We know that $k_w \in ker(M_z^* - \bar{w})$. Let $f \in H^2$ be such that $f(z) = \sum a_n z^n$ and $M_z^* f = \overline{w} f$. This implies

$$a_1 + a_2 z + a_3 z^2 + \cdots = \overline{w}(a_0 + a_1 z + a_2 z^2 + \cdots)$$

Therefore, $a_1 = \overline{w} a_0,\ a_2 = \overline{w} a_1 = \overline{w}^2 a_2, \ldots,$ and $a_n = \overline{w}^n a_0,\ \forall\, n \in \mathbb{N}$ and hence, $f = a_0 k_w$. Consequently, $\ker(M_z^* - \overline{w}) = \text{span}\{k_w\}$. This completes the proof of part (i).

Since $\|M_z^*\| = 1$, we have that $\sigma(M_z^*) \subseteq \overline{\mathbb{D}}$. Also by part (i), $\mathbb{D} \subseteq \sigma(M_z^*)$ and hence $\sigma(M_z^*) = \overline{\mathbb{D}}$. Since $\sigma_a(M_z^*) \supseteq \sigma_p(M_z^*) \supseteq \mathbb{D}$, by the fact above this theorem, we conclude that $\partial \sigma_a(M_z^*) = \mathbb{T}$. Therefore, $\sigma_a(M_z^*) = \overline{\mathbb{D}}$. This completes the proof of (iii).

Finally, it is easy to see that if $f \in H^2$ and $M_z^* f = \lambda f$ for some $|\lambda| = 1$ then that $f = 0$. Therefore, $\sigma_p(M_z^*) = \mathbb{D}$. ■

2.10 Remarks.

1. $\sigma_p(M_z) = \emptyset$.

2. For $T \in L(H), \sigma_p(T) = \overline{\Pi(T^*)}$.

3. (1) and (2) $\Rightarrow \Pi(M_z^*) = \emptyset$.

4. $\forall w \in \mathbb{D}$, $\text{ran}(M_z^* - \overline{w}) = H^2$. [$\overline{\text{range}}$ is not required here because $\forall w \in \mathbb{D}$, $\dim(\ker(M_z^* - \overline{w} I)) = 1$ and $\dim(\ker(M_z - wI)) = 0$. This yields that $M_z^* - \overline{w} I$ is *Fredholm* and ind $(M_z^* - \overline{w} I) = 1$ for all $w \in \mathbb{D}$.]

2.11 Definition. Let $\mathcal{E}$ be a Hilbert space. Define

$$H^2_{\mathcal{E}} := \left\{ \sum_{n=0}^{\infty} a_n z^n \in \text{Hol}(\mathbb{D}, \mathcal{E}) : a_n \in \mathcal{E}, \sum_{n=0}^{\infty} \|a_n\|^2_{\mathcal{E}} < \infty \right\}.$$

Here by $\text{Hol}(\mathbb{D}, \mathcal{E})$ we denote the set of all $\mathcal{E}$-valued holomorphic functions on $\mathbb{D}$.

Also, $l^2_{\mathcal{E}}$ is defined as the set of all square-summable $\mathcal{E}$-valued sequences with the natural inner product

$$\langle \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \rangle = \sum_{n=0}^{\infty} \langle a_n, b_n \rangle_{\mathcal{E}},$$

for all $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \in l^2_{\mathcal{E}}$.

Note that $H^2_{\mathcal{E}}$ is isomorphic to $l^2_{\mathcal{E}}$ in the following sense:
Fact: The map $U : H^2_{\mathcal{E}} \to H^2 \otimes \mathcal{E}$ defined by

$$z^n \eta \xrightarrow{U} z^n \otimes \eta,$$

for all $\eta \in \mathcal{E}$ and $n \geq 0$, is an isometric isomorphism onto $H^2 \otimes \mathcal{E}$ and

$$UM_z = (M_z \otimes I)U.$$

Exercises.

1. Determine M_z-reducing subspace of $H^2_{\mathcal{E}}$.
2. Prove that $M_z^{*n} \longrightarrow 0$ in strong operator topology.
3. $M_z^* M_z = I_{H^2_{\mathcal{E}}}$
4. $M_z M_z^* = I - P_{\mathcal{E}}$, where $P_{\mathcal{E}}$ is the projection of $H^2_{\mathcal{E}}$ onto the $\mathcal{E}$-valued constant functions.
5. $H^2_{\mathcal{E}} \ominus M_z H^2_{\mathcal{E}} = \mathcal{E}$

3 Isometries

Let $T \in L(H)$ and $\mathcal{W}$ be a closed subspace of H. Then $\mathcal{W}$ is said to be a *wandering subspace* of V if

$$T^n \mathcal{W} \perp \mathcal{W},$$

for all $n \geq 1$.

HW:
(1) Prove that a closed subspace $\mathcal{W}$ of H is wandering for an isometry V on H if and only if $V^n \mathcal{W} \perp V^m \mathcal{W}$ for all $m, n \geq 0$ and $m \neq n$.
(2) Let $T \in L(H)$ and $\mathcal{W}_T := H \ominus TH = \ker T^* \neq \{0\}$. Prove that $\mathcal{W}_T$ is a wandering subspace of T.

Given operator $T \in L(H)$, the general question of interest is the following:

What are the wandering subspaces of T?

The above question is related to the invariant subspace problem for operators on Hilbert spaces. For instance, if $\mathcal{W}$ is a wandering subspace of T on H then

$$\bigvee_{n=0}^{\infty} T^n \mathcal{W},$$

is an invariant subspace of T.

Another possible formulation of the wandering subspace problem is the following:

Let $T \in L(H)$ and $H = \bigvee_{n=0}^{\infty} T^n \mathcal{W}_T$, where $\mathcal{W}_T := H \ominus TH$ is the wandering subspace of T. Now consider a non-trivial invariant subspace $\mathcal{S}$ of H and define $R := T|_{\mathcal{S}} \in L(\mathcal{S})$. Does it follows that

$$\mathcal{S} = \bigvee_{n=0}^{\infty} T^n \mathcal{W}_R \quad (= \bigvee_{n=0}^{\infty} R^n \mathcal{W}_R),$$

where $\mathcal{W}_R := \mathcal{S} \ominus R\mathcal{S}$ $(= \mathcal{S} \ominus T\mathcal{S})$ *is the wandering subspace of* R.
The more general question now arises:

For which $T \in L(H)$*, one has* $H = \vee_{n=0}^{\infty} T^n(H \ominus TH)$*?*

The above questions are mostly unknown for many "friend" operators on "friend" Hilbert spaces. However, one has a complete answer for the class of isometries.

Exercise. Let $V \in L(H)$ and $V^*V = I$. Consider $\mathcal{W}_V = H \ominus V(H)$ (assume $\mathcal{W}_V \neq \{0\}$, that is, V is non-unitary). Then $V^n\mathcal{W}_V \perp V^m\mathcal{W}_v$, for all $n \neq m \in \mathbb{N}$.

3.1 Definition. An isometry $V \in L(H)$ is said to be shift of multiplicity $\dim W$ if

$$\bigoplus_{n\geq 0} V^n\mathcal{W}_V = H.$$

3.2 Examples.

1. M_z on H^2 is a shift of multiplicity 1.
2. M_z on $H^2_{\mathcal{E}}$ is a shift of multiplicity $\dim \mathcal{E}$.

3.3 Theorem. *Let* $U \in L(H)$ *and* $V \in L(K)$ *be a pair of shift operators. Then* U *is unitarily equivalent to* V *if and only if* $\dim(\mathcal{W}_U) = \dim(\mathcal{W}_V)$.

Proof. $(\Rightarrow)$ There exists $\Phi : H \longrightarrow K$ such that $\Phi^*V\Phi = U$. Therefore, $\operatorname{Ker} U^*$ is isomorphic to $\operatorname{Ker} V^*$.

$(\Leftarrow)$ There exists unitary $\varphi : \mathcal{W}_U \longrightarrow \mathcal{W}_V$. Define $\Phi : H \longrightarrow K$ by

$$\Phi\left(\sum_{n\geq 0} u^n h_n\right) = \sum_{n\geq 0} V^n \phi h_n; \quad h_n \in \mathcal{W}_U$$

Now check that, $\Phi U = V\Phi$ and that Φ is unitary. ■

In particular, the multiplicity of a shift operator is well defined.

3.4 Corollary. *Let* $V \in L(H)$ *be a shift. Then* V *is unitarly equivalent to* M_z *on* $H^2_{\mathcal{E}}$ *where dim*$\mathcal{E}$ *= multiplicity of* V.

3.5 Corollary. *If* V *is a shift, then* $V^{*n} \longrightarrow 0$ *in strong operator topology.*

3.6 Theorem (Wold Decomposition)**.** *Let* $V \in L(H)$ *is an isometry. Then there exists* V*-reducing subspaces* H_U *and* H_S *such that* $H = H_U \oplus H_S$ *and* $V|_{H_U}$ *is unitary and* $V|_{H_S}$ *is a shift.*

Proof. Let $\mathcal{W}_V := H \ominus V(H)$ and $H_S := \oplus_{n\geq 0} V^n\mathcal{W}_V$. Note that H_S is reducing Define $H_U := H_U \ominus H_S$. We claim that $H_U = \cap_{n\geq 0} V^n H$

$$\begin{aligned}
f \perp H_S &\Leftrightarrow \overset{l}{\underset{n=0}{\oplus}} V^n\mathcal{W}_V; \forall l \geq 0 \\
&\Leftrightarrow f \perp \mathcal{W}_V \oplus V\mathcal{W}_V \oplus ... \oplus V^l\mathcal{W}_V \\
&\Leftrightarrow f \perp H \ominus V^{l+1}H \\
&\Leftrightarrow f \in V^{l+1}H, \ \forall l \geq 0 \\
&\Leftrightarrow f \in \bigcap_{n\geq 0} V^n H \\
&\Rightarrow H_U = \bigcap_{n\geq 0} V^n H
\end{aligned}$$

■

3.7 Corollary. *Any isometry is of the form* $\begin{bmatrix} \text{unitary} & 0 \\ 0 & M_z \end{bmatrix}$.

3.8 Definition. Define $l^2(\mathbb{Z})$ by

$$l^2(\mathbb{Z}) = \{(a_n)_{n\in\mathbb{Z}} : \sum |a_n|^2 < \infty\}.$$

3.9 Definition. The bilateral shift U on $l^2(\mathbb{Z})$ is defined by $Ue_n = e_{n+1}$, where e_n is the sequence with 1 in the n^{th} position and zero elsewhere for all $n \in \mathbb{Z}$.

Note that $U^*e_n = e_{n-1}$ for all $n \in \mathbb{Z}$ and therefore

$$UU^* = U^*U = I.$$

Let $\mathbb{T}$ be the unit circle in the complex plane. By $L^2(\mathbb{T})$ we denote the familiar collection of the square integrable functions on $\mathbb{T}$ with respect to the normalized lebesgue measure dm on $\mathbb{T}$. Define

$$X : L^2(\mathbb{T}) \longrightarrow l^2(\mathbb{Z})$$

$$f \longmapsto \hat{f}$$

where $\hat{f}$ is the Fourier transform of f which is defined by $\hat{f}(n) = \int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta$.
Fact: X defined above is unitary and $XM_{e^{i\theta}} = UX$. Hence, that $M_{e^{i\theta}}$ is the bilateral shift.

3.10 Definition. Define $L^2_+ := \overline{\bigvee}_{n\geq 0} e^{in\theta} \subseteq L^2(\mathbb{T})$.

We observe that $M_{e^{i\theta}} L^2_+ \subseteq L^2_+$. If $V = M_{e^{i\theta}}|_{L^2_+}$ then $V^*V = I_{L^2_+}$. Check that V is a shift of multiplicity 1 and hence, V on L^2_+ is unitarily equivalent to M_z on H^2 (follows from Corollary 3.4).

3.11 Definition. Define $H^\infty(\mathbb{D})$ by

$$H^\infty(\mathbb{D}) = \{\varphi \in L^\infty(\mathbb{T}) : \hat{\varphi}(n) = 0, \forall n < 0\} = L^\infty \cap H^2.$$

3.12 Theorem. $\{M_{e^{i\theta}}\}' = \{M_\varphi : \varphi \in L^\infty\}$.

Proof. See Halmos. ■

3.13 Theorem. *Let* $\varphi \in L^\infty$. *Then* $\varphi H^2 \subseteq H^2$ *if and only if* $\varphi \in H^\infty$.

Proof. ($\Rightarrow$) $\varphi 1 = \varphi = f$ for some $f \in H^2$. Consequently, $\varphi \in H^\infty$.
($\Leftarrow$) We claim that $\varphi H^2 \subseteq H^2$. Indeed, fix $l \in \mathbb{N}$:

$$\begin{aligned} \widehat{\varphi z^l}(n) &= \int_0^{2\pi} \varphi(e^{i\theta})e^{il\theta}e^{-in\theta}d\theta \\ &= \int_0^{2\pi} \varphi(e^{i\theta})e^{-i(n-l)\theta}d\theta \\ &= \widehat{\varphi}(n-l) \\ &= 0 \text{ if } n < 0 \text{ and } l \geq 0 \end{aligned}$$

Therefore, $\varphi p \in H^2$, for every $p \in \mathbb{C}[Z]$. Since the polynomials are dense in H^2 we have that $\varphi H^2 \subseteq H^2$. ■

3.14 Definition. For $\varphi \in H^\infty(\mathbb{D})$, define $T_\varphi := M_\varphi|_{H^2}$, and is called *Toeplitz Operator* with holomorphic symbol φ.

3.15 Theorem. $\{M_z\}' = \{T_\varphi : \varphi \in H^\infty(\mathbb{D})\}$.

Proof. See Halmos.

3.16 Theorem. *$\mathcal{M} \subseteq L^2(\mathbb{T})$ is $M_{e^{i\theta}}$-reducing if and only if*

$$\mathcal{M} = \{f \in L^2(\mathbb{T}) : f = 0 \text{ a.e. on } X\}$$

for some m-measurable set $X \subseteq \mathbb{T}$.

Proof. ($\Leftarrow$) is trivial.
($\Rightarrow$) Let $P_\mathcal{M}$ be the orthogonal projection of $L^2(\mathbb{T})$ onto $\mathcal{M} \subseteq L^2$. That $\mathcal{M}$ is $M_{e^{i\theta}}$-reducing yields that

$$P_\mathcal{M} M_{e^{i\theta}} = M_{e^{i\theta}} P_\mathcal{M}.$$

That is, $P_\mathcal{M} \in \{M_{e^{i\theta}}\}'$ and hence, $P_\mathcal{M} = M_\varphi$ for some $\varphi \in L^\infty$. But $P_\mathcal{M} = P_\mathcal{M}^2 = P_\mathcal{M}^* \Rightarrow \text{range}(\varphi) \subseteq \{0,1\}$ Let $\text{range}(\varphi) = \{0,1\}$ implies $\varphi = \chi_X$ where $X = \{e^{i\theta} \in \mathbb{T} : \varphi(e^{i\theta}) = 0\}$. This implies $\text{range}(P_\mathcal{M}) = \mathcal{M} = \text{range}(M_\varphi) = \{f \in L^2(\mathbb{T}) : f = 0 \text{ a.e. on } X\}$. ■

3.17 Theorem. *Let $\mathcal{M}(\neq \{0\}) \subseteq L^2(\mathbb{T})$. Then $\mathcal{M}$ is a non-reducing invariant subspace of $M_{e^{i\theta}}$ if and only if $\mathcal{M} = \varphi H^2$ for some $\varphi \in L^\infty(\mathbb{T})$ with $|\varphi| = 1$ a.e.*

Proof. Let $\mathcal{W} = \mathcal{M} \ominus e^{i\theta}\mathcal{M}$. Choose $\varphi \in W$ with $\|\varphi\| = 1$ (note that $\mathcal{W} = \{0\}$ is equivalent to the condition that $\mathcal{M}$ is reducing). Since $e^{in\theta}\mathcal{M} \subseteq e^{i\theta}\mathcal{M}$, we have

$$\begin{aligned}
\varphi \perp e^{i\theta}\mathcal{M} &\Rightarrow \varphi \perp e^{in\theta}\mathcal{M}, \ \forall n \geq 1 \\
&\Rightarrow \varphi \perp e^{in\theta}\varphi, \ \forall n \geq 0 \\
&\Rightarrow \int_0^{2\pi} |\varphi(e^{i\theta})|^2 e^{-in\theta} d\theta = 0, \ \forall n \geq 1 \\
&\Rightarrow \widehat{|\varphi|^2}(n) = 0 \ \forall n \neq 0 \\
&\Rightarrow |\varphi|^2 = \text{constant } a.e.
\end{aligned}$$

On the other hand,

$$\|\varphi\|_\infty = 1,$$

and hence

$$|\varphi| = 1 \ a.e.$$

Claim: $\varphi H^2 = \mathcal{M}$.
Note that M_φ is unitary. Therefore $\{\varphi e^{in\theta} : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$. This means

$$L^2(\mathbb{T}) = \overline{\bigvee}_{n \in \mathbb{Z}} \{\varphi e^{in\theta}\},$$

and

$$\varphi H^2 = \overline{\bigvee}_{n \in \mathbb{N}} \{\varphi e^{in\theta}\}.$$

Now $\varphi \in \mathcal{M}$ implies $e^{in\theta}\varphi \in \mathcal{M}$, for $n \geq 0$, and this further implies $\varphi H^2 \subseteq \mathcal{M}$. Now we proceed to prove $\mathcal{M} \subseteq \varphi H^2$. Let $f \in \mathcal{M}$. Claim: $f \perp \varphi e^{in\theta}$, $\forall n < 0$. Now for $n > 0$

$$\begin{aligned}\langle f, \varphi e^{-in\theta}\rangle &= \langle f, M^*_{e^{in\theta}}\varphi\rangle \\ &= \langle M^n_{e^{i\theta}} f, \varphi\rangle \\ &= \langle e^{in\theta} f, \varphi\rangle \\ &= 0.\end{aligned}$$

The converse part is trivial. ■

3.18 Theorem. *(Uniqueness of φ) If $\mathcal{M}$ is as in Theorem 3. 16, then*

$$\mathcal{M} \ominus e^{i\theta}\mathcal{M} = \varphi \cdot \mathbb{C}.$$

In particular, dim$(\mathcal{M} \ominus e^{i\theta}\mathcal{M}) = 1$ and hence, that φ is unique (up to a scalar multiplier of length one).

Proof. First, check that if $\mathcal{M} = \psi H^2$ for some $\psi \in L^\infty(\mathbb{T})$ with $\|\varphi\|_\infty = 1$, then that $\psi \in \mathcal{M} \ominus e^{i\theta}\mathcal{M}$. Now let

$$\mathcal{M} = \varphi H^2 = \psi H^2,$$

for $\varphi, \psi \in L^\infty(\mathbb{T})$ with $\|\varphi\|_\infty = \|\psi\|_\infty = 1$. Then

$$\varphi(f) = \psi,$$

and

$$\psi g = \varphi,$$

for some $f, g \in H^2$. Consequently, $\overline{\varphi}\psi = f$. Also, $\overline{\psi}\varphi = g$. Therefore,

$$\overline{f} = g.$$

Thus, f is a constant, $f = c$ (say) with $|c| = 1$. ■

4 Beurling's Theorem and some Consequences

4.1 Theorem (Beurling's Theorem)**.** *Let $\mathcal{M} \neq \{0\}$ be a closed subspace of H^2. Then, $\mathcal{M}$ is M_z-invariant if and only if $\mathcal{M} = \varphi H^2$ for some $\varphi \in H^\infty$ and*

$$|\varphi(e^{i\theta})| = 1 \text{ a.e.}$$

Moreover, that φ is unique up to a scalar multipliers of length one.

Proof. $(\Rightarrow)$

$$\begin{aligned}M_z(\varphi f) &= M_z T_\varphi f \\ &= T_\varphi(zf) \in \varphi H^2,\end{aligned}$$

since $T_\varphi M_z = M_z T_\varphi$.
$(\Leftarrow)$: Let $\mathcal{M} \neq \{0\}$ be a M_z-invariant subspace. In particular, $\mathcal{M}$ is $M_{e^{i\theta}}$-invariant. This means, $\mathcal{M} = \varphi H^2$ for $\varphi \in L^\infty$ and $|\varphi| = 1$ a.e. Now, $\varphi \cdot 1 \in \mathcal{M}$ which implies that φ is holomorphic, that is, $\varphi \in H^\infty$.
Uniqueness of φ follows from Theorem 3.18. ■

4.2 Corollary. *If $\mathcal{M}$ is a M_z-invariant subspace of H^2, then $M_z|_{\mathcal{M}} \in L(\mathcal{M})$ is an isometry with multiplicity $1(= dim(\mathcal{M} \ominus z\mathcal{M})$.*

4.3 Definition. A function $\varphi \in H^\infty(\mathbb{D})$ is inner if $|\varphi(e^{i\theta})| = 1$ a.e.

Fact. Let $\varphi \in H^\infty$. Then $T_\varphi : H^2 \to H^2$ ($f \mapsto \varphi f$ for all $f \in H^2$) is an isometry if and only if φ is inner. (see Halmos)

4.4 Definition. A function $f \in H^2$ is outer if $\overline{\bigvee}_{n=0}^{\infty} z^n f = H^2$.

Fact. If f is outer, then, $f(z) \neq 0$ for all $z \in \mathbb{D}$.
[If not, then there exists $w \in \mathbb{D}$ such that $f(w) = 0$ for all $f \in H^2$ - a contradiction.]

The following result also follows directly from Corollary 4.2.

4.5 Corollary. *Let $\{0\} \neq \mathcal{M}_1, \mathcal{M}_2 \subseteq H^2$ be M_z-invariant. Then, $M_z|_{\mathcal{M}_1}$ (on $\mathcal{M}_1$) is unitarily equivalent to $M_z|_{\mathcal{M}_2}$ (on $\mathcal{M}_2$).*

Proof. Set $\mathcal{M}_1 = \varphi_1(H^2)$ and $\mathcal{M} = \varphi_2 H^2$. Define

$$\begin{aligned} X : \mathcal{M}_1 &\to \mathcal{M}_2 \\ \varphi_1 f &\mapsto \varphi_2 f, \end{aligned}$$

for all $f \in H^2$. Check that X is unitary and $X(M_z|_{\mathcal{M}_1}) = (M_z|_{\mathcal{M}_2})X$. ■

4.6 Corollary. *Let $\mathcal{M} \subseteq H^2$ be an M_z-invariant subspace. Then, $M_z|_{\mathcal{M}}$ on $\mathcal{M}$ has a cyclic vector.*

Proof. First, note that $(M_z|_{\mathcal{M}})^n = M_z^n|_{\mathcal{M}}$ for all $n \in \mathbb{N}$. Then

$$\mathcal{M} = \varphi H^2 = \bigvee_{n \geqslant 0} z^n \varphi = \bigvee_{n \geqslant 0} (M_z|_{\mathcal{M}})^n \varphi.$$

Thus φ is the required cyclic vector. ■

4.7 Corollary. *Let $\mathcal{M}_1, \mathcal{M}_2$ be a pair of non-zero M_z-invariant subspaces of H^2. Then, $\mathcal{M}_1 \cap \mathcal{M}_2 \neq \{0\}$.*

Proof. Set $\mathcal{M}_i = \varphi_i H^2$. Then, $(\varphi_1\varphi_2)H^2 \subseteq \varphi_1 H^2, \varphi_2 H^2$. ■

4.8 Corollary. *(Riesz Brother's theorem) Let $f \in H^2$ and $E := \{e^{i\theta} \in \mathbb{T} : f(e^{i\theta}) = 0\}$. Then, $m(E) = 0$.*

Proof. Let E be a measurable subspace and let

$$\mathcal{M}_E := \{g \in H^2 : g = 0 \text{ on } E \text{ a.e.}\}.$$

Note that for all $g \in \mathcal{M}_E$ and $w \in E$,

$$(zg)(w) = wg(w) = 0.$$

Therefore, $\mathcal{M}_E$ is shift-invariant. Thus, $\mathcal{M}_E = \varphi H^2$ and $|\varphi| = 1$ a.e. It follows that $\varphi \in \mathcal{M}_E$, which implies that $\varphi = 0$ on E a.e. contradicting the fact that $|\varphi| = 1$ a.e. unless that $m(E) = 0$. ■

4.9 Corollary. *(Inner-outer Factorization) Let $0 \neq f \in H^2$. Then, $f = \varphi_i \varphi_0$ where φ_i is inner and φ_0 is outer. Furthermore, this representation is unique up to scalar multipliers of length one.*

Proof. Let $\mathcal{M}_f = \overline{\bigvee}_{n=0}^{\infty} z^n f$. Since that $\mathcal{M}_f$ is M_z-invariant, we have

$$\mathcal{M}_f = \varphi_i H^2,$$

for some inner function $\varphi_i \in H^\infty(\mathbb{D})$. We are done if $\mathcal{M}_f = \varphi_i H^2 = H^2$. Therefore, we assume that $\varphi_i H^2 \subsetneq H^2$. But, $f \in \mathcal{M}_f = \varphi_i H^2$. Thus

$$f = \varphi_i \varphi_0,$$

for some $\varphi_0 \in H^2$. We claim that φ_0 is outer, that is,

$$\overline{\bigvee} z^n \varphi_0 = H^2.$$

If not, by M_z-invariance of $\overline{\bigvee} z^n \varphi_0$, we have

$$\overline{\bigvee} z^n \varphi_0 = \psi H^2,$$

for some inner function $\psi \in H^\infty(\mathbb{D})$. Therefore,

$$\begin{aligned} \varphi_i H^2 &= \overline{\bigvee} z^n f \\ &= \overline{\bigvee} z^n \varphi_i \varphi_0 \\ &= \varphi_i \overline{\bigvee} z^n \varphi_0 \\ &= \varphi_i \psi H^2. \end{aligned}$$

By the uniqueness part of Beurling's theorem, $\varphi_i = c\varphi_i \psi$ for some c such that $|c| = 1$. Thus, $\psi = \overline{c}$ and φ_0 is outer: $\overline{\bigvee} z^n \varphi_0 = \psi H^2 = H^2$.
Uniqueness part is left to the reader. ∎

4.10 Example. Let $w \in \mathbb{D}$ and consider $\mathcal{M}_w = \{f \in H^2 : f(w) = 0\}$. Then, $\mathcal{M}_w$ is closed, being the kernel of the evaluation functional ev_w. Also, $\mathcal{M}_w$ is shift-invariant.

Now, $\mathcal{M}_w = \varphi H^2$ and $\varphi(w) = 0$. Also, $\varphi : \mathbb{D} \to \mathbb{D}$. Then,

$$\varphi_w(z) = e^{i\phi} \frac{z - w}{1 - \overline{w} z} \text{ for some } \phi$$

We claim that φ is inner:

$$\begin{aligned} \left| \frac{e^{i\theta} - w}{1 - e^{i\theta}\overline{w}} \right| &= |e^{i\theta}| \left| \frac{1 - e^{-i\theta} w}{1 - e^{i\theta}\overline{w}} \right| \\ &= 1 \end{aligned}$$

since the denominator of the last fraction is the conjugate of the numerator. Thus, $\mathcal{M}_w = \varphi_w H^2$.

4.11 Theorem. *Let $w_i \in \mathbb{D}, 1 \leqslant i \leqslant n$. Then,*

$$\mathcal{M}_{w_1 \ldots w_n} := \{f \in H^2 : f(w_i) = 0\} = \left(\prod_{i=1}^{n} \varphi_{w_i} \right) H^2$$

.

5 Bergman space and more

Let $L_a^2(\mathbb{D})$ (see [5]) be the space of all square integrable (with respect to the area measure) holomorphic functions on the open unit disc $\mathbb{D}$, that is,

$$L_a^2(\mathbb{D}) = \{f \in \text{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f|^2 dA < \infty\},$$

where dA is the normalized area measure on $\mathbb{D}$. Then $L_a^2(\mathbb{D})$ is a Hilbert space with

$$\langle f, g \rangle := \int_{\mathbb{D}} f\bar{g} dA,$$

for all $f, g \in L_a^2(\mathbb{D})$.
One checks that the ring of polynomials $\mathbb{C}[z] \subseteq L_a^2(\mathbb{D})$ and

$$\|z^n\| = \sqrt{\frac{1}{n+1}},$$

for all $n \geq 0$. Another way to represent the Bergman space is based on the weighted-square summability condition:

$$L_a^2(\mathbb{D}) = \{f \in \text{Hol}(\mathbb{D}) : f = \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty\},$$

with

$$\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \rangle = \sum_{n=0}^{\infty} \frac{a_n \bar{b}_n}{n+1},$$

for all $\sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \in L_a^2(\mathbb{D})$.

It is easy to see that the multiplication operator $M_z : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ defined by $f \mapsto zf$ (for all $f \in L_a^2(\mathbb{D})$) is bounded. Moreover, for each $w \in \mathbb{D}$, the function

$$k_w : \mathbb{D} \to \mathbb{D},$$

defined by

$$k_w(z) = (1 - z\bar{w})^{-2}, \qquad (\forall z \in \mathbb{D})$$

is in the Bergman space. Also these functions (magic/kernel) reproduces the values of functions of $L_a^2(\mathbb{D})$ in the following sense:

$$f(w) = \langle f, k_w \rangle. \quad (\forall f \in L_a^2(\mathbb{D}), w \in \mathbb{D})$$

The *Bergman kernel* on the open unit disc $\mathbb{D}$ is defined by

$$k(z, w) = (1 - z\bar{w})^{-2},$$

for all $(z, w) \in \mathbb{D} \times \mathbb{D}$.
Some surprising facts:
(1) Beurling type representations of M_z-invariant subspaces of $L_a^2(\mathbb{D})$ fails.
(2) If $\mathcal{M}_1, \mathcal{M}_2 \subseteq L_a^2(\mathbb{D})$ be a pair of M_z-invariant subspaces and that

$$M_z|_{\mathcal{M}_1} \cong M_z|_{\mathcal{M}_2},$$

then, $\mathcal{M}_1 = \mathcal{M}_2$.
(3) For a M_z-invariant subspace $\mathcal{M}$ of $L^2_a(\mathbb{D})$, the dimension of $\mathcal{M} \ominus z\mathcal{M}$ could be any natural number $1, 2, \ldots$, even ∞.
(4) However, M_z-invariant subspaces of the Bergman space obeys the wandering subspace theorem. That is, for any non-zero closed M_z-invariant subspace $\mathcal{M}$ of $L^2_a(\mathbb{D})$, one has

$$\mathcal{S} = \bigvee_{n=0}^{\infty} z^n(\mathcal{M} \ominus z\mathcal{S}).$$

(5) For the weighted Bergman spaces, the above result is still not known.
(6) In several variables, situation is more complicated.
(7) The fact in (2) is known as the *rigidity* property. It seems that except the Hardy space on the unit disc and some pathological examples, all known reproducing kernel Hilbert spaces enjoy the rigidity property!

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