A Non-Commutative Version of Finite Discrete-Time and Finite State Model in Mathematical Finance

Yoshihiro Ryu

Ritsumeikan University, Japan

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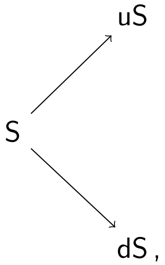
Introduction for Mathematical Finance

Major subjects of Mathematical Finance are the followings:

- To decide the price of options theoretically. (ex. how much is the right of buying a stock tomorrow?)
- To make the "risk" smaller. (ex. finding a safer strategy for buying and selling of stocks.)

Binary Model is described as the diagrams below:



$$1 \longrightarrow (1+r)1\,.$$

Black-Scholes Model is described as follows: the stock price process is the solution of the following stochastic differential equation:

$$\mathrm{d}S_t = bS_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t.$$

Here b and σ are constants, and W_t is Brownian motion.

"Commutative" Arbitrage Theory

This model is largely based on [1, 5]. We consider a security market with N securities and maturity time T. Let

$$\mathbb{T} := \{0, 1, \ldots, T\}, \ \Omega := \{1, 2, \ldots, n\},$$
$$(\Omega, \mathcal{F}, \mathrm{P}) : \text{a probability space}, \ \mathcal{F} := \mathcal{P}(\Omega), \ \mathrm{P} > 0 \text{ on } \mathcal{F} \setminus \{\emptyset\},$$
$$\{\mathcal{F}_t\}_{t \in \mathbb{T}} : \text{a filtration of } \mathcal{F}, \ \mathcal{F}_0 := \{\emptyset, \Omega\}, \ \mathcal{F}_T := \mathcal{F}.$$

Definition 2.1

For $j = 1, 2, \ldots, N$ and $t \in \mathbb{T}$, s_t^j and d_t^j are price and dividend of the j-th security at period t if

$$s_t^j, d_t^j : \Omega \to \mathbb{R} : \mathcal{F}_\mathrm{t}\text{-measurable}.$$

Definition 2.2

A portfolio $\theta \equiv \left(\theta^1, \cdots, \theta^N\right)$ *is*

$$\theta^j := \left(\theta^j_0, \cdots, \theta^j_T\right), \textit{where}$$
$$\theta^j_t : \Omega \to \mathbb{R} : \mathcal{F}_{\mathrm{t}}\text{-measurable.}$$

A portfolio θ^j_t represents the number of holdings of the j-th security at period t.

Definition 2.3

The gain sequence $w(\theta)$ of a portfolio θ is defined by

$$w_t(\theta) := \sum_{j=1}^{N} \left(s_t^j + d_t^j\right) \theta_{t-1}^j - s_t^j \theta_t^j,$$
$$w(\theta) := \left(w_0(\theta), \cdots, w_T(\theta)\right).$$

Here $\theta_{-1}^j := 0$.

The random variable $w_t(\theta)$ is the gain of the portfolio θ at period t.

Definition 2.4

A portfolio θ is called arbitrage if

$$\forall t \in \mathbb{T},\ \ w_t\left(\theta\right) \geq 0 \text{ and } \exists t \in \mathbb{T}\ w_t\left(\theta\right) \neq 0.$$

Existence of an arbitrage portfolio means there is an opportunity to earn some benefit without risk.

Definition 2.5

The market is called no arbitrage if there does not exist an arbitrage portfolio.

This proposition is well-known as (a version of) the first fundamental theorem of mathematical finance.

Proposition 2.6

The market is no arbitrage if and only if there exists a positive adapted sequence $(\psi_t)_{t\in\mathbb{T}}$ *such that for all portfolio* θ*,*

$$\sum_{t=0}^{T} \mathrm{E}\left(\psi_t w_t\left(\theta\right)\right) = 0.$$

"Non-Commutative" Arbitrage Theory in a Model

We borrow a framework of non-commutative (or quantum) probabilities from [2, 4].
For non-commutative probability space $(\mathcal{A}, \varphi)$, we consider

$$\mathcal{A} := \mathrm{M}(n, \mathbb{C}), \ \varphi(X) := \mathrm{tr}\,(\rho X) \ (X \in \mathcal{A}).$$

Here

$$\rho \in \mathrm{Her}_{++}(n, \mathbb{C}), \ \mathrm{tr}\,\rho = 1.$$

Let $\{\mathcal{A}_t\}_{t \in \mathbb{T}}$ be an increasing sequence of sub $*$-algebra of $\mathcal{A}$ (as a filtration).

Definition 3.1

We define the price sequence S^j *and the dividend sequence* D^j *for j-th security by*

$$S^j := \left(S_0^j, \cdots, S_T^j\right), \; D^j := \left(D_0^j, \cdots, D_T^j\right)$$
$$S_t^j, D_t^j \in \mathcal{A}_t \;\; (t \in \mathbb{T}) \, .$$

Definition 3.2

A portfolio Θ *and its gain sequence* $W(\Theta)$ *is defined as follows: let* $\Theta_t^j \in \mathcal{A}_t$*, and put*

$$\Theta_t := \begin{pmatrix} \Theta_t^1 & & \\ & \ddots & \\ & & \Theta_t^N \end{pmatrix}, \ \Theta := \begin{pmatrix} \Theta^1 & & \\ & \ddots & \\ & & \Theta^N \end{pmatrix}.$$

Suppose

$$W_t(\Theta) := \sum_{j=1}^{N} \Theta_{t-1}^j \left(S_t^j + D_t^j \right) - \Theta_t^j S_t^j, \ W(\Theta) := \begin{pmatrix} W_0 & & \\ & \ddots & \\ & & W_T \end{pmatrix}.$$

Here $\Theta_{-1}^j := 0$*.*

The definitions of arbitrage and no arbitrage in non-commutative version are as below.

Definition 3.3

For a portfolio Θ,

$$\Theta : \text{an arbitrage} \overset{\text{def}}{\iff} W(\Theta) \in \mathrm{Her}_+(n(T+1), \mathbb{C}) \setminus \{0\}.$$

Definition 3.4

The market admits no arbitrage opportunity if there is no arbitrage portfolio.

Note that, since W is a linear map the domain of which is the vector space of all portfolios, from the definition we obtain

$$\text{no arbitrage} \iff \mathrm{Im}\ W \cap \mathrm{Her}_+(n(T+1), \mathbb{C}) = \{0\}.$$

Theorem 3.5

The market admits no arbitrage opportunity if and only if there exists $\Psi_t \in \mathcal{A}$ $(t \in \mathbb{T})$, *for all portfolio* Θ,

$$\sum_{t=0}^{T} \varphi\left(\Psi_t W_t\left(\Theta\right)\right) = 0$$

and

$$\forall t \in \mathbb{T},\ \rho\Psi_t \in \mathrm{Re}_{++}\left(n, \mathbb{C}\right).$$

Here

$$\mathrm{Re}_{++}\left(n, \mathbb{C}\right) := \left\{\, A \in \mathcal{A} \mid \text{Real part of } A \text{ is strictly positive} \,\right\}.$$

Proof. Assume that,

$$\begin{aligned} &(\mathrm{Im}\ W)^{\perp} \cap \mathrm{Re}_{++}\left(n(T+1), \mathbb{C}\right) \neq \emptyset \\ &\iff \mathrm{Im}\ W \cap \mathrm{Her}_{+}\left(n(T+1), \mathbb{C}\right) = \{0\} \\ &(\iff \text{no arbitrage}) \end{aligned}$$

is true.

Necessity. From above, there exists $\tilde{\Psi} \in (\mathrm{Im}\ W)^{\perp} \cap \mathrm{Re}_{++}\left(n(T+1), \mathbb{C}\right)$.
If we separate $\tilde{\Psi}$ as

$$\tilde{\Psi} = \begin{pmatrix} \tilde{\Psi}_0 & & * \\ & \ddots & \\ * & & \tilde{\Psi}_T \end{pmatrix} \ \left(\tilde{\Psi}_t \in \mathcal{A}\right),$$

then $\tilde{\Psi}_t^* \in \mathrm{Re}_{++}\left(n, \mathbb{C}\right)$ $(\forall t \in \mathbb{T})$ and

$$0 = \left\langle \tilde{\Psi}, W\left(\Theta\right) \right\rangle = \sum_{t=0}^{T} \mathrm{tr}\left(\tilde{\Psi}_t^* W_t\left(\Theta\right)\right)\ (\forall \Theta).$$

Put $\Psi_t := \rho^{-1}\tilde{\Psi}_t^*$, therefore for all $t \in \mathbb{T}$, $\rho\Psi_t \in \mathrm{Re}_{++}(n, \mathbb{C})$ and

$$0 = \sum_{t=0}^{T} \mathrm{tr}\left(\rho\Psi_t W_t(\Theta)\right) = \sum_{t=0}^{T} \varphi\left(\Psi_t W_t(\Theta)\right) \ (\forall\Theta).$$

Sufficiency. Suppose

$$\Psi := \begin{pmatrix} (\rho\Psi_0)^* & & \\ & \ddots & \\ & & (\rho\Psi_T)^* \end{pmatrix}.$$

We have $\Psi \in \mathrm{Re}_{++}(n(T+1), \mathbb{C})$ and

$$\langle\Psi, W(\Theta)\rangle = \sum_{t=0}^{T} \varphi\left(\Psi_t W_t(\Theta)\right) = 0 \ (\forall\Theta).$$

Thus,

$$
\begin{aligned}
&(\mathrm{Im}\ W)^{\perp} \cap \mathrm{Re}_{++}\left(m(T+1), \mathbb{C}\right) \neq \emptyset \\
\iff\ & \mathrm{Im}\ W \cap \mathrm{Her}_{+}\left(m(T+1), \mathbb{C}\right) = \{0\} \\
\iff\ & \text{no arbitrage. } \square
\end{aligned}
$$

The assumption in the previous proof is a corollary of the next lemma.

Lemma 3.6

Let

$$V : \text{a subspace of } \mathrm{M}(I, \mathbb{C}) \ \ (I \in \mathbb{Z}_{++}).$$

Then

$$V \cap \mathrm{Her}_{+}(I, \mathbb{C}) = \{0\} \iff V^{\perp} \cap \mathrm{Re}_{++}(I, \mathbb{C}) \neq \emptyset.$$

Reference

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[5] 津野 義道, (1999), ファイナンスの数学的基礎-離散モデル-, 共立出版.

Thank you for your kind attention.