Spectral theorem for compact normal operator on Quaternionic Hilbert spaces

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Overview

Quaternion ring

Definition:

$\mathbb{H} := \{q = q_0 + q_1 i + q_2 j + q_3 k : q_\ell \in \mathbb{R} \text{ and } \ell = 0, 1, 2, 3\}$, the ring of all real quaternions.

- Here $i \cdot j = k = -j \cdot i$, $j \cdot k = i = -k \cdot j$, $k \cdot i = j = -i \cdot k$ and $i^2 = j^2 = k^2 = -1$.
- $\mathbb{H}$ is a division ring (non-commutative).

Properties:

- Let $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$. Then $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$, the conjugate of q.
- $|q| = \sqrt{\overline{q} \cdot q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$. This defines norm on $\mathbb{H}$.
- $\text{Re}(\mathbb{H}) := \{q \in \mathbb{H} : \overline{q} = q\}$ and $\text{Im}(\mathbb{H}) := \{q \in \mathbb{H} : \overline{q} = -q\}$.

Quaternion ring

Properties:

- Let $p, q \in \mathbb{H}$. Then
  1. $\overline{p \cdot q} = \overline{q} \cdot \overline{p}$
  2. $|p \cdot q| = |p| \cdot |q|$
  3. $|p| = |\overline{p}|$.

Imaginary unit sphere:

- $\mathbb{S} := \{q \in \mathrm{Im}(\mathbb{H}) : |q| = 1\}$.
- $q \in \mathbb{S} \Leftrightarrow q^2 = -1$.

Relation on $\mathbb{H}$

- $p \sim q$ if and only if $s^{-1}ps = q$, for some $0 \neq s \in \mathbb{H}$.
- It is an equivalance relation on $\mathbb{H}$.
- $[p] = \{s^{-1}ps : 0 \neq s \in \mathbb{H}\}$.
- Since $p = \mathrm{Re}(p) + |\mathrm{Im}(p)|.\frac{\mathrm{Im}(p)}{|\mathrm{Im}(p)|}$, we have

$$\begin{aligned}[p] &= \mathrm{Re}(p) + |\mathrm{Im}(p)|\ s^{-1}\frac{\mathrm{Im}(p)}{|\mathrm{Im}(p)|}s,\ \forall\ 0 \neq s \in \mathbb{H} \\ &= \mathrm{Re}(p) + |\mathrm{Im}(p)| \cdot \mathbb{S}.\end{aligned}$$

Equivalant condition:

- $$\begin{aligned} p \sim q &\Leftrightarrow [p] = [q] \\ &\Leftrightarrow \mathrm{Re}(p) + \mathbb{S} \cdot |\mathrm{Im}(p)| = \mathrm{Re}(q) + \mathbb{S} \cdot |\mathrm{Im}(q)| \\ &\Leftrightarrow \mathrm{Re}(p) = \mathrm{Re}(q) \text{ and } |\mathrm{Im}(p)| = |\mathrm{Im}(q)|. \end{aligned}$$
- Let $\alpha + i\beta \in \mathbb{C}$. Then $\alpha + i\beta \in [q] \Leftrightarrow \alpha = \mathrm{Re}(q)$ and $\beta = \pm|\mathrm{Im}(q)|$.

complex parts:

- For every $q \in \mathbb{H}$, we have

$$\begin{aligned} q &= (q_0 + q_1 i) + (q_2 + q_3 i) \cdot j \\ &= a_1 + a_2 \cdot j, \end{aligned}$$

where $a_1, a_2 \in \mathbb{C}$.

Identification

- Define $\psi\colon \mathbb{H} \longrightarrow \mathrm{M}_2(\mathbb{C})$ by

$$\psi(q = a_1 + a_2 \cdot j) = \begin{pmatrix} a_1 & a_2 \\ -\overline{a_2} & \overline{a_1} \end{pmatrix}$$

- ψ is a bijective onto its range.
- $\det\,(\psi(q)) = |q|^2$
- $\mathrm{Re}(q) \pm |\mathrm{Im}(q)| i$ are the eigenvalues of $\psi(q)$.

Quaternionic Hilbert space

Definition:

Let H be right $\mathbb{H}-$ module with the innerproduct $\langle\ |\ \rangle : \mathrm{H} \times \mathrm{H} \longrightarrow \mathbb{H}$ satisfies

1. $\langle u|u\rangle = 0 \Leftrightarrow u = 0.$
2. $\langle u|v\rangle = \overline{\langle v|u\rangle}.$
3. $\langle u + v \cdot q|w\rangle = \langle u|w\rangle + \overline{q}\,\langle v|w\rangle\,,\ \forall u, v, w \in \mathrm{H}\ \&\ q \in \mathbb{H}.$

Define $\|u\| = \sqrt{\langle u|u\rangle},\ \forall\ u\ \ \in \mathrm{H}$. If $(\mathrm{H}, \|.\|)$ is complete then H is called right quaternionic Hilbert space.

Example:

$$\ell^2(\mathbb{N}, \mathbb{H}) = \{(q_1, q_2, q_3, \cdots) : \sum_{n=1}^{\infty} |q_n|^2 < \infty\}.$$

Definition:

A map T : H $\longrightarrow$ H is said to be

- righ $\mathbb{H}-$ linear, if $T(u + v \cdot q) = Tu + Tv \cdot q$, for all $u, v \in$ H and $q \in \mathbb{H}$.
- bounded, if there exists $K > 0$, such that $\|Tu\| \leq K\|u\|$, for all $u \in$H.
- If T is bounded, then

$$\|\mathrm{T}\| = \sup\{\|\mathrm{T}u\| : u \in \mathrm{H}, \|u\| = 1\}.$$

Notation:

- $\mathcal{B}(\mathrm{H})-$ All bounded right $\mathbb{H}-$ linear operators on H.

Definition

If $T \in \mathcal{B}(H)$ then there exists unique operator $T^* \in \mathcal{B}(H)$ such that

$$\langle u|Tv\rangle = \langle T^*u|v\rangle\,,\ \text{for all } u, v \in H$$

called the adjoint of T.

Definition:

Let $T \in \mathcal{B}(H)$. Then T is said to be

1. self-adjoint, if $T^* = T$.
2. anti self-adjoint, if $T^* = -T$.
3. normal, if $TT^* = T^*T$.
4. unitary, if $T^*T = TT^* = I$.

Definition:

Let T $\in \mathcal{B}$(H). Then T is said to be compact if $\overline{\mathrm{T}(U)}$ is compact for every bounded subset U of H.

Notation:

- $\mathcal{K}$(H)$-$ All compact operators on quaternionic Hilbert sapce H.

Example:

Define $D\colon \ell^2(\mathbb{N},\mathbb{H}) \longrightarrow \ell^2(\mathbb{N},\mathbb{H})$ by

$$D(q_n) = \left(\frac{q_n}{n}\right), \text{ for all } (q_n) \in \ell^2(\mathbb{N},\mathbb{H}).$$

Then $D \in \mathcal{K}(\ell^2(\mathbb{N},\mathbb{H}))$.

Quaternion Matrices

- $M_n(\mathbb{H})$ - Ring of all $n \times n$ quaternion matrices.
- If $A \in M_n(\mathbb{H})$. Then $A = A_1 + A_2 \cdot j$, where $A_1, A_2 \in M_n(\mathbb{C})$.

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How to find eigenvalues of A ?

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How to find eigenvalues of A ?

- Define
$$\chi_A = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix} \in \mathrm{M}_{2n}(\mathbb{C}).$$
- If $\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathrm{M}_n(\mathbb{H})$, then there exists $x = x_1 + x_2 \cdot j \in \mathbb{H}^n$, where $x_1, x_2 \in \mathbb{C}^n$ such that $Ax = x \cdot \lambda$. Then

$$\begin{aligned}(A_1 + A_2 \cdot j)(x_1 + x_2 \cdot j) &= (A_1 x_1 - A_2 \overline{x_2}) + (A_1 x_2 + A_2 \overline{x_1}) \cdot j \\ &= x_1 \cdot \lambda + x_2 \cdot \overline{\lambda} \cdot j\end{aligned}$$

Quaternion Matrices

Upon comparision, we get

$$A_1 x_1 - A_2 \overline{x_2} = x_1 \cdot \lambda \tag{1}$$

$$A_1 x_2 + A_2 \overline{x_1} = x_2 \cdot \overline{\lambda}. \tag{2}$$

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From equations (1) & (2), we have

$$\begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix} \begin{pmatrix} x_1 \\ -\overline{x_2} \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ -\overline{x_2} \end{pmatrix}$$

and

$$\begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix} \begin{pmatrix} x_2 \\ \overline{x_1} \end{pmatrix} = \overline{\lambda} \begin{pmatrix} x_2 \\ \overline{x_1} \end{pmatrix}$$

Standard eigenvalues

- Both λ and $\overline{\lambda}$ are the eigenvalues of χ_A.
- similarly eigenvalues of χ_A are also an eigenvalues of A.

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Eigensphere:

Let $Ax = x \cdot q$ for some $q \in \mathbb{H}$. Then, for every $0 \neq s \in \mathbb{H}$, we have

$$A(x \cdot s) = Ax \cdot s = x \cdot q \cdot s = x \cdot s(s^{-1}qs).$$

Therefore, we have

- $[q]$ is an eigensphere and $E_{[q]} := \{x : Ax = x \cdot [q]\}$ is an eigenspace corresponding to $[q]$.

Standard eigenvalues

- The complex number $\mathrm{Re}(q) + i \cdot |\mathrm{Im}(q)| \in [q]$ called the standard eigenvalue of A.

Standard eigenvalues

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Theorem:

If $A \in M_n(\mathbb{H})$, then A has exactly $n-$ standard eigenvalues.

Corollary:

If $A \in M_n(\mathbb{H})$, then A has exactly $n-$ eigenvalues upto equivalance.

sphereical spectrum

Let $T \in \mathcal{B}(H)$ and $q \in \mathbb{H}$.

- Define $\Delta_q(T) := T^2 - T(q + \overline{q}) + I|q|^2$.
- The spherical spectrum,
  $\sigma_S(T) = \{q \in \mathbb{H} : \Delta_q(T) \text{ is not invertible } \}$
- The spherical point spectrum,
  $\sigma_{pS}(T) = \{q \in \mathbb{H} : N(\Delta_q(T)) \neq \{0\}\}$.

Properties:

- If $T = T^*$, then $\sigma_S(T) \subset \mathbb{R}$.
- If $T = -T^*$, then $\sigma_S(T) \subset Im(\mathbb{H})$.
- If $TT^* = T^*T = I$ and $T^* =$ -T, then $\sigma_{pS}(T) = \sigma_S(T) = \mathbb{S}$.

compact operators

Proposition:(Fashandi, 2014.)

If T$\in \mathcal{K}(H)$, then $N(\Delta_q(T))$ is finite dimensional, for all $0 \neq q \in \mathbb{H}$.

Theorem:(Fashandi, 2014.)

If T$\in \mathcal{K}(H)$ and $\inf\{\|\Delta_q(T)h\| : \|h\| = 1\} = 0$, for $0 \neq q \in \mathbb{H}$, then $q \in \sigma_{p^S}(T)$.

corollary:

If $T \in \mathcal{K}(H)$, then $\sigma_S(T) \setminus \{0\} = \sigma_{p^S}(T) \setminus \{0\}$.

Slice Representation

R. Ghiloni, V. Moretti and A. Perotti
Spectral properties of compact normal quaternionic operators
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Slice:

Fix $m \in \mathbb{S}$, then $\mathbb{C}_m = \{\alpha + m\beta : \alpha, \beta \in \mathbb{R}\}$ called the slice complex plane.

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Slice:

Fix $m \in \mathbb{S}$, then $\mathbb{C}_m = \{\alpha + m\beta : \alpha, \beta \in \mathbb{R}\}$ called the slice complex plane.

- Let $\mathrm{J} \in \mathcal{B}(\mathrm{H})$ be anti self-adjoint, unitary. Then $\mathrm{H}_{\pm}^{\mathrm{J}m} = \{u \in \mathrm{H} : \mathrm{J}u = \pm um\}$ are closed $\mathrm{H}_{\pm}^{\mathrm{J}m}$ are closed $\mathbb{C}_m$-linear subspaces.
- The orthonormal basis of $\mathrm{H}_{+}^{\mathrm{J}m}$ is also an orthonormal basis of H.
- H as $\mathbb{C}_m$- Hilbert space, $\mathrm{H} = \mathrm{H}_{+}^{\mathrm{J}m} \oplus \mathrm{H}_{-}^{\mathrm{J}m}$.

Slice Representation

- Let T $\in \mathcal{B}(\mathrm{H})$ be normal and JT $=$ TJ, JT* $=$ T* J. Then $\mathrm{H}_{\pm}^{Jm}$ are invariant under T.
- Define $\mathrm{T}_{\pm} = T|_{H_{\pm}^{Jm}} : \mathrm{H}_{\pm}^{Jm} \longrightarrow \mathrm{H}_{\pm}^{Jm}$ are $\mathbb{C}_m$ linear.
- If T$\in \mathcal{K}(\mathrm{H})$, then also $\mathrm{T}_{\pm}$.
- Let $\{e_n\}$ be an orthonormal basis of H_{+}^{Jm}. For $u \in \mathrm{H}$, we have

$$\mathrm{T}_{+}u = \sum_{n=1}^{\infty} e_n \lambda_n \langle e_n | u \rangle$$

Then

$$\mathrm{T}u = \sum_{n=1}^{\infty} e_n \lambda_n \langle e_n | u \rangle .$$

Slice Representation

Example:

Let $\mathsf{H} = \ell^2(\mathbb{N}, \mathbb{H})$. Define $T\colon \mathsf{H} \to \mathsf{H}$ by

$$\mathrm{T}(q_n) = \left(i \cdot q_1, j \cdot q_2, k \cdot q_3, \frac{k}{4} \cdot q_4, \frac{k}{5} \cdot q_5, \cdots\right), \text{ for all } (q_n) \in H.$$

Then, we define $\mathrm{J}\colon \mathsf{H} \to \mathsf{H}$ by

$$\mathrm{J}(q_n) = (i \cdot q_1, j \cdot q_2, k \cdot q_3, k \cdot q_4, \cdots), \text{ for all } (q_n) \in H.$$

Slice Representation

Example:

Define $H_{\pm}^{Jj} := \{(q_n)_{n\in\mathbb{N}} : J((q_n)_{n\in\mathbb{N}}) = (q_n \cdot j)_{n\in\mathbb{N}}\}$.

- $$f_1 = (\frac{1+i+j-k}{2}, 0, 0\cdots)\ ;\ f_2 = (0, 1, 0\cdots),$$
  $$f_3 = (0, 0, \frac{1+i+j+k}{2}, 0, 0\cdots)\ ;$$
  $$f_n = (0, 0, \cdots, \frac{1+i+j-k}{n}, 0, 0\cdots),\ \text{for} n \geq 4.$$
- $\{f_n : n \in \mathbb{N}\}$ is a set of eigenvectors of T_+, which forms an orthonormal basis for H_+^{Jj}.

Slice Representation

Example:

For every $u \in$H, we have

$$Tu = T(\sum_{n=1}^{\infty} f_n \langle f_n | u \rangle)$$

$$= f_1 \cdot j \cdot \left(\overline{\frac{1+i+j-k}{2}} \right) \cdot q_1 + f_2 \cdot j \cdot q_2 +$$

$$f_3 \cdot j \cdot \left(\overline{\frac{1+i+j+k}{2}} \right) \cdot q_3 +$$

$$\sum_{n \geq 4} f_n \cdot \frac{j}{n} \cdot \left(\overline{\frac{1+i+j+k}{2}} \right) \cdot q_n$$

Proposition:

If T$=$ T*, then

1. $N(\Delta_r(T)) = N(T - r \cdot I)$, for all $r \in \mathbb{R}$.
2. $\sigma_{p^S}(T) = \sigma_p(T)$.
3. $\sigma_S(T) = \sigma(T)$.

Lemma:

If $T \in \mathcal{K}(H)$ and $T^* = T$. Then $\pm\|T\| \in \sigma_{p^S}(T)$.

compact self-adjoint operator

Theorem:

Let $\mathrm{T} \in \mathcal{K}(\mathrm{H})$ be self-adjoint. Then there exists a system of eigenvectors $\phi_1, \phi_2, \phi_3, \cdots$ correspoding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$ such that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots$ and

$$\mathrm{T}u = \sum_{n=1}^{\infty} \phi_n \lambda_n \langle \phi_n | u \rangle \,, \text{ for all } u \in H.$$

Moreover, if (λ_n) is infinite then $\lambda_n \longrightarrow 0$, as $n \to \infty$

Simultaneous diagnolization

Theorem:

If A, B $\in \mathcal{K}(H)$ be self-adjoint and AB = BA, then there exists a system of eigenvectors $\phi_1, \phi_2, \phi_3, \cdots$ correspoding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots$ of A and $\mu_1, \mu_2, \cdots$ of B such that

$$Au = \sum_{n=1}^{\infty} \phi_n \lambda_n \langle \phi_n | u \rangle \,; \; Bu = \sum_{n=1}^{\infty} \phi_n \mu_n \langle \phi_n | u \rangle \,, \text{ for all } u \in H.$$

Theorem:

If $T \in \mathcal{B}(H)$ be normal, then there exists three mutually commuting operators A, B and J such that

$$T = A + J \cdot B.$$

Moreover, $A = A^*, B \geq 0$ and J is anti self-adjoint, unitary.

Lemma:

If J be anti self-adjoint, unitary and $B \geq 0$ such that $JB = BJ$ then

$$\sigma_{pS}(JB) = \mathbb{S} \cdot \sigma_{pS}(B) = \{m \cdot r : m \in \mathbb{S}, r \in \sigma_{pS}(B)\}.$$

Compact normal operator

Theorem:

Let $\mathrm{T} \in \mathcal{K}(\mathrm{H})$ be normal. Then there exists a system of eigenvectors $\phi_1, \phi_2, \phi_3, \cdots$ correspoding to the eigenvalues $q_1, q_2, q_3, \cdots$ such that

$$\mathrm{T}u = \sum_{n=1}^{\infty} \phi_n \cdot q_n \langle \phi_n | u \rangle , \ \forall \ u \in \mathrm{H}.$$

Moreover,

1. if (q_n) is infinite then $q_n \to 0$.
2. $\sigma_S(\mathrm{T}) = \{[q_n] : n \in \mathbb{N}\} = \alpha + \mathbb{S} \cdot \beta; \alpha \in \sigma_{p^S}(\mathrm{A})$ and $\beta \in \sigma_{p^S}(\mathrm{B})$.
3. $\{\alpha + i \cdot \beta : \alpha \in \sigma_{p^S}(\mathrm{A}), \beta \in \sigma_{p^S}(\mathrm{B})\}$, the standard spectral values.

comparision

- Define

$$B((q_n)) = (q_1, q_2, q_3, \frac{q_4}{4}, \frac{q_5}{5}, \cdots), \text{ for all } (q_n) \in \ell^2(\mathbb{N}, \mathbb{H}).$$

  Then $T = JB$.

- $e_n(j) = \delta_{nj}$ is an orthonormal basis of eigenvectors of B. We have

$$T((q_n)) = \sum_{n=1}^{\infty} e_n \alpha_n \langle e_n | (q_n) \rangle, \text{ for all } (q_n) \in H.$$

  where $\alpha_1 = i, \alpha_2 = j, \alpha_3 = k$ and $\alpha_n = \frac{k}{n}$ for $n \geq 4$.

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Thank you