# On Hilbert Schmidt and Schatten P Class Operators in P-adic Hilbert Spaces

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In this talk, we introduce the well known **Hilbert Schimdt and Schatten-P class** operators on p-adic Hilbert spaces. We also show that the **Trace class** operators in p-adic Hilbert spaces contains the class of completely continuous operators, which contains the Schatten Class operators.

Let $K$ be a complete ultrametric valued field. Classical examples of such a field include the field $\mathbb{Q}_p$ of $p$-adic numbers where $p$ is a prime, and its various extensions .

An ultrametric Banach space $E$ over $\mathbb{K}$ is said to be a free Banach space if there exists a family $(e_i)_{i\in I}$ of elements of $E$ such that each element $x \in E$ can be written uniquely as $\mathbf{x} = \sum_{\mathbf{i}\in\mathbf{I}} \mathbf{x_i e_i}$ that is, $\lim_{\mathbf{i}\in\mathbf{I}} \mathbf{x_i e_i} = \mathbf{0}$ and $\|\mathbf{x}\| = \sup_{\mathbf{i}\in\mathbf{I}} |\mathbf{x_i}|\, \|\mathbf{e_i}\|$ Such $(e_i)_{i\in I}$ is called an "**orthogonal base**" for $E$, and if $\|e_i\| = 1$, for all $i \in I$, then $(e_i)_{i\in I}$ is called an "**orthonormal base**".

Throughout this discussion, we consider free Banach spaces over $\mathbb{K}$ and we shall assume that the index set $I$ is the set of natural numbers $\mathbb{N}$.

For a free Banach space $E$, let $E^*$denote its topological dual and $\mathcal{B}(E)$ the set of all bounded linear operators on $E$. Both $E^*$ and $\mathcal{B}(E)$ are endowed with their respective usual norms.

For $(u, v) \in E \times E^*$, we define $(v \otimes u)$ by:

$$\forall x \in E, \ (v \otimes u)(x) = v(x)\, u = \langle v, x \rangle\, u,$$

then $(v \otimes u) \in \mathcal{B}(E)$ and $\|v \otimes u\| = \|v\| \,.\, \|u\|$.

Let $(e_i)_{i\in\mathbb{N}}$ be an orthogonal base for $E$, then one can define $e_i' \in E^*$ by:

$$x = \sum_{i\in\mathbb{N}} x_i e_i, \quad e_i'(x) = x_i.$$

It turns out that $\|e_i'\| = \dfrac{1}{\|e_i\|}$, furthermore, every $x^{'} \in E^*$ can be expressed as $x^{'} = \sum_{i\in\mathbb{N}} \langle x', e_i\rangle \, e_i'$ and $\|x'\| = \sup_{i\in\mathbb{N}} \dfrac{|\langle x', e_i\rangle|}{\|e_i\|}$.

Each operator $A$ on $E$ can be expressed as a point-wise convergent series, that is, there exists an infinite matrix $(a_{ij})_{(i,j)\in\mathbb{N}x\mathbb{N}}$with coefficients in $\mathbb{K}$, such that:

$A = \sum_{ij} a_{ij}(e'_j \otimes e_i)$,

and for any $j \in \mathbb{N}$ , $\lim\limits_{i\to\infty} |a_{ij}|\,\|e_i\| = 0$.

Moreover, for any $s \in \mathbb{N}$

$$Ae_s = \sum_{i\in\mathbb{N}} a_{is}e_i \text{ and } \|A\| = \sup_{i,j}\frac{|a_{ij}|\,\|e_i\|}{\|e_j\|}.$$

Let $\omega = (\omega_i)_{i\in I}$ be a sequence of non-zero elements in the valued field $\mathbb{K}$ and

$$E_\omega = \left\{ x = (x_i)_{i\in\mathbb{N}} \mid \forall i, \;\; x_i \in \mathbb{K} \;\text{ and } \lim_{i\to\infty} |x_i|\,|\omega_i|^{1/2} = 0 \right\}.$$

Then, it is easy to see that $x = (x_i)_{i\in\mathbb{N}} \in E_\omega$ if and only if $\lim_{i\to\infty} x_i^2\omega_i = 0$. The space $E_\omega$ is a free Banach space over $\mathbb{K}$, with the norm given by:

$$x = (x_i)_{i\in\mathbb{N}} \in E_\omega, \;\; \|x\| = \sup_{i\in\mathbb{N}} |x_i|\,|\omega_i|^{1/2}.$$

In fact, $E_\omega$ is a free Banach space and it admits a *canonical orthogonal base*, namely, $(e_i)_{i\in\mathbb{N}}$, where $e_i = (\delta_{ij})_{j\in\mathbb{N}}$, where $\delta_{ij}$ is the usual Kronecker symbol. We note that for each $i$, $\|e_i\| = |\omega_i|^{1/2}$. If $|\omega_i| = 1$, we shall refer to $(e_i)_{i\in\mathbb{N}}$ as the *canonical orthonormal base.*

Let $\langle , \rangle : E_\omega \times E_\omega \to \mathbb{K}$ be defined by: $\forall x, y \in E_\omega,\ x = (x_i)_{i\in\mathbb{N}},\ y = (y_i)_{i\in N},\ \ \langle x, y\rangle = \sum_{i\in\mathbb{N}} \omega_i x_i y_i.$

Then, $\langle , \rangle$ is a symmetric, bilinear, non-degenerate form on $E_\omega$, with value in $\mathbb{K}$, The space $E_\omega$ endowed with this form $\langle , \rangle$ is called a *p-adic Hilbert space.*

It is not difficult to see that this “inner product” satisfies the Cauchy-Schwarz-Bunyakovsky inequality:

$$x, y \in E_\omega,\ |\langle x, y\rangle| \leq \|x\| \,.\, \|y\| \,.$$

and on the cannonical orthogonal base, we have:

$$\langle e_i, e_j\rangle = \omega_i \delta_{ij} = \begin{cases} 0,\ if\ i \neq j \\ \omega_i,\ if\ i = j. \end{cases}$$

In sharp contrast to classical Hilbert spaces, in p-adic Hilbert space, there exists $x \in E_\omega$, such that

$$|\langle x, x\rangle| = 0 \text{ while } \|x\| \neq 0.$$

Again in sharp contrast to classical Hilbert space, all operators in $\mathcal{B}(E_\omega)$ may not have an adjoint.

So, we denote by

$$\mathcal{B}_0(E_\omega) = \{A \in \mathcal{B}(E_\omega) : \ \exists A^* \in \mathcal{B}(E_\omega)\}.$$

It is well known that the operator

$A = \sum_{ij} a_{ij}(e'_j \otimes e_i) \in \mathcal{B}_0(E_\omega) \Longleftrightarrow \quad \forall i,\ \lim_{j\to\infty} \frac{|a_{ij}|}{|\omega_j|^{1/2}} =$

$0$, and $A^* = \sum_{ij} \omega_i^{-1} \omega_j a_{ji}(e'_j \otimes e_i)$

The space $\mathcal{B}_0(E_\omega)$ is stable under the operation of

taking an adjoint and for any $A \in \mathcal{B}_0(E_\omega) : (A^*)^* =$

$A$ and $\|A\| = \|A^*\|$.

## 1. Schatten-p class Operators

Let us now recall few well known results from the Scahtten class operators in classical Hilbert spaces.

Let $H$ be a Hilbert space and let $T : H \longrightarrow H$ be a compact operator. Then the operator $|T|$, defined as $|T| = (TT^*)^{\frac{1}{2}}$ is also a compact operator hence its spectrum consists of at most countably many distinct Eigenvalues. Let $\sigma_j(|T|)$ denote the eigen values of $|T|$. These numbers are called the singular numbers of $T$. Let $(\sigma_j(|T|)$ denote the non increasing sequence of the singular numbers of $T$, every number counted according to its multiplicity as an eigenvalue of $|T|$.

**Definition 1.1.** *For* $0 < r < \infty$, *let* $B_r(H)$ *denote the following*

$$B_r(H) = \{T \in B(H) : \sum_j \sigma_(j)(|T|^r) < \infty|\}$$

*Then* $B_r(H)$ *is called the Scahtten-r-class operators of* $H$.

The difficulty that arises in defining a *Scahtten r-class* operator in a p-adic Hilbert space is there is no well developed theory of compact operator and its representation and there is no notion of positive operators existing in the literature, the reason being the inner product is defined on the abstract ultramteic field $\mathbb{K}$, as opposed to Real or Complex fields. So, to introduce this class in p-adic Hilbert spaces we first look at the following straightforward observation for $B_r(H)$. For more details, see [14].

**Proposition 1.2.** *Let* $B_r(H)$ *denote the Schatten* $r$*-class operators of* $H$ *and let* $\{e_j\}$ *be the eigenvectors corresponding to the eigenvalues* $\{\sigma_j(|T|)\}$*. Then* $T \in B_r(H)$ *if and only if* $\sum_j \|T(e_j)\|^r < \infty$*.*

**Definition 1.3.** *An operator $A \in B(E_\omega)$ is said to be in Schatten-p class ( $1 \leq p < \infty$) denoted by $B_p(E_\omega)$ if*

$$\|A\|_p = (\sum \frac{\|A(e_s)\|^p}{|\omega_s|^{\frac{p}{2}}})^{\frac{1}{p}} < \infty.$$

**Remark 1.4.** *If $p = 2$, we get the p-adic Hilbert Schimdt operator.*

**Proposition 1.5.** *If* $A \in B_2(E_\omega)$, *then*

(i) $A$ *has an adjoint* $A^*$.

(ii) $\|A\|_2 = \|A^*\|_2$ *and* $A^* \in B_2(E_\omega)$. *In particular,* $B_2(E_\omega) \subseteq B_0(E_\omega)$.

(iii) $\|A\| \leq \|A\|_2$

**Example 1.6.** *Suppose* $\mathbb{K} = \mathbb{Q}_p$. . *Also, let,* $\omega_i =$

$p^{i+1}$, *hence* $|\omega_i| = p^{-i-1} \to 0$ *as* $i \to \infty$.

*We define an operator* $A$ *on* $E_\omega$ *as,* $A = \sum a_{ij} e_j^{'} \otimes$

$e_i$ *where,* $a_{ij} = \omega_i^j$

*Then, clearly* $\lim_i |a_{ij}| \|e_i\| = 0$, *hence* $A \in B(E_\omega)$

*Also,* $\|A(e_s)\| = \|\sum_{i=0}^{\infty} a_{is} e_i\| = \sup_i \frac{1}{p^{is+s+i+1}} =$

$\frac{1}{p^{2s+2}}$ *Hence* $\frac{\|A(e_s)\|}{|\omega_s|^{\frac{1}{2}}} = \frac{p^{\frac{s+1}{2}}}{p^{2s+2}}$ *Therefore,* $\sum_{s=1}^{\infty} \frac{\|A(e_s)\|^r}{|\omega_s|^{\frac{r}{2}}} =$

$\sum \frac{1}{p^{\frac{3r(s+1)}{2}}} < \infty$. *Hence* $A \in B_r(E_\omega)$.

For every $A \in \mathcal{B}_p(E_\omega)$, we define its *Schatten-p norm* as

$$|||A||| = \|A\|_p$$

In the space $\mathcal{B}_2(E_\omega)$, we introduce a symmetric bi-linear form, namely$A, \; B \in \mathcal{B}_2(E_\omega), \; \langle A, B\rangle = \sum_s \frac{\langle Ae_s, Be_s\rangle}{\omega_s}$. Its relationship with the Hilbert-Schmidt norm is through the Cauchy-Schwarz-Bunyakovsky inequality.

**Theorem 1.7.** $A,\ B \in \mathcal{B}_2(E_\omega),\ |\langle A, B\rangle| \leq |||A|||\,.\,|||B|||\,.$

Proof: $A = \sum\limits_{i,j} a_{ij}\left(e'_j \otimes e_i\right)$ and $B = \sum\limits_{i,j} b_{ij}\left(e'_j \otimes e_i\right)$,

then $| < A, B > | = \left|\sum\limits_s \frac{\langle Ae_s, Be_s\rangle}{\omega_s}\right| \leq \sup\limits_s \left|\frac{\langle Ae_s, Be_s\rangle}{\omega_s}\right|$

$$\leq \sup_s \frac{\|Ae_s\|\,\|Be_s\|}{|\omega_s|}$$

$$\leq \left(\sup_s \frac{\|Ae_s\|}{|\omega_s|^{1/2}}\right) \cdot \left(\sup_s \frac{\|Be_s\|}{|\omega_s|^{1/2}}\right)$$

$$= \sup_s \frac{\|Ae_s\|}{\|e_s\|}.\ \sup_s \frac{\|Be_s\|}{\|e_s\|}$$

$$= \|A\|\,.\,\|B\| \leq |||A|||\,.\,|||B||| \quad \square$$

**Proposition 1.8.** *For any padic Hilbert space $E_\omega$, the following is true*

(i) $B_2(E_\omega)$ *is a two ideal of* $B_0(E_\omega)$.

(ii) *If* $S, T \in B_2(E\omega)$, *then* $ST \in B_2(E_\omega)$ *and* $|||ST||| \leq |||S||| \; |||T|||$, *i. e.* $B_2(E_\omega)$ *is a normed algebra with respect to the Hilbert Schmidt norm*

## 2. Completely Continuous operators and Schatten -p class

**Definition 2.1.** *An operator $A \in \mathcal{B}(E_\omega)$ is completely continuous if it is the limit, in $\mathcal{B}(E_\omega)$, (i.e.a uniform limit) of a sequence of operators of finite ranks. We denote by $\mathcal{C}(E_\omega)$ the subspace of all completely continuous operators on $E_\omega$.*

**Proposition 2.2.** *Every Scahtten-p class operator is completely continuous,*

*i.e,* $\mathcal{B}_p(E_\omega) \subset \mathcal{C}(E_\omega)$.

## 3. Trace and Schatten-p class

One important notion in the classical theory is that of *trace.*

**Definition 3.1.** *For $A \in \mathcal{L}(E_\omega)$, we define the trace of $A$ to be* $Tr A = \sum_s \frac{\langle Ae_s, e_s \rangle}{\omega_s}$ *if this series converges in $\mathbb{K}$. We denote by $\mathcal{TC}(E_\omega)$ the subspace of all Trace class operators, namely, those operators for which the trace exists.*

**Remark 3.2.** *Let* $A = \sum\limits_{i,j} a_{ij} \left(e'_j \otimes e_i\right)$ *and suppose that* $A \in \mathcal{TC}\left(E_\omega\right)$ *then, the series* $\sum\limits_{k} a_{kk}$ *converges and* $TrA = \sum\limits_{k} a_{kk}$.

**Theorem 3.3.** $B_p(E_\omega) \subset \mathcal{TC}\left(E_\omega\right)$, *i.e., every Scahtten-p Class operator has a trace.*

**Theorem 3.4.** *$B_2(E_\omega) \subset \mathcal{T}(E_\omega)$, i.e., every Hilbert-Schmidt operator has a trace.*

*Proof.* Let $A = \sum\limits_{i,j} a_{ij} \left(e'_j \otimes e_i\right)$ be a Hilbert-Schmidt operator. $\sum\limits_{k} \frac{\|Ae_k\|^2}{|\omega_k|}$ converges, hence, $\lim\limits_{k} \frac{\|Ae_k\|^2}{|\omega_k|} = 0$. We observe that for any $k$, $|a_{kk}|^2 |\omega_k| \leq \sup\limits_{i} |a_{ik}|^2 |\omega_i| = \|Ae_k\|^2$, therefore, $|a_{kk}|^2 \leq \frac{\|Ae_k\|^2}{|\omega_k|}$, which implies that $\lim\limits_{k} |a_{kk}| = 0, i.e. \lim\limits_{k} a_{kk} = 0$ and the series $\sum\limits_{k} a_{kk}$ converges in $\mathbb{K}$. $\square$

**Remark 3.5.** *The above theorem is in sharp contrast with the classical case, since* $\exists$ *Schatten-p Class operators which do not have traces. In fact, for a classical Hilbert space* $H$*, the trace class operator* $B_1(H)$ *is a subset of the algebra of Hilbert Schimdt operators in* $H$*. In the padic case however, we definitely have* $B_1(E_\omega) \subseteq B_2(E_\omega)$ *but the class of operators with a trace is a larger class.*

We also have the following

**Theorem 3.6.** *Suppose* $A \in B_p(E_\omega)$ *and* $B \in B_0(E_\omega)$, *then* $Tr(AB) = Tr(BA)$.

## 4. Examples

**Example 4.1.** *Assume that the filed* $\mathbb{K} = \mathbb{Q}_p$ *and consider the linear operator on* $E_\omega$ *defined by* $Ae_s = \sum_{k=0}^{+\infty} a_{ks}e_k$*, where* $a_{ks} = \frac{p^{s+k}}{1+p+p^2+....+p^s}$ *if* $k \leq s$ *and* $a_{ks} = 0$ *if* $k > s$*. Suppose that* $|\omega_k| \geq 1$ *for all* $k$ *and that* $\sup_k |\omega_k| \leq M$ *for some positive real number* $M$*, then, the operator* $A$ *defined above is in* $\mathcal{B}_p(E_\omega)$*.*

**Example 4.2.** $\mathbb{K} = \mathbb{Q}_p$ *and suppose that* $|\omega_s| \to \infty$ *as* $s \to \infty$.

*For integers* $m \geq 1$ *and* $n \geq 0$, *let*

$$A^{(m,n)} = \sum_{i.j} \frac{1}{\omega_i^m \omega_j^n} \left(e'_j \otimes e_i\right).$$

*Then,* $A^{(m,n)} \in \mathcal{B}_p\left(E_\omega\right)$.

**Example 4.3.** *Assume that the series* $\sum_s |\omega_s|$ *converges and fix a vector* $x = (x_s)_{s\in\mathbb{N}} \in E_\omega$*. Let* $A$ *be such that* $Ae_s = \langle x, e_s\rangle e_s = x_s\omega_s e_s$*. Then* $A \in \mathcal{B}_2(E_\omega)$*. Now,* $A = \sum_{i,j} (\delta_{ij} x_j \omega_j)\left(e_j' \otimes e_i\right)$*.*

*Since* $\lim_s |\omega_s| = 0$*, then* $A \in \mathcal{B}(E_\omega)$*.*

*Moreover,* $\dfrac{\|Ae_s\|^2}{|\omega_s|} = |x_s|^2 |\omega_s|^2$

$= |\langle x, e_s\rangle|^2$

$\leq \|x\|^2 |\omega_s|$*, and hence* $A \in \mathcal{B}_2(E_\omega)$*.*

# References


[1] S. Albeverio, J.M. Bayod, C. Perez-Garcia, R. Cianci, Y. Khrennikov, *Non-Archimedean analogues of orthogonal and symmetric operators and p-adic Quantization*, Acta Appl. Math. **57** (1999), no. 3, 205–237.

[2] S.Basu, T.Diagana, F.Ramarosan, *A p-adic Version of Hilbert-Schmidt Operators and Applications*, J.Analysis and Appl. 2, no. 3,(2004) 173–188.

[3] B.Diarra, *An Operator on some ultrametric Hilbert Spaces*, J. Analysis 6 (1998), 55-74.

[4] B.Diarra, *Geometry of the p-adic Hilbert Spaces*, Preprint (1999).

[5] A. Y. Khrennikov, *Mathematical methods in non-Archimedean Physics.* (Russian) Uspekhi Math Nauk **45** (1990), no. 4(279), 79-110, 192; translation in Russian Math. Surveys **45** (1990), no. 4, 87-125.

[6] A. Y. Khrennikov, *Generalized functions on a non-Archimedean superspace.*(Russian) Izv. Akad. Nauk SSSR Ser. Math. **55** (1991), no. 6, 1257-1286; English translation in Math. USSR - Izv. **39** (1992), no. 3, 1209-1238.

[7] A. Y. Khrennikov, *p-adic quantum mechanics with p-adic wave functions*, J. Math. Phys, **32** (1991), no. 4, 932–937.

[8] H.A. Keller and H. Ochsensius, *Algebras of bounded Operators*, Jour. Theor. Phys., **33** (1994), 1–11

[9] S. Ludkovsky and B. Diarra, *Spectral Integration and Spectral Theory for Non-Archimedean Banach Spaces*, Int. J. Math. Math. Sci., **31**, no. 7, (2002), 421–442.

[10] H. Ochsensius, W.H. Schikhof, *Banach Spaces over fields withan infinite rank valuation, p-adic Functional Analysis* (Poznan,1998), Dekker, New York, 1999, 233–293.

[11] J. P. Serre, *Endomorphismes completement continus des espaces de Banach p-adiques*, Publ. Math. I.H.E.S., **12** (1962), 69–85.

[12] J. R. Retherford, *Hilbert Space: Compact Operators and the Trace Theorem*, London Math. Soc. Student Texts **27**, Cambridge Univ. Press, New York, 1993.

[13] A.C.M. van Rooij, *Non-arcimedean Functional Analysis*, MarcelDekker Inc, New York (1978).

[14] J. Weidemann, *Linear Operators in Hilbert Spaces*, GTM68, Springer Verlag, New York, 1980.