Cohomology for Super-Product Systems (Joint work with Oliver T. Margetts)

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Definition

A super-product system of Hilbert spaces is a one parameter family of separable complex Hilbert spaces $\{H_t : t > 0\}$, together with isometries

$$U_{s,t} : H_s \otimes H_t \ \mapsto H_{s+t} \text{ for } s, t \in (0, \infty),$$

satisfying the axioms of associativity and measurability.

(i) (Associativity) For any $s_1, s_2, s_3 \in (0, \infty)$

$$\begin{array}{ccc} H_{s_1} \otimes H_{s_2} \otimes H_{s_3} & \xrightarrow{1_{H_{s_1}} \otimes U_{s_2,s_3}} & H_{s_1} \otimes H_{s_2+s_3} \\ {\scriptstyle U_{s_1,s_2} \otimes 1_{H_{s_3}}} \downarrow & & \downarrow {\scriptstyle U_{s_1,s_2+s_3}} \\ H_{s_1+s_2} \otimes H_{s_3} & \xrightarrow[U_{s_1+s_2,s_3}]{} & H_{s_1+s_2+s_3} \end{array}$$

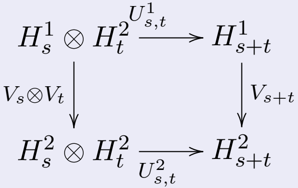
commutes.

(ii) (Measurability)

The super-product system of Hilbert spaces is a generalisation of Arveson's product system of Hilbert spaces. A super-product system is an Arveson product system if the isometries $U_{s,t}$ are unitaries

Definition

By an isomorphism between super-product systems $(H^1_t, U^1_{s,t})$ and $(H^2_t, U^2_{s,t})$ we mean a family of unitary operators $V_t : H^1_t \mapsto H^2_t$ satisfying



$$V_{s+t} U^1_{s,t} = U^2_{s,t}(V_s \otimes V_t).$$

$+$ (Measurability)

$\mathbb{N}$ denotes the set of natural numbers, and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

For $S \subseteq \mathbb{R}$, $L^2(S, \mathrm{k})$ denote the square integrable functions from S taking values in a complex separable Hilbert space k. $L^2_{loc}(S, \mathrm{k})$ denotes the functions which are square integrable on compact subsets.

Throughout we denote by $(T_t)_{t \geq 0}$ the right shift semigroup of isometries on $L^2((0, \infty), \mathrm{k})$ defined by

$$\begin{aligned} (T_t f)(s) &= 0, \quad s < t, \\ &= f(s-t), \quad s \geq t, \end{aligned}$$

for $f \in L^2((0, \infty), \mathrm{k})$.

Example

(*CAR product systems*) $H^{\mathrm{k}}(t) = \Gamma(L^2((0,t),\mathrm{k})$-the anti-symmetric Fock space- and $U_{s,t} : H_s \otimes H_t \mapsto H_{s+t}$ is the extension of

$$U_{s,t}((\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_m)\otimes(\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_n)) \\ = T_s\eta_1 \wedge T_s\eta_2 \wedge \cdots \wedge T_s\eta_n \wedge \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_m$$

where $\xi_1, \xi_2, \ldots, \xi_m \in L^2((0,s),\mathrm{k})$ and $\eta_1, \eta_2, \ldots, \eta_n \in L^2((0,t),\mathrm{k})$. Then $(H^{\mathrm{k}}(t), U_{s,t})$ is product system.

Example

(*Clifford super-product system*) Let k be a separable Hilbert space.

$$H^{e,\mathrm{k}}(t) := \bigoplus_{n \in 2\mathbb{N}_0} L^2([0,t];\mathrm{k})^{\wedge n} \subseteq H^{\mathrm{k}}(t);$$

with the isometries given by the restriction of the unitaries the antisymmetric product systems respectively, these families of Hilbert spaces are super-product systems.

Example

(*CAR super-product systems*) Let k be a separable Hilbert space. Define

$$E^{\mathrm{k}}(t) := \bigoplus_{n_1+n_2 \in 2\mathbb{N}_0} L^2([0,t];\mathrm{k})^{\wedge n_1} \otimes L^2([0,t];\mathrm{k})^{\wedge n_2}.$$

The isometries are given by the restriction of the unitaries of the product system $(H^{\mathrm{k}}(t) \otimes H^{\mathrm{k}}(t), U_{s,t} \otimes U_{s,t})$ respectively, these families of Hilbert spaces are super-product systems.

Example

(*GICAR super-product systems*) Let k be a separable Hilbert space. Define

$$E_0^{\mathrm{k}}(t) := \bigoplus_{n\in\mathbb{N}_0} L^2([0,t];\mathrm{k})^{\wedge n} \otimes L^2([0,t];\mathrm{k})^{\wedge n}.$$

The isometries are given by the restriction of the unitaries of the product system $(H^{\mathrm{k}}(t) \otimes H^{\mathrm{k}}(t), U_{s,t} \otimes U_{s,t})$ respectively, these families of Hilbert spaces are super-product systems.

Definition

A unit for a super-product system $(H_t, U_{s,t})$ is a measurable section $\{u_t : u_t \in H_t\}$ satisfying

$$U_{s,t}(u_s \otimes u_t) = u_{s+t} \ \forall \ s,t \in (0,\infty).$$

A super-product system is called spatial if it admits a unit. We usually fix a special unit, denoted by $\{\Omega_t \in H_t\}$, in a spatial super-product system, and call that as the canonical unit.

For the antisymmetric product system $H^{\mathrm{k}}(t)$, we set the vacuum vector ($1 \in \mathbb{C}$ in the $0-$particle space) as the canonical unit.

Let $H = (H_t, U_{s,t})$ be a spatial super-product system, with canonical unit $\{\Omega_t\}$. The embeddings

$$\iota_{s,t} : H_s \mapsto H_{s+t} \qquad \xi \mapsto U_{s,t}(\xi \otimes \Omega_t)$$

allow us to construct an inductive limit of the family of Hilbert spaces $(H_s)_{s>0}$, which we denote by H_∞, together with $\iota_s : H_s \to H_\infty$.

We can also define a second family of embeddings

$$\kappa_{s,t} : H_t \mapsto H_{s+t} \qquad \xi \mapsto U_{s,t}(\Omega_s \otimes \xi).$$

Thanks to the associativity axiom, the squares

$$\begin{array}{ccc} H_s & \xrightarrow{\iota_{s,t}} & H_{s+t} \\ \downarrow{\scriptstyle \kappa_{r,s}} & & \downarrow{\scriptstyle \kappa_{r,s+t}} \\ H_{r+s} & \xrightarrow[\iota_{r+s,t}]{} & H_{r+s+t} \end{array}$$

commute for all $r, s, t > 0$. So there exist isometries $(\kappa_t : H_\infty \mapsto H_\infty)_{t \geq 0}$, which define an action of the semigroup $\mathbb{R}_+$ on H_∞, satisfying $\kappa_s \iota_t = \iota_{s+t} \kappa_{s,t}$.

Set $\iota(\Omega_s) = \Omega_\infty$ for all $s \in (0, \infty)$. We say a function $f : \mathbb{R}_+^n \to H_\infty \ominus \mathbb{C}\Omega_\infty$ is adapted if $f(s_1, \ldots, s_n) \in \iota(H_{s_1+\cdots+s_n})$ for all $s_1, \ldots, s_n > 0$.

Let $C^n = C^n(H, \Omega)$ denote the space of all adapted continuous maps $f : \mathbb{R}_+^n \to H_\infty \ominus \mathbb{C}\Omega_\infty$, and $d^n : C^n(H, \Omega) \to C^{n+1}(H, \Omega)$ be defined by

$$d^n f(s_1, \ldots, s_{n+1}) := \kappa_{s_1} f(s_2, \ldots, s_{n+1})$$
$$+ \sum_{i=1}^{n} (-1)^n f(s_1, \ldots, s_i + s_{i+1}, \ldots, s_{n+1}) + (-1)^{n+1} f(s_1, \ldots, s_n)$$

.

Lemma

(C, d) *forms a cochain complex.*

Definition

For all $n \geq 0$, the collection of n-cocycles for (H, Ω) is the space $Z^n(H, \Omega) := Ker(d^n)$ and the collection of n-coboundaries is defined by $B^1(H, \Omega) = 0$ and, for $n \geq 2$, $B^n(H, \Omega) = \mathrm{Ran}\,(d^{n-1})$. The n-th cohomology group is the space $\mathcal{H}^n(E, \Omega) := Z^n(E, \Omega)/B^n(E, \Omega)$.

Definition

Let $H = (H_t, U_{s,t})$ be a super-product system with canonical unit Ω. A defective n-cochain for (H, Ω) is a member of $C^n(E, \Omega)$ satisfying

$$a(s_1, \ldots, s_n) \perp \iota_{s_1+\cdots+s_n} U_{s_1,\ldots,s_n}(H_{s_1} \otimes \cdots \otimes H_{s_n})$$

for all $s_1, \ldots, s_n > 0$, where $U_{s_1,\ldots,s_n} : H_{s_1} \otimes \cdots \otimes H_{s_n} \mapsto H_{s_1,\ldots,s_n}$ is canonical unitary map determined uniquely by the associativity axiom.

We denote the space of defective n-cochains by $C^n_{def}(H, \Omega)$ and define, similarly, the collection of defective n-cocycles $Z^n_{def}(H, \Omega)$ and coboundaries $B^n_{def}(H, \Omega)$. The corresponding quotient $\mathcal{H}^n_{def}(H, \Omega)$ is the n-th defective cohomology group.

Corollary

Let $H = (H_t, U_{s,t})$ *and* $H = (K_t, U'_{s,t})$ *be two spatial super-product systems with canonical units* Ω *and* Ω' *respectively. Let* $V_t : H_t \to K_t$ *be an isomorphism of super-product systems taking the unit* $(\Omega_t)_{t \geq 0}$ *to* $(\Omega'_t)_{t \geq 0}$*. Then, for each* $n \geq 0$*, there is an isomorphism* $\Phi : C^n_{def}(H, \Omega) \to C^n_{def}(K, \Omega')$ *which preserves cocycles and coboundaries.*

Definition

A $2-$addit for a spatial super-product system $(H_t, U_{s,t})$, with respect to a canonical unit $\{\Omega_t\}$, is a measurable family of vectors $\{a_{s,t} : s, t \geq 0\}$ satisfying

(i) $a_{s,t} \in H_{s+t} \ \forall \ s, t \geq 0$,

(ii) $U_{r+s,t}(a_{r,s} \otimes \Omega_t) + a_{r,s+t} = U_{r,s+t}(\Omega_r \otimes a_{s,t}) + a_{r+s,t} \ \forall \ \ r, s, t \geq 0.$

A $2-$addit is said to be defective if further $a_{s,t} \in (U_{s,t}(H_s \otimes H_t))^{\perp}$ $\forall \ s, t \geq 0$.

Identify $L^2([0,t],\mathrm{k})^{\otimes n}$ with $L^2([0,t]^n,\mathrm{k}^{\otimes n})$ by the natural isomorphism. Then $L^2([0,t],\mathrm{k})^{\wedge n}$ is the collection of functions $f \in L^2([0,t]^n,\mathrm{k}^{\otimes n})$ satisfying

$$f(s_{\sigma(1)},\dots s_{\sigma(n)}) = \epsilon(\sigma)\Pi_\sigma f(s_1,\dots,s_n),$$

for any permutation $\sigma \in S_n$, where Π_σ the corresponding tensor flip on $\mathrm{k}^{\otimes n}$.

Proposition

Let $\{a_{s,t} : s, t \geq 0\}$ *be a defective* $2-$*addit for* $(H^{e,\mathrm{k}}(t), U_{s,t})$*. Then* $a_{s,t} \in L^2([0, s+t)], \mathrm{k})^{\wedge 2} \subseteq H^{e,\mathrm{k}}(s+t)$*. Further there exists* $f \in L^2_{loc}(\mathbb{R}_+, \mathrm{k}^{\otimes 2})$ *such that*

$$a_{s,t}(x,y) = 1_{[s,s+t]\times[0,s]}(x,y) f(x-y) - 1_{[0,s]\times[s,s+t]}(x,y)\Pi_{\mathrm{k}^{\otimes 2}} f(y-x),$$

for all $s, t, x, y \in (0, \infty)$*, where* $\Pi_{\mathrm{k}^{\otimes 2}}$ *is the usual tensor-flip.*

Proposition

Let $\{a_{s,t} : s,t \geq 0\}$ *be a defective* $2-$*addit for* $(E^{\mathrm{k}}(t), U_{s,t})$. *Then* $a_{s,t} = (a^1_{s,t} \otimes \Omega_2) + (\Omega_1 \otimes a^2_{s,t}) + a^{12}_{s,t}$, *with* $a^i_{s,t} \in L^2([0,s+t)], \mathrm{k})^{\wedge 2}$ *for* $i = 1,2$ *and* $a^{12}_{s,t} \in L^2([0,s+t)], \mathrm{k}) \otimes L^2([0,s+t)], \mathrm{k})$, *where* Ω_1 *and* Ω_2 *are vacuum vectors of the first and second Fock spaces respectively. Further there exist* $f^1, f^2, f^{12}_1, f^{12}_2 \in L^2_{loc}(\mathbb{R}_+, \mathrm{k}^{\otimes 2})$ *such that*

$$a^i_{s,t}(x,y) = 1_{[s,s+t]\times[0,s]}(x,y) f^i(x-y) - 1_{[0,s]\times[s,s+t]}(x,y)\Pi_{\mathrm{k}^{\otimes 2}} f^i(y-x),$$

$$a^{12}_{s,t}(x,y) = 1_{[s,s+t]\times[0,s]}(x,y) f^{12}_1(x-y) + 1_{[0,s]\times[s,s+t]}(x,y) f^{12}_2(y-x),$$

for all $s,t,x,y \in (0,\infty)$, $i = 1,2$.

Proposition

Let $\{a_{s,t} : s,t \geq 0\}$ *be a defective* $2-$*addit for* $(E_0^{\mathrm{k}}(t), U_{s,t})$*. Then* $a_{s,t} \in L^2([0,s+t)], \mathrm{k}) \otimes L^2([0,s+t)], \mathrm{k})$*. Further there exist* $f_1, f_2 \in L^2_{loc}(\mathbb{R}_+, \mathrm{k}^{\otimes 2})$ *such that*

$$a_{s,t}(x,y) = 1_{[s,s+t]\times[0,s]}(x,y) f_1(x-y) + 1_{[0,s]\times[s,s+t]}(x,y) f_2(y-x),$$

for all $s,t,x,y \in (0,\infty)$*,* $i = 1,2$*.*

Definition

A $2-$addit $\{a^1_{s,t} : s, t \geq 0\}$ is said to be orthogonal to another $2-$addit $\{a^2_{s,t} : s, t \geq 0\}$ if $a^1_{s,t} \perp a^2_{s,t}$ for all $s, t \geq 0$.

Definition

Let $(H_t, U_{s,t})$ be spatial super-product system with canonical unit Ω. The $2-$index with respect to Ω is defined as the supremum of the cardinality of all sets containing mutually orthogonal $2-$addits.

If the automorphism group acts transitively on the set of all units, then the $2-$index with respect to any unit is an invariant of the super-product system. In particular when the super product system is type II_0 (means there exists a unique unit up to scalars), $2-$index is an invariant.

The function $f \in L^2_{loc}(\mathbb{R}_+; \mathrm{k}^{\otimes 2})$ associated with a given 2-cocycle will be referred to as its symbol and, for a given $f \in L^2_{loc}(\mathbb{R}_+; \mathrm{k}^{\otimes 2})$, we denote the 2-cocycle with symbol f by $(a^f_{s,t})_{s,t>0}$.

Lemma

Let $f, g \in L^2_{loc}(\mathbb{R}_+, \mathrm{k}^{\otimes 2})$ and $T \in (0, \infty]$, then $f(r) \perp g(r)$ for almost all $r \in (0, T)$ if and only if $a^f_{s,t} \perp a^g_{s,t}$ for all $s, t \in (0, \infty)$ with $s + t \leq T$.

Theorem

The super-product system $(H^{e,\mathrm{k}}(t), U_{s,t})$ *has* $2-$*index equals to* $(dim(\mathrm{k}))^2$. *Hence these super-product systems are non-isomorphic if* $dim(\mathrm{k})$ *differs.*

Corollary

Clifford flows on hyperfinite II_1 *factors are non-cocycle-conjugate for different ranks.*

Theorem

The super-product system $(E^{\mathrm{k}}(t), U_{s,t})$ *has* $2-$*index equals to* $4(dim(\mathrm{k}))^2$.
The super-product system $(E_0^{\mathrm{k}}(t), U_{s,t})$ *has* $2-$*index equals to* $2(dim(\mathrm{k}))^2$.

These super-product systems are non-isomorphic if $dim(\mathrm{k})$ *differs.*

Corollary

CAR flows on hyperfinite type III_λ *factors are non-cocycle-conjugate for different ranks. They are not cocycle conjugate to any of the GICAR flows.*

Definition

Let M be any von Neumann algebra. An E_0-semigroup on M is a semigroup $\{\alpha_t : t \geq 0\}$ of 'normal' unital $*-$endomorphisms, which are $\sigma-$weakly continuous, (i.e) the map $t \mapsto \rho(\alpha_t(x))$ is continuous as a complex valued function, for every fixed $\rho \in \mathrm{M}_*$ and $x \in \mathrm{M}$.

We further assume $\alpha_t(\mathrm{M}) \neq M$, (i.e) α_t is not an automorphism.

Definition

A cocycle for an E_0-semigroup α on M is a strongly continuous family of unitaries $U = (U_t)_{t \geq 0}$ satisfying $U_s \alpha_s(U_t) = U_{s+t}$ for all $s, t \geq 0$.

The family of endomorphisms $\alpha_t^U(x) := U_t \alpha_t(x) U_t^*$ defines an E_0-semigroup. This leads to the following equivalence relations on E_0-semigroups.

Definition

Let α and β be E_0-semigroups on von Neumann algebras M and N.

(i) α and β are *conjugate* if there exists a *-isomorphism $\theta : \mathrm{M} \to \mathrm{N}$ such that $\beta_t = \theta \circ \alpha_t \circ \theta^{-1}$ for all $t \geq 0$.

(ii) α and β are *cocycle conjugate* if there exists a cocycle U for α such that β is conjugate to α^U.

Let $\mathrm{M} \subseteq B(L^2(\mathrm{M}, \varphi))$ be a factor, where φ is a faithful normal state and Ω cyclic and separating vector (i.e) M is in standard form. J be the modular conjugation operator associated to the vector Ω by the Tomita-Takesaki theory. We can define the dual (or complementary) E_0-semigroup on M' by

$$\alpha_t'(x') = J\alpha_t(Jx'J)J \ \forall x' \in \mathrm{M}'.$$

The dual E_0-semigroup is well-defined up to cocycle conjugacy.

Proposition

If the E_0-semigroups α and β on M are cocycle conjugate, then the dual E_0-semigroups α' and β' are also cocycle conjugate.

Theorem

Let $\mathrm{M} \subseteq B(H)$ *be a factor in standard form and* α *an* E_0*-semigroup on* M*. For each* $t > 0$*, Let*

$$H_t^\alpha = \{X \in B(H) : \forall_{m \in \mathrm{M}, m' \in \mathrm{M}'} \ \alpha_t(m)X = Xm, \ \alpha_t'(m')X = Xm'\}.$$

Then $H^\alpha = \{H_t^\alpha : t > 0\}$ *is a super-product system with respect to the family of isometries* $U_{s,t}\,(X \otimes Y) = XY$*.*

Let α *and* β *be* E_0*-semigroups acting on respective factors* M *and* N *in standard form. If* α *and* β *are cocycle conjugate then* H^α *and* H^β *are isomorphic.*

Let K be a complex Hilbert space. We denote by $\mathcal{A}(K)$ the CAR algebra over K, which is the universal C^*-algebra generated by $\{a(x) : x \in K\}$, where $x \mapsto a(x)$ is an antilinear map satisfying the CAR relations:

$$a(x)a(y) + a(y)a(x) = 0,$$

$$a(x)a(y)^* + a(y)^*a(x) = \langle x, y \rangle 1,$$

for all $x, y \in K$.

The *quasi-free state* ω_A on $\mathcal{A}(K)$, associated with a positive contraction $A \in B(K)$, is the state determined by its $2n$-point function as

$$\omega_A(a(x_n) \cdots a(x_1)a(y_1)^* \cdots a(y_m)^*) = \delta_{n,m} \det(\langle x_i, Ay_j \rangle),$$

where $\det(\cdot)$ denotes the determinant of a matrix.

(H_A, π_A, Ω_A) be the GNS triple associated with ω_A on $\mathcal{A}(K)$, and set $\mathrm{M}_A := \pi_A(\mathcal{A}(K))''$, which is a factor.

Let $K = L^2((0,\infty), \mathrm{k})$ and $A \in B(K)$ be a positive contraction satisfying $Ker(A) = Ker(1-A) = \{0\}$. We further assume that A is a Toeplitz operator, meaning $T_t^* A T_t = A$ for all $t \geq 0$.

There exists a unique E$_0$-semigroups $\alpha^A = \{\alpha_t^A : t \geq 0\}$ on M_A, determined by

$$\alpha_t^A(\pi_A(a(f))) = \pi_A(a(T_t f)), \quad \forall f \in K.$$

We call α^A as the *Toeplitz CAR flow* on M_A associated with A.

Now assume A is of the form $1_{L^2(\mathbb{R}_+)} \otimes R$ for some $R \in B(\mathrm{k})$. Then when $R \neq \frac{1}{2}$, M_A is of type III, and when $R = 1/2$ M_A is of type II$_1$.

Example

We denote the E$_0$-semigroup associated with $1_{L^2(\mathbb{R}_+)} \otimes R$ by α^R. As the Toeplitz part is trivial, we just call these E$_0$-semigroups as *CAR flows* on M_A.

Let M_A^e denotes the von Neumann subalgebra generated by the even products of $\pi_A(a(f)), \pi_A(a^*(g))$. When $tr(A^2 - A) = \infty$ this action is outer and hence M_A is a factor. The restriction of α^A to M_A^e is called as the *even CAR flow* on M_A. We denote it by β^R, when $A = 1_{L^2(\mathbb{R}_+)} \otimes R$.

The gauge group action on M_A is given by $\pi_A(a(f)) \mapsto e^{it}\pi_A(a(f))$ for $t \in \mathbb{R}$. We denote the von Neumann subalgebra fixed by the gauge group action by M_A^0. Again since $tr(A^2 - A) = \infty$, M_A^0 is factor. The restriction of α^A to M^0 is called as the *GICAR flow* on M_A. We denote it by γ^R, when $A = 1_{L^2(\mathbb{R}_+)} \otimes R$.