A RESULT ON NIJENHUIS OPERATOR

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In 1984 Magri and Morosi worked on Poisson Nijenhuis manifolds via deformation of Hamiltonian system, which was again studied by Kosmann and Magri in the year 1990.

In 1990 T.J. Courant introduced Dirac structures and around at the same time I. Dorfman, one student of Gelfand, independently studied Dirac structure in a different set up.

Dorfman studied Nijenhuis operator via deformation of Lie algebra. Introduction of Dirac structures by her gave new interpretations to the already existing Nijenhuis set ups.

In 2004 Gallardo and Nunes da Costa introduced Dirac Nijenhuis structures.

Guang and Kang separately developed Dirac Nijenhuis manifolds in 2004.

In 2011 Kosmann-Schwarzbach studied Dirac Nijenhuis structures on Courant algebroid.
Aim of this talk is

- To construct Nijenhuis operator on $\Gamma(TM \oplus T^*M)$ in the same sense Irene Dorfman has constructed Nijenhuis operator on $\Gamma(TM)$.
- To study the deformation of Dirac structures on $\Gamma(TM \oplus T^*M)$.
- To define Nijenhuis relation on the new set up.
Some pre-requirements

- Let $M$ be a differentiable manifold. $TM$ be its Tangent bundle.

- $\Gamma(TM)$, the space of section of $TM$ is endowed with a bilinear operation on it, i.e. $[,] : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ defined by $[X, Y] = XY - YX$, $X, Y \in \Gamma(TM)$, which is known as Lie bracket (Named in the honour of Sophus Lie).

- $(\Gamma(TM), [,])$ is a Lie algebra with the bracket operation on it.

- A vector space $G$ with a bilinear operation on it $[,] : G \times G \to G$ is said to be a Lie algebra if the bracket satisfies two properties, i.e.
  1. $[X, Y]$ is skew symmetric.
  2. $[X, Y]$ satisfies Jacobi identity property, i.e.

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$
Dorfman’s Construction

Dorfman has started her calculation choosing a deformed bracket on $\Gamma(TM)$ with a parameter $\mu$, i.e. $[X, Y]_\mu = [X, Y] + \mu \omega(X, Y)$, where $\omega$ is a bilinear map on $\Gamma(TM)$, i.e.

$$
\omega : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM).
$$

The given Lie algebra has a deformation w.r.t. $\omega$ if $[X, Y]_\mu$ has a Lie bracket structures. To have a Lie bracket structure we must have

1. $\omega$ is skew symmetric, i.e. $\omega(X, Y) = -\omega(Y, X)$.
2. $\omega$ must satisfy Jacobi identity, i.e.

$$
\omega(\omega(X, Y), Z) + \text{c.p.} = 0.
$$

Let $N : \Gamma(TM) \rightarrow \Gamma(TM)$ be an endomorphism on $\Gamma(TM)$. 

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For a fixed $N : \Gamma(TM) \rightarrow \Gamma(TM)$ the deformation of Lie algebra is said to be trivial if for $T_\mu = id + \mu N$ the following condition hold:

$$T_\mu[X, Y]_\mu = [T_\mu X, T_\mu Y].$$

(1)

Now expanding L.H.S. and R.H.S. both and comparing them we get

$$\omega(X, Y) = [X, N(Y)] + [N(X), Y] - N[X, Y]$$

(2)

$$N\omega(X, Y) = [N(X), N(Y)]$$

(3)

From the above two equations (2), (3) we have


(4)

A linear operator $N$ satisfying (4) is called a **Nijenhuis Operator**.
Theorem ([4])

Let $N : \Gamma(TM) \to \Gamma(TM)$ be a Nijenhuis Operator. Then a trivial deformation of $\Gamma(TM)$ can be obtained by putting

$$\omega(X, Y) = [Na, b] + [a, Nb] - N[a, b].$$
The space $\Gamma(TM \oplus T^*M)$

Let us consider $TM \oplus T^*M$ on $M$. $\Gamma(TM \oplus T^*M)$, be the space of sections of $TM \oplus T^*M$ defined by $\Gamma(TM \oplus T^*M) = \Gamma(TM) \oplus \Gamma(T^*M) = \{(X, \xi)|X \in \Gamma(TM), \xi \in \Gamma(T^*M)\}$. $\Gamma(TM \oplus T^*M)$ is naturally endowed with one symmetric and skew symmetric pairing:

$$\langle(X, \alpha), (Y, \beta)\rangle_{\pm} = \frac{1}{2}i_Y \alpha \pm i_X \beta.$$

And this is a algebra with a bracket operation on it, which is known as Courant bracket.

**Definition (Courant Bracket)**

For differentiable manifold $M$ the Courant bracket $[X, Y]_c$ on $\Gamma(TM \oplus T^*M)$ is a bilinear operation defined by

$[X + \xi, Y + \eta]_c = ([X, Y], L_Y \xi - L_X \eta - d\langle X + \xi, Y + \eta\rangle_+)$, where $[X + \xi, Y + \eta] \in \Gamma(TM) \oplus T^*M$ and $L_X$ is the Lie derivative and $\langle X + \xi, Y + \eta\rangle_+$ is the symmetric pairing on $\Gamma(TM \oplus T^*M)$. 
Nijenhuis Operator on $TM \oplus T^*M$

Let us assume a $\lambda$-parametrised family of brackets on $\Gamma(TM \oplus T^*M)$, i.e. $[Y_1, Y_2]_\lambda = [Y_1, Y_2]_c + \lambda \varphi(Y_1, Y_2)$, where $\varphi$ is a bilinear operator on $\Gamma(TM \oplus T^*M)$, $Y_1, Y_2 \in \Gamma(TM \oplus T^*M)$ and $[Y_1, Y_2]_c$ is the Courant bracket on $\Gamma(TM \oplus T^*M)$. Now we have to check the Courant bracket structure of $\varphi(Y_1, Y_2)$. And to have a Courant bracket structure $\varphi(Y_1, Y_2)$ must satisfy:

1. Skew symmetric property.
2. Jacobi Anomaly.

Let $\mathcal{N} : \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$ be a linear map and define $\mathcal{T}_\lambda = id + \lambda \mathcal{N}$ on $\Gamma(TM \oplus T^*M)$. For $\mathcal{T}_\lambda$ this deformation is said to be trivial if $\mathcal{T}_\lambda[X, Y]_\lambda = [\mathcal{T}_\lambda X, \mathcal{T}_\lambda Y]$. 
Comparing both sides of the above equation we have

$$\varphi(X, Y) = [X, \mathcal{N}Y]_c + [\mathcal{N}X, Y]_c - \mathcal{N}[X, Y]_c$$  \hspace{1cm} (5)$$

$$\mathcal{N}\varphi(X, Y) = [\mathcal{N}X, \mathcal{N}Y]_c$$  \hspace{1cm} (6)$$

From the equation (6) we can conclude that

$$\mathcal{N}\varphi(X, Y) - [\mathcal{N}X, \mathcal{N}Y]_c = 0$$

$$\mathcal{N}([X, \mathcal{N}Y]_c + [\mathcal{N}X, Y]_c - \mathcal{N}[X, Y]_c) - [\mathcal{N}X, \mathcal{N}Y]_c = 0$$  \hspace{1cm} (7)$$

With the equation (7), \( \mathcal{N} \) is called a Nijenhuis Operator on \( \Gamma(TM \oplus T^*M) \).
We know that $\mathcal{N}$ is a linear mapping from $\Gamma(TM \oplus T^*M)$ to itself. And it is skew symmetric if $\mathcal{N} = -(\mathcal{N})^t$ because $\mathcal{N}$ can be written as

$$\begin{pmatrix} N & \pi \\ \omega & N^* \end{pmatrix},$$

where $N : \Gamma(TM) \to \Gamma(TM)$, $N^* : \Gamma(T^*M) \to \Gamma(T^*M)$, $\pi : \Gamma(T^*M) \to \Gamma(TM)$ and $\omega : \Gamma(TM) \to \Gamma(T^*M)$. Here $N^2 = -Id_M$. Kosmann-Schwarzbach in her article [8] has shown that $\Gamma(TM \oplus T^*M)$ has a weak deformation with respect to a Nijenhuis Operator $\mathcal{N}$ only if $\mathcal{N}$ is equivalent to a almost complex structure, i.e. $\mathcal{N}^2 = -Id_M$. $\mathcal{N}$ satisfies almost complex structure iff $\pi, \omega$ vanishes on $\Gamma(TM \oplus T^*M)$. 
Here my aim is not to study the deformation of $\Gamma(TM \oplus T^*M)$. It is to study the deformation of Dirac structure on it.

1. Consider $TM \oplus T^*M$, a bundle on $M$.
2. $L \subset TM \oplus T^*M$, a subbundle of $TM \oplus T^*M$ is said to be a Dirac structure on $M$ if
   - $L = L^\perp$.
   - $[\chi_1, \chi_2]_c = \chi_3$, where $\chi_1, \chi_2, \chi_3 \in \Gamma(L)$ (Integrability condition)
3. It is also called a maximally isotropic subbundle of $TM \oplus T^*M$, as the symmetric pairing on $TM \oplus T^*M$ vanishes on $L$.
4. On $TM \oplus T^*M$, Courant bracket does not satisfy Jacobi identity, it satisfies an anomaly known as Jacobi Anomaly, which gives rise to a non vanishing three tensor $T(Z_1, Z_2, Z_3)$ on $TM \oplus T^*M$.
5. $T(Z_1, Z_2, Z_3) = \langle [Z_1, Z_2], Z_3 \rangle_+ + c.p.$
Some authors like Gallardo, Nunes da costa[1], Longguang, Baokang[11] have studied independently on deformation of Dirac structure on 2004. I have tried to give a new approach following Dorfman's construction.

**Theorem**

Let $\mathcal{N} : \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$ be a Nijenhuis Operator, then the deformation of $\Gamma(TM \oplus T^*M)$ is not a weak deformation if and only if both $\mathcal{N}$ and $\varphi(X,Y)$ are restricted to Dirac structures, otherwise the deformation is weak deformation on the whole space.
Proof.

We have \( \varphi(X, Y) = [X, \mathfrak{N}Y]_c + [\mathfrak{N}X, Y]_c - \mathfrak{N}[X, Y]_c \). \( \varphi \) is skew symmetric as Courant bracket is skew symmetric. 

\[ \delta \varphi(X, Y, Z) = [X, \varphi(Y, Z)] + \varphi([X, Y], Z) + c.p. \neq 0 \] as Courant bracket does not satisfy Jacobi identity property. Therefore on \( \Gamma(TM \oplus T^*M) \) deformation is not trivial. Suppose both \( \mathfrak{N}, \varphi(X, Y) \) are restricted to the section of Dirac structure \( L \) on \( \Gamma(TM \oplus T^*M) \). That means \( \mathfrak{N} : \Gamma(L) \to \Gamma(L) \) and \( \varphi : \Gamma(L) \times \Gamma(L) \to \Gamma(L) \).

As we know that Dirac Structure \( L \) is a maximally isotropic subbundle of \( (TM \oplus T^*M) \), the natural symmetric pairing on it vanishes. Courant bracket is restricted to the Dirac structures satisfies Jacobi Identity. Therefore on \( L \), \( \delta \varphi = 0 \) as \( \delta \varphi \) is the representation of Jacobi identity on the given space.

The weak deformation of \( \Gamma(TM \oplus T^*M) \) has been studied by Kosmann-Schwarzbach in [8].
Nijenhuis Relation on $\Gamma(TM)$:

- Let $A \subset \Gamma(TM) \oplus \Gamma(TM)$ and $A^* \subset \Gamma(T^*M) \oplus \Gamma(T^*M)$.
- Let us take $a_1 \oplus a_2 \in \Gamma(TM) \oplus \Gamma(TM)$ and $\zeta_1 \oplus \zeta_2 \in \Gamma(T^*M) \oplus \Gamma(T^*M)$.
- Choose $\zeta_1 \oplus \zeta_2 \in \Gamma(T^*M) \oplus \Gamma(T^*M)$ such that $(\zeta_1, a_2) = (\zeta_2, a_1)$ for arbitrary $a_1 \oplus a_2 \in A$.
- A relation $A \subset \Gamma(TM) \oplus \Gamma(TM)$ is said to be a Nijenhuis relation for arbitrary $a_1, a_2, b_1, b_2 \in \Gamma(TM)$ and $\zeta_1, \zeta_2, \zeta_3 \in \Gamma(T^*M)$ satisfying $a_1 \oplus a_2, b_1 \oplus b_2 \in A$ and $\zeta_1 \oplus \zeta_2, \zeta_2 \oplus \zeta_3 \in A^*$ if the following holds:

  $$(\zeta_1, [a_2, b_2]) - (\zeta_2, [a_2, b_1] + [a_1, b_2]) + (\zeta_3, [a_1, b_1]) = 0.$$ 

- Proposition:(Dorfman) The graph of a Nijenhuis Operator $N : \Gamma(TM) \to \Gamma(TM)$ is a Nijenhuis relation. Conversely if a graph of some operator $N : \Gamma(TM) \to \Gamma(TM)$ is a Nijenhuis relation, then $N$ is a Nijenhuis Operator.
Dirac pairs in previous set up:

- Two Dirac structures $L, M \subset (TM \oplus T^*M)$ are said to be a pair of Dirac structures, if the set

$$A_{L,M} = \{ a_1 \oplus a_2 : \exists \zeta \in \Gamma(T^*M), a_1 \oplus \zeta \in M, a_2 \oplus \zeta \in L \}$$

is a Nijenhuis relation.
Nijenhuis relation on $TM \oplus T^*M$:

- $\mathcal{A} \subset \Gamma(TM \oplus T^*M) \oplus \Gamma(TM \oplus T^*M)$ and $A^* \subset \Gamma(T^*M) \oplus \Gamma(T^*M)$.

- Let us take $\alpha_1 \oplus \alpha_2 \in \Gamma(TM \oplus T^*M) \oplus \Gamma(TM \oplus T^*M)$ and $\eta_1 \oplus \eta_2 \in \Gamma(T^*M) \oplus \Gamma(T^*M)$.

- As per the above calculation here the pairing is defined as

  \[
  (\eta_1, \alpha_2) = (\eta_2, \alpha_1)
  \Rightarrow (\eta_1, X_2 + \xi_2) = (\eta_2, X_1 + \xi_1)
  \Rightarrow i_{X_2} \eta_1 + \eta_1 \wedge \xi_2 = i_{X_1} \eta_2 + \eta_2 \wedge \xi_1.
  \]

- A relation $\mathcal{A} \subset \Gamma(TM \oplus T^*M) \oplus \Gamma(TM \oplus T^*M)$ is said to be a Nijenhuis relation for arbitrary $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma(TM \oplus T^*M)$ and $\eta_1, \eta_2, \eta_3 \in \Gamma(T^*M)$ satisfying $\alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2 \in A$ and $\eta_1 \oplus \eta_2, \eta_2 \oplus \eta_3 \in A^*$ if the following holds:

  \[
  (\eta_1, [\alpha_2, \beta_2]) - (\eta_2, [\alpha_2, \beta_1] + [\alpha_1, \beta_2]) + (\eta_3, [\alpha_1, \beta_1]) = 0.
  \]
Future work

- I am now trying to associate a pair of some geometric structures like Dirac structures with this above defined Nijenhuis relation.
- Deformation of Kahler manifold with respect to Nijenhuis Operator may be seen.
- Hamiltonian pairs and Symplectic pairs can be associated to the Nijenhuis Operator on $\Gamma(TM \oplus T^*M)$.
- One can study the deformation of the space $\Gamma(\Lambda(TM) \oplus \Lambda(T^*M))$ through Nijenhuis Operators.


Reference II


Reference III


Thank You