The equality $C^{*n}C^n = (C^*C)^n$ is not sufficient for quasinormality of a composition operator C in L^2-space

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Definitions of quasinormality

Kaufman's definition of quasinormality

We say that a closed densely defined operator C in $\mathcal{H}$ is quasinormal if C commutes with $E_{|C|}$, i.e $CE_{|C|} \subset E_{|C|}C$

J. Stochel, F. H. Szafraniec definition of quasinormality

A closed densely defined operator C in $\mathcal{H}$ is quasinormal if and only if $U|C| \subset |C|U$, where $C = U|C|$ is the polar decomposition of C

- Z. J. Jablonski, I. B. Jung, J. Stochel proved that this defnitions are equivalent.

Charakterization of quasinormal operators

Theorem

Let C be a closed densely defined operator in $\mathcal{H}$. Then the following conditions are equivalent:

- C is quasinormal
- $C^{*n}C^n = (C^*C)^n$ for every $n \in \mathbb{Z}_+$,
- there exists a (unique) spectral Borel measure E on $\mathbb{R}_+$ such that
  $C^{*n}C^n = \int_{\mathbb{R}_+} x^n E(dx)$ for $n \in \{1, 2, 3\}$
- $C^{*n}C^n = (C^*C)^n$ for every $n \in \{2, 3\}$

Composition operators in L^2-spaces

- $(X, \mathcal{A}, \mu)$ is a σ-finite measure space
- $\phi : X \to X$ is an $\mathcal{A}$-measurable transformation, i.e., $\phi^{-1}(\Delta) \in \mathcal{A}$ for every $\Delta \in \mathcal{A}$
- If the measure $\mu \circ \phi^{-1}$ given by $\mu \circ \phi^{-1}(\Delta) = \mu(\phi^{-1}(\Delta))$ for $\Delta \in \mathcal{A}$ is absolutely continuos with respect to μ (we say that μ is **nonsingular**), then the operator C_ϕ in $L^2(\mu)$ given by
  $\mathcal{D}(C_\phi) = \{f \in L^2(\mu) : f \circ \phi \in L^2(\mu)\}$,
  $C_\phi f = f \circ \phi, f \in \mathcal{D}(C_\phi)$
  is well-defined
- We call it a **composition** operator with **symbol** ϕ

Weighted shifts on directed trees

- $\mathcal{T} = (V; E)$ is a directed tree (V and E are the sets of vertices and edges of $\mathcal{T}$, respectively)
- $V^\circ = V \setminus \{\mathrm{root}\}$ if $\mathcal{T}$ has a root and $V^\circ = V$ if $\mathcal{T}$ is rootles.
- $l^2(V)$ is the Hilbert space of square summable complex functions on V equipped with the standard inner product
- For $u \in V$, we define $e_u \in l(V)$ to be the characteristic function of the one-point set $\{u\}$.

Given a system $\lambda = \{\lambda_v\}_{v \in V^\circ}$ of complex numbers, we define the operator S_λ in $l^2(V)$, which is called a *weighted shift* on $\mathcal{T}$ with weights λ, as follows

$$\mathcal{D}(S_\lambda) = \{f \in l^2(V) : \Lambda_T f \in l^2(V)\} \tag{1}$$

$$S_\lambda = \Lambda_T f \quad \text{for} \quad f \in \mathcal{D}(S_\lambda); \tag{2}$$

where,

$$(\Lambda f)(v) = \begin{cases} \lambda_v f(par(v)) & \text{if } v \in \ell^2(V), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

Let S_λ be a weighted shift on a directed tree $\mathcal{T} = (V, E)$ with weights $\lambda = \{\lambda_v\}_{v \in V^\circ}$. Then the following assertions hold:

(i) S_λ is a closed operator,

(ii) $e_u \in \mathcal{D}(S_\lambda)$ if and only if $\sum_{v \in Chi(u)} |\lambda_v|^2 < \infty$ and in this case

$$S_\lambda e_u = \sum_{v \in Chi(u)} \lambda_v e_v, \qquad \|S_\lambda e_u\| = \sum_{v \in Chi(u)} |\lambda_v|^2 \qquad (3)$$

(iii) S_λ is densely defined if and only if $e_u \in D(S_\lambda)$ for every $u \in V$.

Theorem

Let S_λ be a densely defined weighted shift with weights λ and let $S_\lambda = U|S_\lambda|$ be its polar decomposition. Then $U = S_\pi$ where,

$$\pi_v = \begin{cases} \frac{\lambda_v}{||S_\lambda e_{par(v)}||} & \text{if } par(u) \in V_\lambda^+ \\ 0 & \text{otherwise} \end{cases}$$

Theorem

Let S_λ be a weighted shift on a directed tree $\mathcal{T}$ with weights $\lambda = \{\lambda_u\}_{u \in V^\circ}$. Then the following conditions are equivalent:

(i) $\mathcal{D}(S_\lambda) = \ell^2(V)$,

(ii) $S_\lambda \in B(\ell^2(V))$,

(iii) $\sup_{u \in V} \sum_{v \in Chi(u)} |\lambda_v|^2 < \infty$ If $S_\lambda \in B(l^2(V))$, then

$$||S_\lambda|| = \sup_{u \in V} ||S_\lambda e_u|| = \sqrt{\sup_{u \in V} \sum_{v \in Chi(u)} |\lambda_v|^2} \tag{4}$$

Theorem

Let $n \in \mathbb{Z}_+$. If $S_\lambda \in B(l^2(V))$ is a weighted shift on a directed tree $\mathcal{T} = (V; E)$ with weights $\lambda = \{\lambda_v\}_{v \in V^\circ}$, then the following two conditions are equivalent:

(i) $(S_\lambda^* S_\lambda)^n = (S_\lambda^*)^n S_\lambda^n$,

(ii) $\| S_\lambda e_u \|^n = \| S_\lambda^n e_u \|$ for all $u \in V$.

Transcendentality of $\ln(\alpha)$

Theorem (Lindemann-Weierstrass)

For any finite system of distinct algebraic numbers $\alpha_1, ..., \alpha_n$, the numbers $e_1^\alpha, ..., e_n^\alpha$ are lineary independent over $\mathbb{A}$

Corollary

$\ln(\alpha)$ is transcendental for any algebraic number $\alpha \neq 0, 1$.

- Suppose $\ln(\alpha)$ is algebraic. Then, by Theorem with $\alpha_1 = 0$, $\alpha_2 = \ln(\alpha)$, we see that e^0 and $e^{\ln(\alpha)}$ are lineary independent over $\mathbb{A}$ thus $e^{\ln(\alpha)}$ is transcendental. But $e^{\ln(\alpha)} = \alpha \in \mathbb{A}$ thus we have a contradiction.

A question

- Is the equality $C^{*n}C^n = (C^*C)^n$ sufficient for quasinormality of a composition operator C in L^2-space?

Main Theorem

Theorem

For every integer $n \geq 2$, there exist an injective, non-quasinormal composition operator C in L^2-space over a σ-finite measure such that

$$(C^*C)^n = C^{*n}C^n \qquad (C^*C)^k \neq C^{*k}C^k \tag{5}$$

for all $k \in \{2, 3, ...\} \setminus \{n\}$.

Special directed tree

Leafless and rootless directed trees with one branching vertex of valency $\aleph_0$

textupLet $\mathcal{T} = (V, E)$ be a directed tree with

$$V = \{-k : k \in \mathbb{Z}_+\} \sqcup \{(i,j) : i,j \in \mathbb{N}\} \tag{6}$$

and

$$E = \{(-k, -k+1) : k \in \mathbb{N}\} \sqcup \{(0, (i, 1)) : i \in \mathbb{N}\} \sqcup \{((i,j), (i,j+1)) : i,j \in \tag{7}$$

(the symbol "$\sqcup$" detonates disjoint union of sets).

Special directed tree

Leafless and rootless directed trees with one branching vertex of valency $\aleph_0$

Define the system of weights $\lambda = \{\lambda_v\}_{v \in V}$ by

$$\lambda_v = \begin{cases} \alpha_i & \text{if} \quad v = (i,1), i \in \mathbb{N} \\ \beta_i & \text{if} \quad v = (i,j), i \in \mathbb{N}, j \geq 2 \\ \gamma_i & \text{if} \quad v = i, i \in \mathbb{Z}_+ \end{cases}$$

Theorem

If $n \geq 2$ and S_λ is a weighted shift on the directed tree $\mathcal{F}$ then,$(S_\lambda^* S_\lambda)^n = (S_\lambda^*)^n S_\lambda^n$ if and only if the following conditions holds

(i) $|\gamma_{k+n-1}|^n = |\gamma_{k+n-1}\gamma_{k+n-2}...\gamma_k| \ k \in \mathbb{Z}_+,$

(ii) $|\gamma_{n-i-1}|^n = |\gamma_{n-i-1}...\gamma_0|\sqrt{\sum_{k=1}^\infty |\alpha_k \beta_k^{i-1}|^2} \ i = 1, 2, ..., n-1,$

(iii) $(\sqrt{\sum_{k=1}^\infty |\alpha_k|^2})^n = \sqrt{\sum_{k=1}^\infty |\alpha_k \beta_k^{n-1}|^2}.$

Special sequence of function

we will consider some sequences of special functions For $k \in \mathbb{Z}$, we define $S_k : (0,1) \to (0,\infty)$ by

$$S_k(x) = 1^k + 2^k x + 3^k x^2 + \dots \tag{8}$$

Lemma

The following assertions are valid.

(i) $S_k(x) \in \mathbb{Q}(x)$ for every $k \in \mathbb{Z}_+$

(ii) $S_0(x) = \frac{1}{1-x}, S_1(x) = \frac{1}{(1-x)^2}$, and $S_k(x) = \frac{m_k(x)}{(1-x)^{k+1}}$ for $k = 2, 3, \ldots$ where m_k is a polynomial of degree $k-1$ and 1 is not a root of m_k for $k \geq 1$.

(iii) $S_{-1}(x) = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \ldots = \frac{\ln(1-x)}{x}$

Theorem (Z. J .Jablonski, I. B. Jung, J. Stochel)

Let S_λ be a weighted shift on a rootless directed tree $\mathcal{T} = (V; E)$ with positive weights. Then S_λ is unitarily equivalent to a composition operator C in an L^2-space over a σ-finite measure space. Morover, if the directed tree is leafless, then C can be made injective.

Define

$$\alpha_k = \sqrt{k^{n-1} q^{k-1}}, \qquad \beta_k = \sqrt{\frac{1}{k} c^{\frac{1}{n-1}}}, \tag{9}$$

where $q, c \in \mathbb{Q}$ are chosen as follows

$$(S_{n-1}(q))^n = cS_0(q) \tag{10}$$

and

$$c^{\frac{k}{n-1}} \notin \mathbb{Q} \quad \text{for all} \quad k \in \{1, 2, ..., n-2\} \tag{11}$$

- $(S_\lambda^* S_\lambda)^p \neq (S_\lambda^*)^p S_\lambda^p$ for $p \in \{2, 3, ..., n-1\}$

Suppouse, contrary to our claim that for some $p \in \{2, 3, ..., n-1\}$ the equality $(S_\lambda^* S_\lambda)^p = (S_\lambda^*)^p S_\lambda^p$ holds. In view of Theorem, this equality implies that

$$\Big(\sqrt{\sum_{k=1}^{\infty} |\alpha_k|^2}\Big)^p = \sqrt{\sum_{k=1}^{\infty} |\alpha_k \beta_k^{p-1}|^2} \tag{12}$$

We verify that for the directed tree the last equation is of the form

$$(\sum_{k=1}^{\infty} k^{n-1} q^k)^p = c^{\frac{p-1}{n-1}} (\sum_{k=1}^{\infty} k^{n-p} q^k) \tag{13}$$

which we can write as

$$S_{n-1}(q) = c^{\frac{p-1}{n-1}} S_{n-p} q \tag{14}$$

Recall that by the condition (i) Lemma $S_{n-1}(q) \in \mathbb{Q}$ and $S_{n-p}(q) \in \mathbb{Q}$. But this is a contradiction since c was such that $c^{\frac{p-1}{n-1}} \notin \mathbb{Q}$.

- $(S_\lambda^* S_\lambda)^p \neq (S_\lambda^*)^p S_\lambda^p$ for $p = n+1$

As in the previous case we see that the equality $(S_\lambda^* S_\lambda)^{n+1} = (S_\lambda^*)^{n+1} S_\lambda^{n+1}$ implies that

$$\left(\sum_{k=1}^{\infty} \alpha_k^2\right)^{n+1} = \sum_{k=1}^{\infty} \alpha_k \beta_k^{n2}, \qquad (15)$$

which is equivalent in this case with

$$\left(\sum_{k=1}^{\infty} k^{n-1} q^k\right)^{n+1} = c^{\frac{n}{n-1}} \sum_{k=1}^{\infty} \frac{1}{k} q^k \qquad (16)$$

which one can note as
$S_{n-1}(q)^{n+1} = c^{\frac{n}{n-1}} S_{-1}(q)$ This is a contradiction because $S_{n-1}(q) \in \mathbb{A}$ and $c^{\frac{n}{n-1}} \in \mathbb{A}$ but $S_{-1}(q) = \frac{\ln(1-q)}{q}$ is transcendental

- $(S_\lambda^* S_\lambda)^p \neq (S_\lambda^{*p}) S_\lambda^p$. for $p \in \{n+2, n+3, ...\}$

Otherwise, we have

$$\gamma_{p-1-i}^2 p = (\gamma_{p-1-i} ... \gamma_0)^2 \sum_{k=1}^{\infty} \alpha_k {\beta_k^{i-1}}^2 \tag{17}$$

for $i = 1, 2, ..., p-1$, which implies that to

$$\gamma_{k-1-i}^{2n} = (\gamma_{k-1-i} ... \gamma_0)^2 c^{\frac{2k}{n-1}} \sum_{k=1}^{\infty} \frac{1}{k} q^k \tag{18}$$

for $i = n + 1$. But this is a contradiction as in the previous case, because γ_i is an algebraic number and $\sum_{k=1}^{\infty} \frac{1}{k} q^k$ is transcendental.

This completes the proof.

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