On the classification and modular extendability of E_0-semigroups on factors

Joint work with Daniel Markiewicz

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The content of my talk is based on the following papers.
Joint work with Daniel Markiewicz, On the classification and modular extendability of E_0-semigroups on factors, rXiv:1409.6675 [math.OA], 2014.

Introduction and Motivation :
Our primary focus is the study of E_0-semigroups of non-type I factors. This case has received considerably less attention than the case of E_0-semigroups on type I factors and recently this area has got attention with a considerable success.

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Our primary focus is the study of E_0-semigroups of non-type I factors. This case has received considerably less attention than the case of E_0-semigroups on type I factors and recently this area has got attention with a considerable success.

A weak* continuous one-parameter semigroup of unital $*$-endomorphisms on a von Neumann algebra is called an E_0-semigroup, and there has been considerable interest in their classification up to the equivalence relation called cocycle conjugacy. Most of the progress has focused on the case of E_0-semigroups on type I_∞ factors: those are divided into types I, II and III, and every such E_0-semigroup gives rise to a product system of Hilbert spaces. In fact, the classification theory of E_0-semigroups of type I_∞ factors is equivalent to the classification problem of product systems of Hilbert spaces up to isomorphism.

In 1988, E_0-semigroups on II_1 factors were first studied by Powers, who introduced an index for their study. In 2004, Alevras computed the Powers index of the Clifford flows on type II_1 factors, however the index is not known to be a cocycle conjugacy invariant. On the other hand, also showed that a product system of W*-correspondences can be associated to every E_0-semigroup on a type II_1 factor, and the isomorphism class of the product system is a cocycle conjugacy invariant. In fact, the association of product systems of W*-correspondences to E_0-semigroups on general von Neumann algebras has been established by Bhat and Skeide. However product systems are difficult to compute in practice.

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In 2001, Amosov, Bulinskii and Shirokov were the first to examine the issue of extendability of E_0-semigroups on general factors. Bikram, Izumi, Srinivasan and Sunder introduced the concept of equimodularity for endomorphisms, and applied it to obtain convenient criteria for the existence of extensions. As an application, it was proved in Bikram, Izumi, Srinivasan and Sunder that the CAR flows are not extendable on the hyperfinite factor of type II_1. Similarly, Bikram showed that the CAR flows are not extendable on hyperfinite III_λ factors, for $\lambda \in (0, 1)$, for a certain class of quasi-free states.

In 2012, Srinivasan and Margetts introduced new invariants for E_0-semigroups on type II_1 factors, especially the coupling index, and as an application showed that the Clifford flows are non-cocycle conjugate. Subsequently Margetts and Srinivasan considered more general factors, and they showed that by varying the quasi-free states appropriately, the CCR flows in hyperfinite type III_λ factors are non-cocycle conjugate, for for $\lambda \in (0, 1]$. They also proved that there are uncountably many non-cocycle conjugate E_0-semigroups on all hyperfinite II_∞ and III_λ factors, for $\lambda \in (0, 1]$.

Plan of the talk;

1. **Modularly extendable endomorphism:**
2. **Equimodular endomorphism:**
3. **Type Classification of E_0 Semigroup-:**
4. **Invariants:**
5. **Examples:**

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Let M be a von Neumann algebra and let ϕ be a faithful normal semi-finite weight on M (in the continuation we will often use the abbreviation *f.n.s. weight*). Such a pair (M, ϕ) will be called a *non-commutative measure space.* Recall that ϕ has an associated GNS representation. Let $\mathcal{H}_\phi$ be the quotient and completion of $\mathfrak{N}_\phi = \{x \in M;\ \phi(x^*x) < +\infty\}$, and let $\mathfrak{N}_\phi \ni x \mapsto x_\phi \in \mathcal{H}_\phi$ denote the canonical map. The GNS $*$-representation $\pi_\phi : M \to \mathcal{B}(\mathcal{H}_\phi)$ is uniquely determined by the identity

$$\langle \pi_\phi(a)x_\phi, y_\phi \rangle = \phi(y^* ax), \qquad a \in M,\ x, y \in \mathfrak{N}_\phi.$$

We denote by J_ϕ, Δ_ϕ and $\{\sigma_t^\phi\}$ the modular conjugation operator, modular operator and modular automorphism group, respectively, for M associated to ϕ. When the weight is determined by the context, we will often suppress the subindex, and write J and Δ instead of J_ϕ and Δ_ϕ. We will also often identify M with $\pi_\phi(M)$, and identify $\pi_\phi(a)$ with a, for $a \in M$.

Extendability of endomorphisms

Definition

Let (M, ϕ) be factorial noncommutative measure space. Let $\mathcal{H}_\phi$ be the GNS space corresponding to ϕ, and let us identify M with its image under the GNS representation in $\mathcal{B}(\mathcal{H}_\phi)$. Suppose that $\theta : M \to M$ is a normal unital endormorphism, and let $\theta'_\phi : M' \to M'$ be the endomorphism given by

$$\theta'_\phi(y) = J_\phi \theta(J_\phi y J_\phi) J_\phi, \qquad y \in M'$$

We will say that θ is *ϕ-modularly extendable* if and only if there exists a normal endomorphism $\widetilde{\theta}_\phi$ of $\mathcal{B}(\mathcal{H}_\phi)$ satisfying

$$\widetilde{\theta}_\phi(xy') = \theta(x)\theta'_\phi(y), \qquad \forall x \in M, \forall y \in M'. \tag{0.1}$$

where J_ϕ is the modular conjugation operator. We note that by normality, such an extension is unique if it exists, and it will be called the *ϕ-modular extension* of θ.

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From the theory of standard von Neumann algebra it can be shown that this notion of extendability of the endomorphism θ on M does not depend on the choice of weight. So we simply call it as modularly extendable endomorphism. It also follows that θ is modularly extendable if and only if it's conjugation by an automorphism is also modularly extendable.

Equimodular endomorphisms

We now consider the concept of equimodularity of an endomorphism on a von Neumann algebra with respect to an f.n.s. weight. We exhibit a convenient necessary and sufficient condition for the equimodularity of an endomorphism with respect a fixed weight on von Neumann algebra.

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Given a noncommutative measure space (M, ϕ), let θ be a unital normal endomorphism of M which is ϕ-preserving, i.e.

$$\phi(\theta(x)) = \phi(x), \qquad x \in M^+.$$

This invariance assumption implies that there exists a unique well-defined isometry $u_\theta \in B(\mathcal{H}_\phi)$ given by $u_\theta(x_\phi) = (\theta(x))_\phi$, for $x \in \mathfrak{N}_\phi$. It is clear that $u_\theta x = \theta(x) u_\theta$, for all $x \in \mathfrak{N}_\phi$. Futhermore, since ϕ is semi-finite, $\mathfrak{N}_\phi$ is dense in M in the weak operator topology, so we have

$$u_\theta x = \theta(x) u_\theta, \qquad \forall x \in M.$$

Definition

Given a noncommutative measure space (M, ϕ), a unital endomorphism $\theta : M \to M$ will be called *equimodular* if ϕ is θ-invariant and $u_\theta J_\phi = J_\phi u_\theta$.

Given a unital normal endomorphism θ on a factor, we now describe a necessary and sufficient condition for the existence of an f.n.s. weight ϕ with respect to which θ is equimodular.

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Theorem

Let (M, ϕ) be a non-commutative measure space. Suppose θ is a unital normal endomorphism on M which is ϕ-preserving, i.e. $\phi(\theta(x)) = \phi(x)$ for all $x \in M^+$. Then θ is equimodular if and only if there exists a faithful normal conditional expectation $E : M \to \theta(M)$ which is ϕ-preserving, i.e.

$$\phi(E(x)) = \phi(x), \qquad \forall x \in M^+.$$

Furthermore, such a conditional expectation is unique if it exists since $E(x)e_\theta = e_\theta x e_\theta$ for all $x \in M$, where e_θ is the projection onto the closure of $(\theta(M) \cap \mathfrak{N}_\phi)_\phi$, and moreover $e_\theta = u_\theta u_\theta^*$.

Some Remarks regarding Examples

We now provide the convenient condition for the modular extendibility of equimodular endomorphisms.

Theorem

Let (M, ϕ) be a noncommutative factorial measure space and let θ be a unital normal endomorphism of M which is equimodular. If

$$(\theta(M) \cup (M \cap \theta(M)'))'' = M$$

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However, it is unclear to us whether the converse of the above theorem is true in the case of f.n.s. weights. It certainly holds in the case of faithful normal *states*,

Now we study modular extendability for E_0-semigroups on arbitrary factors. This leads to a classification scheme for E_0-semigroups based on the well-established classification of E_0-semigroups on type I_∞ factors due to Powers and Arveson, as well as several cocycle-conjugacy invariants.

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An *E_0-semigroup* on a W*-algebra M is a family $\alpha = \{\alpha_t : t \geq 0\}$ of normal unital *-homomorphism of M such that $\alpha_0 = id_M$ and $\alpha_s \circ \alpha_t = \alpha_{s+t}$ for all $t, s \geq 0$ which is weak*-continuous, i.e. for every $\rho \in M_*$ and $x \in M$, the map $[0, \infty) \ni t \mapsto \rho(\alpha_t(x))$ is continuous.

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Given an E_0-semigroup α on M, a strongly continuous family of unitary $U = \{U_t : t \geq 0\}$ in M will be called an α-cocycle if $U_{s+t} = U_t\alpha_t(U_s)$, for all $s, t \geq 0$. Notice that for an α-cocycle U we automatically have $U_0 = I$. An E_0-semigroup β on M is said to be *conjugate* to α if there exists an automorphism $\gamma \in \mathrm{Aut}\, M$ such that

$$\gamma \circ \beta_t \circ \gamma^{-1} = \alpha_t, \qquad t \geq 0.$$

We will say that β is *cocycle equivalent* to α if there exists an α-cocycle U such that

$$\beta_t(x) = U_t\alpha_t(x)U_t^*, \qquad t \geq 0, x \in M.$$

Finally, we will say that β is *cocycle conjugate* to α if there exists an E_0-semigroup β' of M which is conjugate to β such that β' is cocycle equivalent to α.

Definition

An E_0-semigroup α on a factor M is said to *modularly extendable* if α_t is modularly extendable for every $t \geq 0$.

Definition

An E_0-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ on a factorial noncommutative measure space (M, ϕ) is said to be *equimodular* if every $t \geq 0$, α_t is an equimodular endomorphism with respect to ϕ.

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We are naturally led to the following classification scheme.

Definition

Let M be a factor, and let α be an E_0-semigroup on M. If α has modular extension $\widetilde{\alpha}$ on some $\mathcal{B}(\mathcal{H}_\phi)$ for some f.n.s. weight ϕ (and hence all f.n.s. weights) , we will say that α has type EI, EII or EIII, respectively if $\widetilde{\alpha}$ has type I, II or III, respectively, in the sense of Arveson and Powers. Otherwise, we will simply say that α is not modularly extendable.

It can be proved that the type of an E_0-semigroup on a factor M is a cocycle conjugacy invariant.
We note that when M is a type I_∞ factor, i.e. M is isomorphic to $\mathcal{B}(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$, the type of E_0-semigroups generalizes the classification of Arveson and Powers. More precisely, if α is an E_0-semigroup of $\mathcal{B}(\mathcal{H})$, then

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We note that automorphism groups are modularly extendable, hence we always have trivial examples of E_0-semigroups of type EI on every factor. At present it is still unclear to us whether any II_1 factor has nontrivial E_0-semigroups of types EI and EII (of course type EIII is impossible in this case). But properly infinite factor, contains E_0-semigroups of type EII and type EIII on M.
Since every modularly extendable equimodular E_0-semigroup α on a properly infinite factor M has a joint unit with α', its modular extension cannot be of type EIII. Therefore, there exist E_0-semigroups which are modularly extendable yet not equimodular with respect to any weight.

We introduce an invariant for certain E_0-semigroups on a factorial noncommutative measure space (M, ϕ), which is a generalization of the invariant defined by Alevras for the context of II_1 factors.

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Let N be a subfactor of a factor M and let $E : M \to N$ be a faithful normal conditional expectation. Haagerup proved that there exists a faithful normal operator-valued weight $E^{-1} : N' \to M'$ which is characterized by the following identity: if ϕ is an f.n.s. weight on N and ψ is an f.n.s. weight on M',

$$\frac{d(\phi \circ E)}{d(\psi)} = \frac{d(\phi)}{d(\psi \circ E^{-1})},$$

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where $d(\phi \circ E)/d(\psi)$ and $d(\phi)/d(\psi \circ E^{-1})$ are Connes spatial derivatives.

The Kosaki index of E, which is a scalar, is defined by

$$(\mathrm{Ind}\, E)\, 1 = E^{-1}(1)$$

Let $\mathcal{E}(M, N)$ be the collection of all faithful normal conditional expectations from M onto N. Then the *minimal index of the pair* $N \subseteq M$ is defined to be

$$[M : N] = \min\{\operatorname{Ind} E \ : \ E \in \mathcal{E}(M, N)\}.$$

We note that if $\operatorname{Ind} E = \infty$ for some $E \in \mathcal{E}(M, N)$, then it is infinite for all elements of $\mathcal{E}(M, N)$, in which case $[M : N] = \infty$. In fact, there exists $E_0 \in \mathcal{E}(M, N)$ such that $[M : N] = \operatorname{Ind} E_0$. We note that if γ is an automorphism of M, then

$$[M : \gamma(N)] = [M : N] \tag{0.2}$$

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Definition

Let M be a factor, and let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup on M. For every $t \geq 0$, let $N_\alpha(t) = (\alpha_t(M)' \cap M) \vee \alpha_t(M)$ be the von Neumann algebra generated by $\alpha_t(M)' \cap M$ and $\alpha_t(M)$. We denote by $\mathcal{I}_\alpha$ the set of all $t \geq 0$ such that $N_\alpha(t)$ is a subfactor of M and $\mathcal{E}(M, N_t) \neq \varnothing$. For every $t \in \mathcal{I}_\alpha$, let

$$c_\alpha(t) = [M : N_\alpha(t)].$$

If $\mathcal{I}_\alpha \neq \varnothing$, then we define the *relative commutant index of* α to be the family $(c_\alpha(t))_{t \in \mathcal{I}_\alpha}$.

Relative commutant index

Theorem

Let M and N be factors and let α and β be E_0-semigroups on M and N, respectively. Suppose that $\mathcal{I}_\alpha$ and $\mathcal{I}_\beta$ are nonempty. Then

1. *The relative commutant index of α, that is the family $\{c_\alpha(t)\}_{t\in\mathcal{I}_\alpha}$, is invariant under conjugacy and cocycle conjugacy.*
2. *For all $t \in \mathcal{I}_\alpha \cap \mathcal{I}_\beta$ we have that $c_{\alpha\otimes\beta}(t) = c_\alpha(t)\cdot c_\beta(t)$.*

We determine the modular extendability and relative commutant index for the following examples of E_0-semigroups: q-CCR flows for $q \in (-1, 1)$ and CAR flows.
Here we discuss examples of E_0-semigroups arising from the q-canonical commutation relations.
Let $\mathcal{K}_{\mathbb{R}}$ be a *real* Hilbert space and let $\mathcal{K} = \mathcal{K}_{\mathbb{R}} + i\mathcal{K}_{\mathbb{R}}$ be its complexification. Let $\mathcal{H}_{\mathbb{R}} = L^2(0, \infty; \mathcal{K}_{\mathbb{R}})$ be the real Hilbert space of square integrable functions taking values in $\mathcal{K}_{\mathbb{R}}$, and let $\mathcal{H} = L^2(0, \infty; \mathcal{K})$, which is the complexification of $\mathcal{H}_{\mathbb{R}}$.

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Let $q \in (-1, 1)$ is a fixed real number. Let $\mathcal{F}_f(\mathcal{H})$ be the linear span of vectors of the form $f_1 \otimes f_2 \otimes \cdots \otimes f_n \in \mathcal{H}^{\otimes n}$ (with varying $n \in \mathbb{N}$), where we set $\mathcal{H}^{\otimes 0} \cong \mathbb{C}\Omega$ for some distinguished vector, called the vacuum vector. On $\mathcal{F}_f(\mathcal{H})$, we consider the sesquilinear form $\langle \cdot, \cdot \rangle_q$ given by the sesquilinear extension of

$$\langle f_1 \otimes f_2 \otimes \cdots f_n,\ g_1 \otimes g_2 \otimes \cdots g_m \rangle_q := \delta_{mn} \sum_{\pi \in S_n} q^{i(\pi)} \langle f_1, g_{\pi(1)} \rangle \cdots \langle f_n, g_{\pi(n)} \rangle$$

where S_n denotes the symmetric group of permutations of n elements and $i(\pi)$ is the number of inversions of the permutation $\pi \in S_n$, defined by

$$i(\pi) := \#\{(i,j) \,|\, 1 \leq i < j \leq n,\ \pi(i) > \pi(j)\}.$$

The q-Fock space $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathcal{F}_f(\mathcal{H})$ with respect to $\langle\cdot,\cdot\rangle_q$. Given $f \in \mathcal{H}$, the creation operator $l(f)$ on $\mathcal{F}_q(\mathcal{H})$ is the bounded operator defined by

$$l(f)\Omega = f,$$
$$l(f)f_1 \otimes \cdots \otimes f_n = f \otimes f_1 \otimes \cdots \otimes f_n,$$

and its adjoint is the annihilation operator $l(f)^*$ given by

$$l(f)^*\Omega = 0,$$
$$l(f)^* f_1 \otimes \cdots \otimes f_n = \sum_{i=1}^{n} q^{i-1}\langle f, f_i\rangle f_1 \otimes \cdots \otimes \check{f_i} \otimes \cdots \otimes f_n,$$

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We have that the following q-canonical commutation relation is satisfied:

$$l(f)^* l(g) - q l(g) l(f)^* = \langle f, g\rangle \cdot 1 \qquad f, g \in \mathcal{H}.$$

For $f \in \mathcal{H}_{\mathbb{R}}$, we define the self-adjoint operator $W(f) = l(f) + l(f)^*$, and we define the von Neumann algebra

$$\Gamma_q(\mathcal{H}_{\mathbb{R}}) = \{W(f) \mid f \in \mathcal{H}_{\mathbb{R}}\}''.$$

Definition

Suppose that $q \in (-1,1)$. Let $\{S_t\}_{t\geq 0}$ denote the shift semigroup on $\mathcal{H}$, and also its restriction to $\mathcal{H}_\mathbb{R}$. The *q-CCR flow* of rank $\dim \mathcal{K}_\mathbb{R}$ is the unique E_0-semigroup α^q on $\Gamma_q(\mathcal{H}_\mathbb{R})$ such that

$$\alpha_t^q(W(f)) = W(S_t f), \quad f \in \mathcal{H}_\mathbb{R}.$$

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Theorem

Suppose that $q \in (-1,1)$ and let α^q be q-CCR flow corresponding to a real Hilbert space $\mathcal{K}_{\mathbb{R}}$.

1. *The q-CCR flow α^q is equimodular with respect to the trace $\tau(x) = \langle x\Omega, \Omega\rangle_q$.*
2. $\Gamma_q(\mathcal{H}_{\mathbb{R}}) \cap \alpha_t^q(\Gamma_q(\mathcal{H}_{\mathbb{R}}))' = \mathbb{C}\cdot 1$ *for all* $t \geq 0$.
3. *the q-CCR flow α^q is not modularly extendable.*
4. *The relative commutant index of α^q is the constant family equal to ∞.*

Let $\mathcal{K}$ be a Hilbert space and let $\mathcal{H} = L^2(0, \infty) \otimes \mathcal{K}$. Let $\mathcal{F}_-(\mathcal{H})$ denote the anti-symmetric Fock space with vacuum vector Ω. For $f \in \mathcal{H}$, let $c(f) \in \mathcal{B}(\mathcal{F}_-(\mathcal{H}))$ be the creation operator given by

$$c(f)\Omega = f, \qquad c(f)f_1 \wedge \cdots \wedge f_n = f \wedge f_1 \wedge \cdots \wedge f_n, \qquad f_1, \ldots, f_n \in \mathcal{H}$$

We note that the map $\mathcal{H} \to \mathcal{B}(\mathcal{F}_-(\mathcal{H}))$ given by $f \mapsto c(f)$ is $\mathbb{C}$-linear, and it satisfies the canonical commutation relations

$$c(f)c(g)+c(g)c(f) = 0 \quad \text{and} \quad c(f)c(g)^*+c(g)^*c(f) = \langle f, g\rangle 1, \qquad f, g \in \mathcal{H}.$$

where of course 1 denotes the identity operator.

Let $\mathcal{K}$ be a Hilbert space and let $\mathcal{H} = L^2(0,\infty) \otimes \mathcal{K}$. Let $\mathcal{F}_-(\mathcal{H})$ denote the anti-symmetric Fock space with vacuum vector Ω. For $f \in \mathcal{H}$, let $c(f) \in \mathcal{B}(\mathcal{F}_-(\mathcal{H}))$ be the creation operator given by

$$c(f)\Omega = f, \qquad c(f)f_1 \wedge \cdots \wedge f_n = f \wedge f_1 \wedge \cdots \wedge f_n, \qquad f_1, \ldots, f_n \in \mathcal{H}$$

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The CAR algebra $\mathcal{A}(\mathcal{H})$ is the unital C^*-algebra generated by $\{a(f) : f \in \mathcal{H}\}$ in $\mathcal{B}(\mathcal{F}_-(\mathcal{H}))$. We note that $||a(f)|| = ||f||$ for $f \in \mathcal{H}$. Now suppose $R \in \mathcal{B}(\mathcal{H})$ satisfies $0 \leq R \leq 1$. Every such operator R determines a unique state ω_R on $\mathcal{A}(\mathcal{H})$, called the quasi-free state with two-point function R, which satisfies the following condition:

$$\omega_R(c^*(f_m)\cdots c^*(f_1)c(g_1)\cdots c(g_n)) = \delta_{mn} \det(\langle g_i, Rf_j\rangle).$$

Definition

Suppose that $R \in \mathcal{B}(\mathcal{H})$ satisfies $0 \leq R \leq 1$ and $S_t^* R S_t = R$ for all $t \geq 0$, and let π_R be the GNS representation for ω_R. Then the unique E_0-semigroup α^R on $M_R = \pi_R(\mathcal{A}(\mathcal{H}))''$ satisfying

$$\alpha_t^R(\pi_R(c(f)) = \pi_R(c(S_t f)), \qquad f \in \mathcal{H}, t \geq 0$$

is called the *CAR flow of rank dim* $\mathcal{K}$ (on M_R) associated to the operator R.

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Theorem

Let $R \in \mathcal{B}(\mathcal{H})$ be an operator such that $0 \leq R \leq 1$, $S_t^* R S_t = R$ for all $t \geq 0$, and R and $1 - R$ are invertible. Furthermore, suppose that $RS_t\mathcal{H} \subseteq S_t\mathcal{H}$ and $Tr(R|_{S_t\mathcal{H}} - R|^2_{S_t\mathcal{H}}) = \infty$. Then the CAR flow α^R has the following properties:

1. *it is equimodular with respect to ω_R.*
2. *it is not modularly extendable.*
3. *the relative commutant index $(c_\alpha(t))_{t \geq 0}$ satisfies $1 < c_\alpha(t) \leq 2$ for all $t \geq 0$.*

Corollary

Let $\mathcal{K}$ be a Hilbert space of any dimension and let $R \in \mathcal{B}(\mathcal{H})$ be an operator satisfying the following properties: $0 \leq R \leq 1$, R and $1 - R$ are invertible, $\mathrm{Tr}(R|_{S_t\mathcal{H}} - R|^2_{S_t\mathcal{H}}) = \infty$ and moreover $S_t^* R S_t = R$ and $RS_t\mathcal{H} \subseteq \mathcal{H}$ for all $t \geq 0$. Let α be the corresponding CAR flow on M_R. Then for $k \neq \ell \in \mathbb{N}$, we have that $\alpha^{\otimes k}$ and $\alpha^{\otimes \ell}$ are not cocycle conjugate when considered as E_0-semigroups on M_R.

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Let $\mathcal{K}$ be a Hilbert space of any dimension and let $R \in \mathcal{B}(\mathcal{H})$ be an operator satisfying the following properties: $0 \leq R \leq 1$, R and $1 - R$ are invertible, $\mathrm{Tr}(R|_{S_t\mathcal{H}} - R|^2_{S_t\mathcal{H}}) = \infty$ and moreover $S_t^* R S_t = R$ and $RS_t\mathcal{H} \subseteq \mathcal{H}$ for all $t \geq 0$. Let α be the corresponding CAR flow on M_R. Then for $k \neq \ell \in \mathbb{N}$, we have that $\alpha^{\otimes k}$ and $\alpha^{\otimes \ell}$ are not cocycle conjugate when considered as E_0-semigroups on M_R.

Remark : It follows from Corollary that by varying $R \in \mathcal{B}(\mathcal{H})$ we obtain on every hyperfinite factor of types II_1, II_∞ and III_λ for $\lambda \in (0, 1)$ a countably infinite family of E_0-semigroups which are not modularly extendable and pairwise non-cocycle conjugate.

THANK YOU