Nilpotent Completely Positive Maps

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OTOA
ISI Bangalore
December 16, 2014

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- Take $V_i = \ker(L^i)$ for $1 \leq i \leq p$.
- Then $\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_p = V$.
- $l_i := \dim V_i / V_{i-1}$, $V_0 = \{0\}$.
- $l_1 + l_2 + \cdots + l_p = n$ and $l_1 \geq l_2 \geq \cdots \geq l_p$.
- $(l_1, l_2, \cdots, l_p)$ is a partition of n.
- We call $(l_1, l_2, \cdots, l_p)$ as nilpotent type of L.

Nilpotent maps and invariant subspaces

- $M \subseteq V$ invariant subspace of L.
- $R := L|_M$.
- Define $S : V/M \to V/M$ by $S(v + M) = L(v) + M$.

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- Littlewood-Richardson rules, Horn's inequalities.
- **Majorization inequalities:**
  $l_1 + \cdots + l_k \leq (r_1 + \cdots + r_k) + (s_1 + \cdots + s_k)$ for all $1 \leq k < p$
  and
  $l_1 + \cdots + l_p = (r_1 + \cdots + r_p) + (s_1 + \cdots + s_p)$.

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Definition

A linear map $\alpha : B(H) \to B(H)$ is said to be CP-map if

$$\sum_{i,j=1}^{k} Y_i^* \alpha(X_i^* X_j) Y_j \geq 0 \qquad \forall X_i, Y_i \in B(H), k \geq 1.$$

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- What happens if α is nilpotent ?
- We want to define "nilpotency type" of α.
- We will show that nilpotency order is not bigger than n

Choi-Kraus decomposition

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- The decomposition is not unique.
- But if $\alpha(X) = \sum_{j=1}^{d'} M_j^* X M_j$ is another decomposition, then
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  - $\text{ran}(\alpha(1)) = \overline{\text{span}}\{L_i u : u \in H\}$.
  - $\alpha^p(X) = 0$ for all $X \in B(H)$ iff $L_{i_1} L_{i_2} \ldots . L_{i_p = 0}$ for all $i_1, i_2 \ldots ., i_p \geq 1$.

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- If $\alpha : B(H) \to B(H)$ is a completely positive map of nilpotent order p, then $p \leq \dim(H)$.

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- Let $a^k := \dim(H^k)$ for $1 \leq k \leq p$.
- Then $a_{p-i+1} + a_{p-i+2} + \cdots + a_p \leq a^1 + a^2 + \cdots + a^i$ for $1 \leq i \leq p$ and $a_1 + a_2 + \cdots + a_p = a^1 + a^2 + \cdots + a^p$.

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- We get $\alpha(X) = \sum L_i^* X L_i$ with $L_i = \begin{bmatrix} B_i & 0 \\ D_i & C_i \end{bmatrix} \in B(M \oplus N)$ for some operators $B_i \in B(M), C_i \in B(N), D_i \in B(M, N)$, $1 \leq i \leq d$.

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- Note that for any $i_1, i_2, \ldots, i_p$,

$$L_{i_1} L_{i_2} \ldots L_{i_p} = \begin{bmatrix} B_{i_1} B_{i_2} \ldots B_{i_p} & 0 \\ D_{i_1, i_2, \ldots, i_p} & C_{i_1} C_{i_2} \ldots C_{i_p} \end{bmatrix}$$

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for some operator $D_{i_1, i_2, \ldots, i_p}$.
- Thus if α is nilpotent of order p, then β and γ are nilpotent of order at most p.

Majorization

Theorem

Suppose $(a_1, \cdots, a_p), (b_1, \cdots, b_p)$ and $(c_1, \cdots, c_p)$ are CP nilpotent type of α, β and γ respectively. Then

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i + \sum_{i=1}^{k} c_i$$

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Proof:

- For the inequality part, first consider the case $k = 1$.
  Let $\{u_1, u_2, \ldots, u_r\}$ be a basis for $(\bigcap_i \ker(B_i)) \bigcap (\bigcap_i \ker(D_i))$.
- Let $\{v_1, v_2, \ldots, v_{c_1}\}$ be a basis for $\bigcap_i \ker(C_i)$.

Proof continues:

- $\{\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} u_r \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ v_{c_1} \end{pmatrix}\}$ is linearly independent in $\bigcap_i \ker(L_i)$.
- Extend this collection to:

  $$\{\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} u_r \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ v_{c_1} \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \ldots,$$

  a basis of $\bigcap_i \ker(L_i)$.
- In particular, $a_1 = r + c_1 + s$.
- Now we observe that $x_1, x_2, \ldots, x_s$ are vectors in $\bigcap_i \ker(B_i)$.
- We claim that $\{u_1, u_2, \ldots, u_r, x_1, x_2, \ldots, x_s\}$ are linearly independent in $\bigcap_i \ker(B_i)$.
- We have $r + s \leq b_1$ and hence $a_1 \leq b_1 + c_1$.

Proof continues:

- Suppose $\sum_j p_j u_j + \sum_j q_j x_j = 0$ for some scalars p_j, q_j.
- Fix $1 \leq i \leq d$.
- Then as $u_j \in \bigcap_i \ker(D_i)$ for all j, $\sum_j q_j x_j \in \ker(D_i)$. Further as $\begin{pmatrix} x_j \\ y_j \end{pmatrix}$ is in $\ker(L_i)$, we have $D_i x_j + C_i y_j = 0$ or $C_i y_j = -D_i x_j$ for all j.
- Consequently $\sum_j C_i q_j y_j = -\sum_j D_i q_j x_j = 0$. Therefore, $\sum_j q_j y_j \in \bigcap_i \ker(C_i)$.
- So there exist scalars $t_i, 1 \leq i \leq c_1$, such that $\sum_i t_i v_i = -\sum_j q_j y_j$.
- Then, $\sum_j p_j \begin{pmatrix} u_j \\ 0 \end{pmatrix} + \sum_j t_j \begin{pmatrix} 0 \\ v_j \end{pmatrix} + \sum_j q_j \begin{pmatrix} x_j \\ y_j \end{pmatrix} = 0.$
- Now due to linear independence of these vectors, $p_j \equiv 0, q_j \equiv 0$ and $t_j \equiv 0$.
- Consider $\alpha^k(1), \beta^k(1)$ and $\gamma^k(1)$, $\sum_{j=1}^k a_j \leq \sum_{j=1}^k b_j + \sum_{j=1}^k c_j$ for $1 \leq k < p$ as $\sum_{j=1}^k a_j =$dim $\ker(\alpha^k(1))$.

Roots of states

Definition

Let H be a finite dimensional Hilbert space and let $u \in H$ be a unit vector in H. Consider the pure state $X \mapsto \langle u, Xu \rangle I$ on $B(H)$. Then a unital completely positive map $\tau : B(H) \to B(H)$ is said to be an n^{th} root of this state if

$$\tau^n(X) = \langle u, Xu \rangle I \quad \forall X \in B(H).$$

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$$\tau^n(X) = \langle u, Xu\rangle I \quad \forall X \in B(H).$$

Theorem

Let $\tau : B(H) \to B(H)$ be a unital CP-map such that $\tau^p(X) = \langle u, Xu\rangle I$ where u is a unit vector of H. Set $H_0 = \{x \in H : \langle x, u\rangle = 0\}$ so that $H = \mathbb{C}u \oplus H_0$. Suppose $\alpha : B(H_0) \to B(H_0)$ is the compression of τ to $B(H_0)$, then α is nilpotent CP map of order at most p.

Roots of states

Theorem

Let $\alpha : \mathcal{B}(H_0) \to \mathcal{B}(H_0)$ be a contractive CP-map such that $\alpha^p(Y) = 0$ for all $Y \in \mathcal{B}(H_0)$. Take $H = \mathbb{C} \oplus H_0$ and $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Suppose $\tau : \mathcal{B}(H) \to \mathcal{B}(H)$ is a map defined by

$$\tau(\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}) = \begin{pmatrix} X_{11} & 0 \\ 0 & \alpha(X_{22}) + X_{11}(I - \alpha(I)) \end{pmatrix}$$

for all $X = [X_{ij}] \in \mathcal{B}(\mathbb{C} \oplus H_0)$. Then τ is a CP-map and $\tau^p(X) = \langle u, Xu \rangle I$ for all $X \in \mathcal{B}(H)$.

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THANKS