C-symmetric operators and its preanihilator

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Joint work with Kamila Kliś-Garlicka

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C – isometric antilinear involution in $\mathcal{H}$

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Example

C-symmetry in $l^2(\mathbb{N})$ is given by

$$C(z_0, z_1, z_2, \dots) = (\overline{z}_0, \overline{z}_1, \overline{z}_2, \dots).$$

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C-symmetry in $\mathbb{C}^n$ is given by

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Jordan block $J_n(\lambda)$ is C–symmetric.

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H^2 – the classical Hardy space , u a nonconstant inner function

$$Cf = u\overline{zf} \quad f \in H^2_u$$

C-symmetry on $H^2_u = H^2 \ominus uH^2$

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$$\varphi \in L^\infty \qquad\qquad T_\varphi f = P_{H^2_u}(\varphi f), \quad f \in H^2_u$$

$T_\varphi \in \mathcal{B}(H^2_u)$ Truncated Toeplitz Operator is C–symmetric,

Example

ϱ a bounded, positive continuous weight on $[-1,1]$,
$\varrho(t) = \varrho(-t)$ for $t \in [0,1]$.
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Example

$Cf(t) = \overline{f(1-t)}$ is C–symmetry on $L^2([0, 1], dt)$.
Voltera operator $Vf(x) = \int_0^x f(t)dt$ is C–symmetric

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Connections with von Neumann Algebras

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norm, WOT, SOT, weak* – closed subspace

Describe preanihilator of $\mathcal{C}$

Theorem

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$x \otimes y \in {}_{\perp}\mathcal{C}$ $\quad 0 =< e_i \otimes e_i, x \otimes y >= \langle (e_i \otimes e_i)x, y \rangle = \langle x, e_i \rangle \langle e_i, y \rangle$

$x \perp e_i$ or $y \perp e_i$ for all $i \in \mathbb{N}$
$k \in \mathbb{N}$ – smallest number such that $\langle x, e_k \rangle \neq 0$
$l \in \mathbb{N}$ – smallest number such that $\langle y, e_l \rangle \neq 0$
Hence $k \neq l$, $\langle x, e_l \rangle = 0$, $\langle y, e_k \rangle = 0$

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Hence $k \neq l$, $\langle x, e_l\rangle = 0$, $\langle y, e_k\rangle = 0$
$\alpha e_l + \beta e_k$, $\alpha, \beta \neq 0$, then $C(\alpha e_l + \beta e_k) = \overline{\alpha} e_l + \overline{\beta} e_k$
$(\alpha e_l + \beta e_k) \otimes (\overline{\alpha} e_l + \overline{\beta} e_k) \in \mathcal{C}$ for any $\alpha, \beta \neq 0$

$$0 =< (\alpha e_l + \beta e_k) \otimes (\overline{\alpha} e_l + \overline{\beta} e_k), x \otimes y >= \langle x, \overline{\alpha} e_l + \overline{\beta} e_k\rangle\langle \alpha e_l + \beta e_k, y\rangle = \beta\langle x, e_k\rangle\alpha\langle e_l, y\rangle$$

Since $\alpha, \beta \neq 0$ and $\langle x, e_k\rangle \neq 0$, $\langle e_l, y\rangle \neq 0$ we get the contradiction.
Hence $x = 0$ or $y = 0$.

Theorem

Let $\mathcal{H}$ be a complex separable Hilbert space with isometric anti-linear involution C. Let $\mathcal{C}$ be the set of C-symmetric operators. Then $F_2 \cap {}_\perp\mathcal{C} = \{h \otimes g - Cg \otimes Ch : h, g \in \mathcal{H}\}$.

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To show the inclusion „$\supset$" note that for $T \in \mathcal{C}$ we have

$$< T, h \otimes g - Cg \otimes Ch >= \langle Th, g\rangle - \langle TCg, Ch\rangle =$$
$$\langle Th, g\rangle - \langle C^2h, CTCg\rangle = \langle h, T^*g\rangle - \langle h, CTCg\rangle = 0$$

Example

C-symmetry in $l^2(\mathbb{N})$

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Theorem

Let $\mathcal{H}$ be a complex separable Hilbert space with isometric antilinear involution C. Let $\mathcal{C}$ be the set of C-symmetric operators. The subspace $\mathcal{C} \subset \mathcal{B}(\mathcal{H})$ of C-symmetric operators is 2-reflexive.

$A \in \mathcal{B}(\mathcal{H}) \qquad \mathcal{W} \subset \mathcal{B}(\mathcal{H})$ – algebra

$\operatorname{dist}(A, \mathcal{W}) = \inf\{\|A - T\| : T \in \mathcal{W}\}$

$\alpha(A, \mathcal{W}) = \sup\{\|P^{\perp} A P\| : P \in \operatorname{Lat} \mathcal{W}\}$

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Arveson(for algebras), Larson, Kraus(for subspaces)

$\mathcal{M}$ is called *hyperreflexive* if there is κ such that

$$\operatorname{dist}(A, \mathcal{M}) \leqslant \kappa\, \alpha(A, \mathcal{M})$$

The smallest constant $\kappa(\mathcal{M})$ is called hyperreflexive constant.

$$\mathrm{dist}(A, \mathcal{M}) = \sup\{|\langle A, t\rangle| : t \in \tau c,\, t \in \mathrm{ball}_{\perp}\mathcal{M}\}$$

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$$\alpha_k(A, \mathcal{M}) = \sup\{|tr(Af)| : f \in F_k(\mathcal{H}) \cap \mathrm{ball}_{\perp}\mathcal{M}\}$$

$\mathcal{M}$ is *k–hyperreflexive* if there is κ such that for any $A \in \mathcal{B}(\mathcal{H})$:

$$d(A, \mathcal{M}) \leqslant \kappa\, \alpha_k(A, \mathcal{M}).$$

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$\mathcal{M}$ is norm–closed
$\mathcal{M}$ is hyperreflexivite $\Longrightarrow$ $\mathcal{M}$ is reflexivite

$\alpha_k(A, \mathcal{M}) = \sup\{|tr(Af)| : f \in F_k(\mathcal{H}) \cap \text{ball}_\perp \mathcal{M}\}$

$\mathcal{M}$ is *k–hyperreflexive* if there is κ such that for any $A \in \mathcal{B}(\mathcal{H})$:

$$d(A, \mathcal{M}) \leqslant \kappa\, \alpha_k(A, \mathcal{M}).$$

The smallest constant $\kappa_k(\mathcal{M})$ is called k–hyperreflexive constant.

Theorem

Let $\mathcal{H}$ be a complex separable Hilbert space with isometric antilinear involution C. Let $\mathcal{C}$ be the set of C-symmetric operators. The subspace $\mathcal{C}$ is 2-hyperreflexive with constant 1.

Proof.

$A \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned}\alpha_2(A,\mathcal{C}) = &\sup\{|\mathrm{tr}(A(\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)))| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\} = \\ &\tfrac{1}{2}\sup\{|\langle Ah, g\rangle - \langle ACg, Ch\rangle| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\} = \\ &\tfrac{1}{2}\sup\{|\langle h, A^*g\rangle - \langle h, CACg\rangle| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\} = \\ &\tfrac{1}{2}\sup\{|\langle h, (A^* - CAC)g\rangle| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\} \geqslant \\ &\tfrac{1}{2}\sup\{\|\langle h, (A^* - CAC)g\rangle| : \|h\| \leqslant 1,\ \|g\| \leqslant 1\} = \tfrac{1}{2}\|A^* - CAC\|\end{aligned}$$

Proof.

$A \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned}\alpha_2(A,\mathcal{C}) = &\sup\{|\mathrm{tr}(A(\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)))| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\} = \\ &\tfrac{1}{2}\sup\{|\langle Ah, g\rangle - \langle ACg, Ch\rangle| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\} = \\ &\tfrac{1}{2}\sup\{|\langle h, A^*g\rangle - \langle h, CACg\rangle| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\} = \\ &\tfrac{1}{2}\sup\{|\langle h, (A^* - CAC)g\rangle| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\} \geqslant \\ &\tfrac{1}{2}\sup\{\|\langle h, (A^* - CAC)g\rangle| : \|h\| \leqslant 1,\ \|g\| \leqslant 1\} = \tfrac{1}{2}\|A^* - CAC\|\end{aligned}$$

$$\langle CACh, g\rangle = \langle Cg, C^2ACh\rangle = \langle Cg, ACh\rangle = \langle A^*Cg, Ch\rangle = \langle C^2h, CA^*Cg\rangle = \langle h, CA^*Cg\rangle$$

Hence $(A + CA^*C)^* = A^* + CAC$,

$$C(A + CA^*C)C = CAC + C^2A^*C^2 = CAC + A^* = (A + CA^*C)^*$$

Thus $A + CA^*C \in \mathcal{C}$,

Proof.

$A \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned}\alpha_2(A,\mathcal{C}) = &\sup\{|\mathrm{tr}(A(\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)))| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\} = \\ &\tfrac{1}{2}\sup\{|\langle Ah,g\rangle - \langle ACg,Ch\rangle| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\} = \\ &\tfrac{1}{2}\sup\{|\langle h,A^*g\rangle - \langle h,CACg\rangle| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\} = \\ &\tfrac{1}{2}\sup\{|\langle h,(A^*-CAC)g\rangle| : \|\tfrac{1}{2}(h\otimes g - Cg\otimes Ch)\|_1 \leqslant 1\} \geqslant \\ &\tfrac{1}{2}\sup\{\||\langle h,(A^*-CAC)g\rangle| : \|h\|\leqslant 1,\ \|g\|\leqslant 1\} = \tfrac{1}{2}\|A^*-CAC\|\end{aligned}$$

$$\langle CACh,g\rangle = \langle Cg,C^2ACh\rangle = \langle Cg,ACh\rangle = \langle A^*Cg,Ch\rangle = \langle C^2h,CA^*Cg\rangle = \langle h,CA^*Cg\rangle$$

Hence $(A+CA^*C)^* = A^*+CAC$,

$$C(A+CA^*C)C = CAC + C^2A^*C^2 = CAC + A^* = (A+CA^*C)^*$$

Thus $A + CA^*C \in \mathcal{C}$, which implies that

$$d(A,\mathcal{C}) \leqslant \|A - \tfrac{1}{2}(A+CA^*C)\| = \tfrac{1}{2}\|A-CA^*C\| = \tfrac{1}{2}\|A^*-CAC\| \leqslant \alpha_2(A,\mathcal{C}).$$

Hence $\mathcal{C}$ is 2-hyperreflexive with constant 1.

Thank you!