Subnormality of composition operators over directed graphs with one circuit: exotic examples

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Operators

- $\mathcal{H}$ is a complex Hilbert space.
- By an **operator** in $\mathcal{H}$ we mean a linear mapping

$$A\colon \mathcal{H} \supseteq \mathcal{D}(A) \to \mathcal{H}$$

  defined on a vector subspace $\mathcal{D}(A)$ of $\mathcal{H}$, called the **domain** of A.
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- An operator S in $\mathcal{H}$ is **subnormal** if S is densely defined and there exists a complex Hilbert space $\mathcal{K}$ and a normal operator N in $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $Sh = Nh$ for every $h \in \mathcal{D}(S)$, or simply

$$S \subseteq N.$$

- An operator A in $\mathcal{H}$ is **hyponormal** if A is densely defined, $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|A^* f\| \leqslant \|Af\|$ for every $f \in \mathcal{D}(A)$.
- An operator A in $\mathcal{H}$ is **paranormal** if $\|Af\|^2 \leqslant \|f\| \|A^2 f\|$ for all $f \in \mathcal{D}(A^2)$.
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Subnormal operators

- The theory of unbounded subnormal operators subsumes the theories of bounded subnormal operators and unbounded symmetric operators.
- bounded operators:
  Halmos (1950), Bram (1955), . . .
  J. Conway (two monographs)
- unbounded operators:
  Bishop (1957), Foiaş (1962), McDonald & Sundberg (1986), JS & Szafraniec (1985-89), . . .

The creation operator of quantum mechanics

- The **creation operator** a_+ is defined in $L^2(\mathbb{R})$ by

$$a_+ = \frac{1}{\sqrt{2}}\Big(x - \frac{d}{dx}\Big).$$

- a_+ is subnormal.
- a_+ is unitarily equivalent to the operator of multiplication by the independent variable "z" in the Segal-Bargmann space

  ($=$ the Hilbert space of entire functions that are square integrable with respect to the Gaussian measure on the complex plane [Segal, Bargmann 1961]).

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Symmetric operators

- An operator S in $\mathcal{H}$ is said to be **symmetric** (resp., **selfadjoint**) if S is densely defined and $S \subseteq S^*$ (respectively, $S = S^*$).
- A symmetric operator S in $\mathcal{H}$ is subnormal because it has a selfadjoint extension possibly in a larger Hilbert space [Naimark].

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Formally normal operators

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- ? $\rightsquigarrow$ normal
- formally normal $\rightsquigarrow$ normal
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- Formally normal operators **may not be** subnormal [Coddington 1965].
- There exist a **nonsubnormal** formally normal operator A and a polynomial $p \in \mathbb{C}[Z, \bar{Z}]$ **of degree** 3 such that $\mathcal{D}(A)$ is invariant for A and A^*, and

$$p(A, A^*)f = 0 \text{ for every } f \in \mathcal{D}(A);$$

  3 is the smallest possible degree [JS 1991].
- $p = Y(Y - X^2)$ where $X = \frac{1}{2}(Z + \overline{Z})$ and $Y = \frac{1}{2\mathrm{i}}(Z - \overline{Z})$.

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Hamburger and Stieltjes moment sequences

- A sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ of real numbers is called a **Hamburger moment sequence** if there exists a (positive) Borel measure μ on $\mathbb{R}$ such that

$$\gamma_n = \int_{\mathbb{R}} x^n \, \mathrm{d}\,\mu(x), \quad n \geqslant 0;$$

such a μ is called an **H-representing measure** of γ.

- We say that a Hamburger moment sequence is **H-determinate** if it has a unique H-representing measure; otherwise, we call it **H-indeterminate**.
- Replacing the real line $\mathbb{R}$ by the half-line $[0, \infty)$ in the above definitions, we get the notions of a **Stieltjes moment sequence**, **S-representing measure**, **S-determinacy** and **S-indeterminacy**.

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- We say that an operator S in $\mathcal{H}$ **generates Stieltjes moment sequences** if the set $\mathscr{D}^\infty(S) := \bigcap_{n=0}^\infty \mathscr{D}(S^n)$ is dense in $\mathcal{H}$ and $\{\|S^n f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $f \in \mathscr{D}^\infty(S)$.

Theorem (Lambert 1976)

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A question

- Lambert's theorem is not true for unbounded operators.
- Recall that there are nonsubnormal formally normal (hence hyponormal) operators which generate Stieltjes moment sequences.
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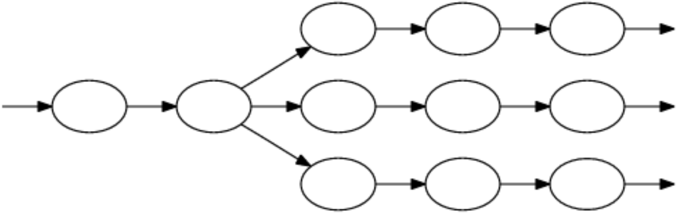
The directed tree $\mathscr{T}_{\eta,\kappa}$.

$\eta \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ and $\kappa \in \{0, 1, 2, \ldots\} \cup \{\infty\}$.



$\mathscr{T}_{\eta,\kappa}$ is a directed tree with one branching vertex and η branches; its trunk consists of $\kappa + 1$ vertices (counting the branching vertex).

A counterexample - the directed tree $\mathscr{T}_{\infty,\kappa}$

Theorem (Jabłoński, Jung & JS – J. Funct. Anal. 2012)

For every $\kappa \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ there exists an injective weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}_{\infty,\kappa}$ such that:

- *$S_{\boldsymbol{\lambda}}$ generates Stieltjes moment sequences,*
- *$S_{\boldsymbol{\lambda}}$ is not hyponormal, hence it is not subnormal,*
- *$S_{\boldsymbol{\lambda}}$ is a paranormal operator,*
- *$\mathcal{D}^{\infty}(S_{\boldsymbol{\lambda}})$ is a core for $S_{\boldsymbol{\lambda}}^n$ for every $n \geqslant 0$.*

♣ The proof of the above theorem depends heavily on some subtle properties of N-extremal measures.

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Composition operators in L^2-spaces

- $(X, \mathcal{A}, \mu)$ is a σ-finite measure space.
- $\phi\colon X \to X$ is an $\mathcal{A}$-**measurable** transformation, i.e., $\phi^{-1}(\varDelta) \in \mathcal{A}$ for every $\varDelta \in \mathcal{A}$.
- If ϕ is **nonsingular**, i.e., the measure $\mu \circ \phi^{-1}$ given by $\mu \circ \phi^{-1}(\varDelta) = \mu(\phi^{-1}(\varDelta))$ for $\varDelta \in \mathcal{A}$ is absolutely continuous with respect to μ, then the operator C_ϕ in $L^2(\mu)$ given by

$$\mathcal{D}(C_\phi) = \{f \in L^2(\mu)\colon f \circ \phi \in L^2(\mu)\},$$
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is well-defined.
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Subnormality of bounded composition operators

Theorem (Lambert 1988)

Let C_ϕ be a **bounded** *composition operator on $L^2(\mu)$. Then the following two conditions are equivalent:*

- *C_ϕ is subnormal,*
- *for μ-a.e. $x \in X$, $\{h_n(x)\}_{n=0}^{\infty}$ is a Stieltjes moment sequence, where*

$$h_n := \frac{\mathrm{d}\,\mu \circ \phi^{-n}}{\mathrm{d}\,\mu} \qquad \textit{(the Radon-Nikodym derivative).}$$

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A problem

- Does Lambert's theorem remain true for unbounded composition operators in L^2-spaces?
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A counterexample - the composition operator case

Theorem (Jabłoński, Jung & JS – J. Funct. Anal. 2012)

There exists an injective composition operator C in an L^2-space over a σ-finite measure space such that:

- *for μ-a.e. $x \in X$, $\{h_n(x)\}_{n=0}^\infty$ is a Stieltjes moment sequence,*
- *C is not hyponormal, thus it is not subnormal,*
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- The above can be derived from the previous counterexample (the weighted shift S_λ on $\mathscr{T}_{\infty,\infty}$) by using the fact that weighted shifts on **rootless** directed trees with nonzero weights are unitary equivalent to some composition operators in L^2 spaces.

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A problem

- Find a criterion for subnormality of unbounded composition operators in L^2 spaces.
- It should cover the case of bounded composition operators.
- No restrictions on domains of powers of operators in question.
- The main difficulty: the **known criteria** for subnormality of general Hilbert space operators do not help us to solve the problem.

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The conditional expectation

- We assume that the transformation ϕ is nonsingular and C_ϕ is densely defined.
- If $f\colon X \to \mathbb{R}_+$ is an $\mathcal{A}$-measure function, then there exists a unique (up to sets of μ-measure zero) $\phi^{-1}(\mathcal{A})$-measurable function $\mathsf{E}(f)\colon X \to \mathbb{R}_+$ such that

$$\int_{\phi^{-1}(\Delta)} f \,\mathrm{d}\,\mu = \int_{\phi^{-1}(\Delta)} \mathsf{E}(f) \,\mathrm{d}\,\mu, \quad \Delta \in \mathcal{A}.$$

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The consistency condition

- $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ is said to be an $\mathcal{A}$**-measurable family of probability measures** if the set-function $P(x, \cdot)$ is a probability measure for every $x \in X$ and the function $P(\cdot, \sigma)$ is $\mathcal{A}$-measurable for every $\sigma \in \mathfrak{B}(\mathbb{R}_+)$.
- We say that an $\mathcal{A}$-measurable family of probability measures $P\colon X \times \mathfrak{B}(\mathbb{R}_+) \to [0,1]$ satisfies the **consistency condition** if

$$\mathsf{E}(P(\cdot,\sigma))(x) = \frac{\int_\sigma t\, P(\phi(x), \mathrm{d}\, t)}{\mathsf{h}_\phi(\phi(x))} \text{ for } \mu\text{-a.e. } x \in X, \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),$$

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A criterion for subnormality

- If C_ϕ is subnormal, then C_ϕ is densely defined and injective.

Theorem (Budzyński, Jabłoński, Jung & JS 2013)

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Generating moment sequences

- Find the relationship between the consistency condition and moments.

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$$\mathsf{h}_{\phi^n}(x) = \int_0^\infty t^n P(x, \mathrm{d}\, t) \text{ for } \mu\text{-a.e. } x \in X, \quad n = 0, 1, 2, \ldots.$$

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A class of composition operators

- Let X be a nonempty set and $\phi\colon X \to X$ be a mapping. Set

$$E_\phi = \{(x, y) \in X \times X\colon x = \phi(y)\}.$$

  Then (X, E_ϕ) is a directed graph.

- Note that for every $y \in X$, $\phi(y)$ is the parent of y. Hence, $\phi^{-1}(\{x\})$ can be thought of as the set of all children of x.
- Connected directed graphs (X, E_ϕ) whose vertices, all but one, have valency one can be described explicitly.

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(X, E_ϕ) with one branching vertex

Theorem (Budzyński, Jabłoński, Jung & JS 2014)

Let (X, E_ϕ) be as above and let $\eta \in \{1, 2, 3, \ldots\} \cup \{\infty\}$. Then the following two conditions are equivalent:

(i) *the directed graph (X, E_ϕ) is connected and there exists $\omega \in X$ such that $\mathrm{card}(\phi^{-1}(\{\omega\})) = \eta + 1$ and $\mathrm{card}(\phi^{-1}(\{x\})) = 1$ for every $x \in X \setminus \{\omega\}$,*

(ii) *(X, E_ϕ) takes one of the following two forms:*

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(ii-a) *there exist* $\kappa \in \{0, 1, 2, \ldots\}$ *and two disjoint systems* $\{x_i\}_{i=0}^{\kappa}$ *and* $\{x_{i,j}\}_{i=1}^{\eta}{}_{j=1}^{\infty}$ *of distinct points of* X *such that*

$$X = \{x_0, \ldots, x_\kappa\} \cup \{x_{i,j} \colon i \in J_\eta, j \geqslant 1\},$$

$$\phi(x) = \begin{cases} x_{i,j-1} & \text{if } x = x_{i,j} \text{ with } i \in J_\eta \text{ and } j \geqslant 2, \\ x_\kappa & \text{if } x = x_{i,1} \text{ with } i \in J_\eta \text{ or } x = x_0, \\ x_{i-1} & \text{if } x = x_i \text{ with } i \in J_\kappa, \end{cases}$$

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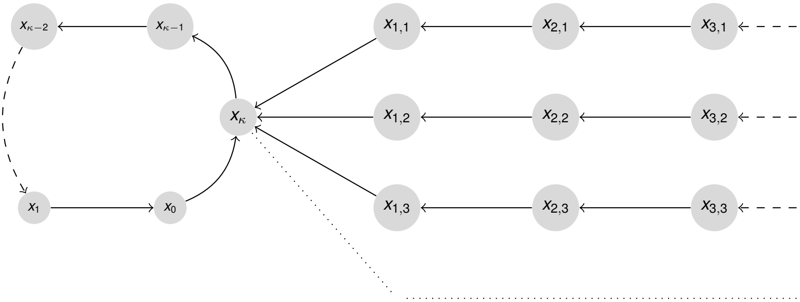

- $\eta \in \{1, 2, 3, \ldots\} \cup \{\infty\}$, $\kappa \in \{0, 1, 2, \ldots\}$. X and ϕ defined in (ii-a) will be denoted by $X_{\eta,\kappa}$ and $\phi_{\eta,\kappa}$, respectively.
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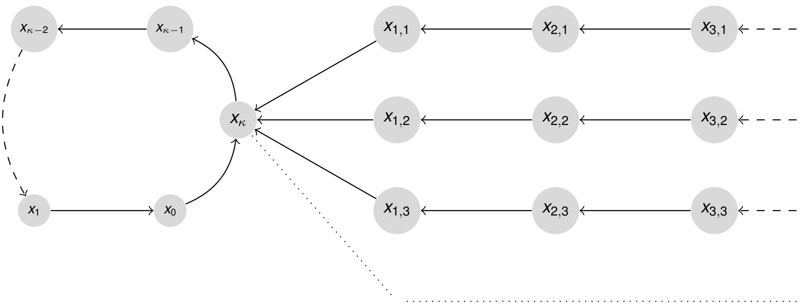

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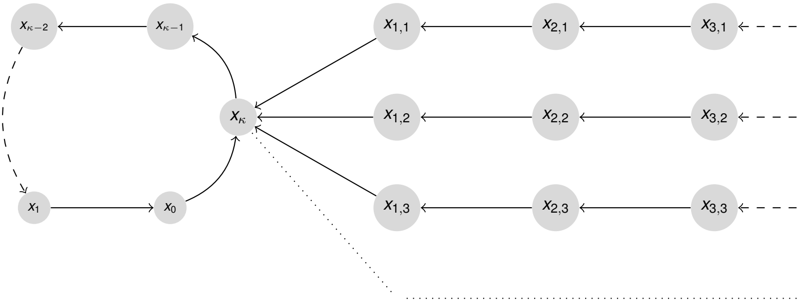

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- **Comment.** The class of composition operators C_ϕ in $L^2(X, \mu)$ with a symbol ϕ as in (ii-b), where μ is a discrete measure[1] on X, coincides with the class of weighted shifts on the directed tree $\mathscr{T}_{\eta+1,\infty}$ with positive weights.
- The latter class was studied earlier. So we can concentrate on composition operators $C_{\phi_{\eta,\kappa}}$ in $L^2(X_{\eta,\kappa}, \mu)$ with the symbol $\phi_{\eta,\kappa}$, where μ is a discrete measure on $X_{\eta,\kappa}$.

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There exists a discrete measure μ on $X_{2,0}$ such that

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- The proof of the above theorem depends heavily on the theory of classical moment problems especially on deep results due to Berg-Valent [1994] and Berg-Durán [1995].

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Index of H-determinacy

- A Borel measure ν on $\mathbb{R}$ is said to be **H-determinate** if the sequence of its moments $\{\int_{\mathbb{R}} t^n \, \mathrm{d}\, \nu(t)\}_{n=0}^{\infty}$ is H-determinate.
- Following Berg-Durán [1995], we define the quantity $\mathrm{ind}_z(\rho) \in \mathbb{Z}_+ \cup \{\infty\}$, called the **index of H-determinacy** of an H-determinate measure ρ at a point $z \in \mathbb{C}$, by

$$\mathrm{ind}_z(\rho) = \sup\Big\{k \in \mathbb{Z}_+ \colon |t - z|^{2k} \, \mathrm{d}\, \rho(t) \text{ is H-determinate}\Big\}.$$

Index of H-determinacy

- A Borel measure ν on $\mathbb{R}$ is said to be **H-determinate** if the sequence of its moments $\{\int_{\mathbb{R}} t^n \,\mathrm{d}\,\nu(t)\}_{n=0}^{\infty}$ is H-determinate.
- Following Berg-Durán [1995], we define the quantity $\mathrm{ind}_z(\rho) \in \mathbb{Z}_+ \cup \{\infty\}$, called the **index of H-determinacy** of an H-determinate measure ρ at a point $z \in \mathbb{C}$, by

$$\mathrm{ind}_z(\rho) = \sup\Big\{k \in \mathbb{Z}_+ \colon |t - z|^{2k} \,\mathrm{d}\,\rho(t) \text{ is H-determinate}\Big\}.$$

A more general result

Theorem (Budzyński, Jabłoński, Jung & JS 2014)

$\boxed{\eta \in \mathbb{N}_2}$. *There exists a discrete measure μ on $X_{\eta,0}$ such that*

1. $C_{\phi_{\eta,0}}$ *generates Stieltjes moment sequences,*
2. $C_{\phi_{\eta,0}}$ *is not hyponormal,*
3. $\{\mathsf{h}_{\phi_{\eta,0}^n}(x)\}_{n=0}^\infty$ *is a Stieltjes moment sequence for every* $x \in X_{\eta,0}$*,*
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6. *for every* $1 \leqslant j \leqslant 2\eta - 4$, $\{\mathsf{h}_{\phi_{\eta,0}^n}(x_{\eta,j})\}_{n=0}^\infty$ *is H-determinate and its unique H-representing measure* $P(x_{\eta,j}, \cdot)$ *satisfies*

$$\eta - 2 - \lceil j/2 \rceil \leqslant \mathrm{ind}_0(P(x_{\eta,j}, \cdot)) \leqslant \eta - 2 - \lfloor j/2 \rfloor.$$

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Sketch of the proof (I)

- Fix $a \in (1, \infty)$. Then using Euler's pentagonal-number theorem, we show that there exists $q \in (0, 1/a)$ such that

$$(q/a; q)_\infty + (aq; q)_\infty > 1, \qquad (1)$$

- where $(z; q)_n$ is the q-Pochhammer symbol defined for $z \in \mathbb{C}$ and $n \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ by

$$(z; q)_n = \prod_{j=1}^{n} (1 - zq^{j-1}) \qquad \boxed{(z; q)_0 = 1 \text{ for all } z \in \mathbb{C}}.$$

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Sketch of the proof (II)

- Define the Borel measures $\widetilde{\alpha}$ and $\widetilde{\beta}$ on $\mathbb{R}$ by

$$\widetilde{\alpha} = \sum_{n=0}^{\infty} (aq; q)_\infty \frac{a^n q^{n^2}}{(aq; q)_n (q; q)_n} \delta_{q^{-n}-1},$$

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- The measures $\widetilde{\alpha}$ and $\widetilde{\beta}$ are probability measures due to a result of Ismail [1985].
- It was proved by Berg and Valent [1994] that $\widetilde{\alpha}$ and $\widetilde{\beta}$ are **N-extremal measures** of the same Stieltjes moment sequence, say γ.
- In fact, one can show that $\widetilde{\alpha}$ is the Krein measure of γ, and $\widetilde{\beta}$ is the Friedrichs measure of γ.
- Set $\alpha = r\widetilde{\alpha}$ and $\beta = r\widetilde{\beta}$ with $r = (1 - \widetilde{\alpha}(\{0\}))^{-1}$. Then α and β are N-extremal measures of $r \cdot \gamma$ such that $0 = \inf \operatorname{supp}(\alpha) < \inf \operatorname{supp}(\beta)$ and $\alpha(\mathbb{R}_+) = 1 + \alpha(\{0\}) > 1$.
- Now, combining the definitions of $\widetilde{\alpha}$ and $\widetilde{\beta}$ with (1), we get

$$\beta\Big(\Big\{\inf \operatorname{supp}(\beta)\Big\}\Big) = \frac{(q/a;q)_\infty}{1-(aq;q)_\infty} > 1. \tag{2}$$

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- Let $\{\theta_i\}_{i=1}^{\infty}$ be a strictly increasing sequence such that $\operatorname{supp}(\beta) = \{\theta_1, \theta_2, \ldots\}$. By (2), we have $\beta(\{\theta_1\}) > 1$. Hence $\beta(\{\theta_1, \ldots, \theta_{\eta-1}\}) > 1$.
- One can show that there exists $\varepsilon > 0$ such that

$$\sum_{i=1}^{\eta-1} \frac{\theta_i^{(\varepsilon)} - 1}{\theta_i^{(\varepsilon)}} \beta^{(\varepsilon)}(\{\theta_i^{(\varepsilon)}\}) > \frac{\int_0^\infty (t-1)\,\mathrm{d}\,\beta^{(\varepsilon)}(t)}{1 + \int_0^\infty (t-1)\,\mathrm{d}\,\beta^{(\varepsilon)}(t)},$$

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$$\theta_i^{(\varepsilon)} = \psi_\varepsilon(\theta_i) \quad \text{and} \quad \beta^{(\varepsilon)}(\sigma) = \beta(\psi_\varepsilon^{-1}(\sigma)) \text{ for } \sigma \in \mathfrak{B}(\mathbb{R}),$$

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$$\psi_\varepsilon(t) = \varepsilon^{-1} t + 1 \text{ for } t \in \mathbb{R}.$$

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Sketch of the proof (V)

- Set $\nu = \alpha^{(\varepsilon)}$ and $\tau = \beta^{(\varepsilon)}$.
- Then we verify that

$$\left.\begin{array}{l}\nu \text{ and } \tau \text{ are N-extremal measures of} \\ \text{the same Stieltjes moment sequence}\end{array}\right\}, \tag{1♣}$$

$$1 = \inf \operatorname{supp}(\nu) < \inf \operatorname{supp}(\tau), \tag{2♣}$$

$$\nu(\mathbb{R}) = 1 + \nu(\{1\}). \tag{3♣}$$

- Let $\{\Delta_i\}_{i=1}^{\eta}$ be a partition of $\operatorname{supp}(\tau)$ given by

$$\Delta_i = \begin{cases} \left\{\theta_i^{(\varepsilon)}\right\} & \text{if } 1 \leqslant i \leqslant \eta - 1, \\ \left\{\theta_\eta^{(\varepsilon)}, \theta_{\eta+1}^{(\varepsilon)}, \ldots\right\} & \text{if } i = \eta. \end{cases}$$

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Sketch of the proof (VI)

- Define Borel probability measures $\{P(x_{i,1},\cdot)\}_{i=1}^{\eta}$ on $\mathbb{R}$ by

$$P(x_{i,1},\sigma) = c_i \int_{\Delta_i \cap \sigma} (t-1)\,\mathrm{d}\,\tau(t), \quad \sigma \in \mathfrak{B}(\mathbb{R}),$$

where

$$c_i = \frac{1}{\int_{\Delta_i} (t-1)\,\mathrm{d}\,\tau(t)}.$$

Sketch of the proof (VII)

- Take any $\mu(x_0) \in (0, \infty)$ and define a sequence $\{\mu(x_{i,1})\}_{i=1}^{\eta}$ of positive real numbers by

$$\mu(x_{i,1}) = \frac{1}{c_i}\,\mu(x_0), \quad 1 \leqslant i \leqslant \eta. \tag{3}$$

- Define a family $\{\mu(x_{i,j})\}_{i=1}^{\eta}{}_{j=2}^{\infty}$ of positive real numbers and a family $\{P(x_{i,j}, \cdot)\}_{i=1}^{\eta}{}_{j=2}^{\infty}$ of Borel probability measures on $\mathbb{R}$ by

$$\mu(x_{i,j}) = \mu(x_{i,1}) \int_0^\infty t^{j-1} P(x_{i,1}, \mathrm{d}\,t),$$

$$P(x_{i,j}, \sigma) = \frac{\mu(x_{i,1})}{\mu(x_{i,j})} \int_\sigma t^{j-1} P(x_{i,1}, \mathrm{d}\,t), \quad \sigma \in \mathfrak{B}(\mathbb{R}).$$

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- Let $P(x_0, \cdot)$ be a Borel measure on $\mathbb{R}$ given by

$$P(x_0, \sigma) = \nu(\sigma) - \nu(\{1\})\delta_1(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}).$$

  By the third property (3♣) of the measure ν, $P(x_0, \cdot)$ is a probability measure.
- Finally, let μ be a (unique) discrete measure on $X_{\eta,0}$ such that $\mu(\{x\}) = \mu(x)$ for every $x \in X$.
- Then the corresponding composition operator $C_{\phi_{\eta,0}}$ in $L^2(\mu)$ has the required properties. Since the rest of the proof is quite long, I stop at this point.

Sketch of the proof (VIII)

- Let $P(x_0, \cdot)$ be a Borel measure on $\mathbb{R}$ given by

$$P(x_0, \sigma) = \nu(\sigma) - \nu(\{1\})\delta_1(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}).$$

  By the third property (3♣) of the measure ν, $P(x_0, \cdot)$ is a probability measure.
- Finally, let μ be a (unique) discrete measure on $X_{\eta,0}$ such that $\mu(\{x\}) = \mu(x)$ for every $x \in X$.
- Then the corresponding composition operator $C_{\phi_{\eta,0}}$ in $L^2(\mu)$ has the required properties. Since the rest of the proof is quite long, I stop at this point.

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Sources

Talk is based on the following papers:

[1] Z. J. Jabłoński, I. B. Jung, J. Stochel, Weighted shifts on directed trees, *Mem. Amer. Math. Soc.* **216** (2012), no. 1017, viii+107pp.

[2] Z. J. Jabłoński, I. B. Jung, J. Stochel, A non-hyponormal operator generating Stieltjes moment sequences, *Journal of Functional Analysis* **262** (2012), 3946-3980.

[3] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Unbounded subnormal composition operators in L^2-spaces (arXiv:1303.6486), submitted.

[4] P. Budzyński, Z. J. Jabłoński, I. B. Jung, J. Stochel, Subnormality of composition operators in L^2 spaces over directed graphs with one circuit, work in progress almost completed.

Thank you!