The Jiang-Su absorption for C*-algebras (Joint work with Tamotsu Teruya)

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Under the assumption that $B \otimes \mathcal{Z} \cong B$, when $A \otimes \mathcal{Z} \cong A$?
Here Jiag-Su algebra $\mathcal{Z}$ is a simple unital projectionless C*-algebra with a unique trqcial state constructed by the inductive limite of dimension drop algebras $I(k, k+1)$, where
$I(k, k+1) = \{f \in C[0,1] \otimes M_k(\mathbf{C}) \otimes M_{k+1}(\mathbf{C}) \mid f(0) \in M_k(\mathbf{C}) \otimes I_{k+1}, f(1) \in I_k \otimes M_{k+1}(\mathbf{C})\}$.

Contents

The structure of simple C*-algebras

Definition

1. Two projections p and q in a C*-algebra A are said to be **Murray-von Neumann equivalent** if $p = v^*v$ and $q = vv^*$ for some v in A. (Write $p \sim q$), and p is subequivalent to q, written $p \preceq q$, if p is equivalent to a subprojection of q.
2. A projection is a C*-algebra A to be **infinite** if it is equivalent to a proper subprojection of itself, and it is called to be **finite** otherwise.
3. A simple C*-algebra A is called **stably infinite** if its stabilization $A \otimes \mathcal{K}$ contains an infinite projection, and it is called **stably finite** others.
4. A simple C*-algebra A is said to be **purely infinite** if every non-zero hereditary subalgebra of A contains an infinite projection.

Definition (Nuclear C*-algebras)

A C*-algebra A is daid to be nuclear if the canonical surjection $A \otimes_{max} B \to A \otimes_{min} B$ is injective (thatis, an isomorphism) for every C*-algebra B.

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Theorem (Lance: '73, Connes: '78, Choi-Effros: '77, 78)

Let A be a C*-algebra. TFAE:

1. A is nuclear.
2. The identity map from A to A can be approximated pointwise in norm by sompletely positive finite-rank contractions.
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- All commutative C*-algebras and finite dimensional C*-algebras are nuclear.
- The nuclearity is stable under the stability isomorphism, inductive limits, C*-extensions, crossed products by amanable groups, C*-tensor products.

Theorem (Kirchberg)

Let A and B be simple non-type I C*-algebras. If A or B is stably infinite, then $A \otimes_{min} B$ is purely infinite.
If A and B are both stably finite and exact (i.e. $A \otimes_{min}$ is exact), then $A \otimes_{min} B$ is stably finite.

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Theorem (Rørdam: 2003)

There is a simple, separable, nuclear C*-algebra that is stably infinite but not purely infinite, and there is a simple, separable, nuclear, unital, finite C*-algebra that is not stably finite.

Dimension for C*-algebras

In the commutative case since a C*-algebra A is isomorphic to $C_0(X)$ for sme locally compact Hausdorff space X we can define the dimension of A (writtendim A) to be the classical dimension of the space X (i.e. $\dim X$).
In the case of non-commutative case Rieffel and Brown-Pedersen introduced topological stable rank (written $\mathrm{tsr}(A)$) and real rank (written $\mathrm{RR}(A)$) as follows:

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Definition (Rieffel: '83, Brown-Pedersen: '91)

If the set of invertible elements in a C*-algebra A (or in the unitization of A, if A is non-unital) is dense in A, then A is said to be of **topological stable rank 1**, that is, $\mathrm{tsr}(A) = 1$.
If the set of **self-adjoint element** invertible elements in the set of self-adjoint elements in A, then A is said to be of **real rank zero**, that is, $\mathrm{RR}(A) = 0$.

When X is a compact Hausdorff space X, $\mathrm{tsr}(C_0(X)) = 1$ if and only if $\dim X \leq 1$, and $\mathrm{RR}(C_0(X)) = \dim X$.

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Proposition (Cuntz: '81, Zhang: '90)

The following three conditions are equivalent:

1. A is purely infinite,
2. for all non-zero positive elements $a, b \in A$ there exists $x \in A$ such that $b = x^* a x$,
3. $\mathrm{RR}(A) = 0$ and all non-zero projections in A is infinite.

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Winter-Zacharias (2010) introduced another non-commutative dimension, that is, **nuclear dimension**, which is weeker one than decomposition rank by Winter- Kirchberg (2004).

Nuclear dimension

Definition (Kirchberg-Winter: 2004)

Let A be a separable C*-algebra.

(1) A completely positive map $\varphi\colon \bigoplus_{i=1}^{s} M_{r_i} \to A$ has order zero,if it preserves orthogonality, i.e., $\varphi(e)\varphi(f) = \varphi(f)\varphi(e) = 0$ for $e, f \in \oplus_{i=1}^{s} M_{r_i}$ with $ef = fe = 0$.

(2) A completely positive map $\varphi\colon \bigoplus_{i=1}^{s} M_{r_i} \to A$ is n-decomposable, there is a decomposition $\{1, \ldots, s\} = \coprod_{j=0}^{n} I_j$ such that the restriction of φ to $\oplus_{i \in I_j} M_{r_i}$ has ordere zero for each $j \in \{0, \ldots, n\}$.

Definition (Kirchberg-Winter: 2004)

(3) A has decomposition rank n, $\mathrm{dr}A = n$, if n is the least integer such that the following holds : Given $\{a_1, \ldots, a_m\} \subset A$ and $\varepsilon > 0$, there is a completely positive approximation property (F, ψ, φ) for $a_1, \ldots, a_m$ within ε, i.e., F is a finite dimensional F, and $\psi\colon A \to F$ and $\varphi\colon F \to A$ are completely positive contruction ($=$ c. p. c.) such that

1. $\|\varphi\psi(a_i) - a_i\| < \varepsilon$,
2. φ is n-decomposable.

If no such n exists, we write $\mathrm{dr}A = \infty$.

Definition (Winter-Zacharias: 2010)

A has nuclear dimension n, $\dim_{\mathrm{nuc}} A = n$, if n is the least integer such that the following holds : Given $\{a_1, \ldots, a_m\} \subset A$ and $\varepsilon > 0$, there is a completely positive approximation property (F, ψ, φ) for $a_1, \ldots, a_m$ within ε, i.e., F is a finite dimensional F, and $\psi\colon A \to F$ and $\varphi\colon F \to A$ are completely positive such that

1. $\|\varphi\psi(a_i) - a_i\| < \varepsilon$
2. $\|\psi\| \leq 1$
3. φ is n-decomposable and each restriction $\varphi|_{\oplus_{i \in I_j} M_{r_i}}$ is c. p. c.

If no such n exists, we write $\dim_{\mathrm{nuc}} A = \infty$.

The followings are basic facts about finite decomposition and nuclear dimension by [Kirchberg-Winter: 2004], [Winter: 2010], [Winter-Zacharias: 2010]:

(1) If $\dim_{\mathrm{nuc}}(A) \leq n < \infty$, then A is nuclear.
(2) For any C*-algebras $\dim_{\mathrm{nuc}} A \leq \mathrm{dr} A$.
(3) $\dim_{\mathrm{nuc}} A = 0$ if and only if $\mathrm{dr} A = 0$ if and only if A is an AF algebra.
(4) Nuclear dimension and decomposition rank in general do not coincide. Indeed, the Toeplitz algebra $\mathcal{T}$ has nuclear dimension at most 2, but its decomposition rank is infinity. Note that if $\mathrm{dr} A \leq n < \infty$, A is quasidiagonal, that is , stably finite. The Toeplitz algebra $\mathcal{T}$ has an isometry, and we know that $\mathcal{T}$ is infinite.
(5) Let X be a locally compact Hausdorff space. Then

$$\dim_{\mathrm{nuc}} C_0(X) = \mathrm{dr}\, C_0(X).$$

In particular, if X is second countable,

$$\dim_{\mathrm{nuc}} C_0(X) = \mathrm{dr}\, C_0(X) = \dim X.$$

(6) For any $n \in \mathbf{N}$ $\dim_{\mathrm{nuc}} A = \dim_{\mathrm{nuc}}(M_n(A)) = \dim_{\mathrm{nuc}}(A \otimes \mathcal{K})$ and $\mathrm{dr}(A) = \mathrm{dr}(M_n(A)) = \mathrm{dr}(A \otimes \mathcal{K})$.

(7) If $B \subset A$ is full hereditary C*-algebra, then $\dim_{\mathrm{nuc}}(B) = \dim_{\mathrm{nuc}}(A)$ and $\mathrm{dr}(B) = \mathrm{dr}(A)$.

(8) $\dim_{\mathrm{nuc}}(\mathcal{O}_n) = 1$ for $n = 2, 3, \ldots$ and $\dim_{nuc}(\mathcal{O}_\infty) \leq 2$.

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Question

If a C*-algebra with $\dim_{\mathrm{nuc}}(A) < +\infty$ and A has a faithful trace, $\dim_{\mathrm{nuc}}(A) = \mathrm{dr}(A)$?

Comparison Theory for C*-algebras

The comparison properties for a C*-algebra A are contained in the ordered monoid $V(A)$ (consisting of equivalent classes of projections) and $W(A)$ (consisting of equivalent classes of positive elements) respectively, in the *-algebra $M_\infty(A) = \cup_{n=1}^\infty M_n(A)$. Following Cuntz, comparison of positive elements $a, b \in M_\infty(A)$ is defined as follows: $a \preceq b$ if there is a sequence $\{x_n\}$ in $M_\infty(A)$ such that $x_n^* b x_n \to a$, and by $a \sim b$ iff $a \preceq b$ and $b \preceq a$ one defines equivalence relations on the positive elements. The set $V(A)$ and $W(A)$ become ordered abelian semigroups by defining addition to be "orthogonal addition".
If A is generated as an ideal by its projections (in particular, simple C*-algebras with non-trivial projection), $K_0(A)$ is the Grothendieck group of $V(A)$, and the positive cone, $K_0(A)^+$, is the image of $V(A)$ in $K_0(A)$.

Definition

An ordered abelian positive semigroup $(W(A), +, \leq)$ is said to be **almost unperforated** if

$$\forall n, m \in \mathbf{N}, \forall x, y \in W(A) \text{ such that } nx \leq my, n > m \Rightarrow x \leq y$$

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It is know that $V(A)$ and $W(A)$ are almost unperforated for many C*-algebras:

- All purely infinite simple C*-algebras
- All C*-algebras of the form $A \otimes \mathcal{Z}$, that is, which absorb $\mathcal{Z}$

Definition

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1. A is said to have **the comparison of projections** if for any projections $p, q \in M_\infty(A)$ $p \preceq q$ if $\tau(p) < \tau(q)$ for all traces τ on A.
2. A is said to have **the strict compariosn** if for any positive elements $a, b \in M_\infty$ $a \preceq b$ if $\lim_{n\to\infty} \tau(a^{1/n}) < \lim_{n\to\infty} \tau(b^{1/n})$ for all tracial states τ on A.

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Theorem (Rørdam: 2004)

Let A be a $\mathcal{Z}$-absorbing C*-algebra. Then $W(A)$ has the strict comparison.

Elliott's classification conjectures

1960: Glimm: UHF algebras by supernatural numbers.
1976: Elliott: AF algebras by K_0 groups.
late 1980s Elliott: AT algebras A of real rank zero by $K_*(A)$
(simple case by $(K_0(A), K_0(A)^+, [1]_0, K_1(A))$)
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Conjecture (Elliott: stably finite case)

Let A and B be separable, simple unital nuclear, stably finite C*-algebras. Then

$$A \cong B \Leftrightarrow (K_0(A), K_0(A)^+, [1_A]_0, K_1(A), T(A), r_A : T(A) \to S(K_0(A)))$$
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Theorem (Kirchberg-Phillips: 1994, 2000)

Let A and B be separable, nuclear, simple, purely infinite, K-amenable, unital C*-algebras. Then

$$A \cong B \Leftrightarrow (K_0(A), [1_A]_0, K_1(A)) \cong (K_0(B), [1_B]_0, K_1(B))$$

A C*-algebra A is K-amenable if it is KK-equivalent to an abelian C*-algebra.

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A C*-algebra A is K-amenable if it is KK-equivalent to an abelian C*-algebra.
The following result says that K-data is not enough for classification.

Example (Rørdam: 2005)

There are simple, separable, nuclear, stably infinite unital C*-algebras A and B such that

$$(K_0(A), [1_A]_0, K_1(A)) \cong (K_0(B), [1_B]_0, K_1(B)) \text{ and } A \not\cong B$$

Example (Toms: 2005)

There is a simple, unital, nuclear, separable, infinite dimensional, stably finite C*-algebra A such that $Ell(A)$ is isomorphic to $Ell(A \otimes \mathcal{Z})$, while A and $A \otimes \mathcal{Z}$ are not isomorphic.

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Theorem (Lin-Niu: 2008)

Let A and B be unital separable, simple $\mathcal{Z}$-stable C*-algebras with unique tracial states which are inductive limites of type I C*-algebras. Suppose that $(K_0(A), K_0(A)_+, [1_A]_0, K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B]_0, K_1(B))$. Then $A \cong B$.

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Then $A \cong B$.

Very recently, Sato rported that for simple, separable, unital, nuclear, QD, C*-algebras A, B with unique tracial states satifying the strict comparson and the UCT, then $A \cong B$ if and only if $(K_0(A), K_0(A)_+, [1_A]_0, K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B]_0, K_1(B))$.

Toms-Winter conjecture

Conjecture (Toms-Winter conjecture 2010)

Let A be a simple, unital, separable, infinite-dimensional, nuclear C*-algebra. TAFA:

1. A has the strictly comparison property.
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It is known that (2) $\rightarrow$ (1), cf. [Rørdam:2004], and (3) $\rightarrow$ (2), cf. [Winter:2011].

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Theorem (Sato-White-Winter: 2014)

Let A be a simple, unital, separable, infinite-dimensional, nuclear C*-algebra with a unique tracial state. If A has the strict cmparison, then $\dim_{nuc}(A) \leq 3$. Hence, Toms-Winter conjecture is affirmative.

Strongly self-absorbing

Definition (Toms-Winter: 2005)

A C*-algebra $\mathcal{D}$ is called *strongly self-absorbing* if $\mathcal{D} \ncong \mathbf{C}$ and there is an isomorphism $\phi\colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ satisfying ϕ and the map $id_{\mathcal{D}} \otimes I_{\mathcal{D}}$ are approximately unitarily equivalent, that is ,

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Definition (Hirshberg-Winter 2007)

For a strongly self-absorbing C*-algebra $\mathcal{D}$ we say that a C*-algebra A is *$\mathcal{D}$-absorbing* if the tensor product $A \otimes \mathcal{D}$ is isomorphic to A.

Remark

Known examples of strongly self-absorbing C*-algebras are UHF-algebras of infinite type, the Jiang-Su algebra $\mathcal{Z}$, the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, and tensor products of $\mathcal{O}_\infty$ by UHF algebras of infinite type. Note that they belong to the class of inductive limits of weakly semiprojective C*-algebras.

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Theorem (Toms-Winter: 2008 and Winter: 2009)

A unital C*-algebra $\mathcal{D}$ is isomorphic to $\mathcal{Z}$ if and only if $\mathcal{D}$ is strongly self-absorbing and $\mathcal{D}$ is KK-equivalent to $\mathbf{C}$.

Rokhlin property

For a C*-algebra A we set

$$C_0(A) = \{(a_n) \in \ell^\infty(\mathbf{N}, A) \colon \lim_{n\to\infty} \|a_n\| = 0|\},$$
$$A^\infty = \ell^\infty(\mathbf{N}, A)/C_0(A).$$

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Definition (Izumi: 2004)

Let α be an action of a finite group G on a unital C^*-algebra A. α is said to have the *Rokhlin property* if there exists a partition of unity $\{e_g\}_{g\in G} \subset A' \cap A^\infty$ consisting of projections satisfying $(\alpha_g)_\infty(e_h) = e_{gh} \quad$ for $g, h \in G$. We call $\{e_g\}_{g\in G}$ Rokhlin projections.

C*-index theory

Definition (Watatani: '90)

Let $P \subset A$ be an inclusion of unital C*-algebras with a conditional expectation E from A onto P.

1. A *quasi-basis* for E is a finite set $\{(u_i, v_i)\}_{i=1}^{n} \subset A \times A$ such that for every $a \in A$,

$$a = \sum_{i=1}^{n} u_i E(v_i a) = \sum_{i=1}^{n} E(a u_i) v_i.$$

2. When $\{(u_i, v_i)\}_{i=1}^{n}$ is a quasi-basis for E, we define $\mathrm{Index} E$ by

$$\mathrm{Index} E = \sum_{i=1}^{n} u_i v_i.$$

   When there is no quasi-basis, we write $\mathrm{Index} E = \infty$. $\mathrm{Index} E$ is called the Watatani index of E.

Definition (Kodaka-Osaka-Teruya: 2008)

A conditional expectation E of a unital C^*-algebra A with a finite index is said to have the *Rokhlin property* if there exists a projection $e \in A' \cap A^\infty$ satisfying

$$E^\infty(e) = (\mathrm{Index} E)^{-1} \cdot 1$$

and a map $A \ni x \mapsto xe$ is injective. We call e a Rokhlin projection.

Proposition (Kodaka-Osaka-Teruya: 2008)

Let α be an action of a finite group G on a unital C^*-algebra A and E the canonical conditional expectation from A onto the fixed point algebra $P = A^\alpha$ defined by

$$E(x) = \frac{1}{\#G} \sum_{g \in G} \alpha_g(x) \quad \text{for } x \in A,$$

where $\#G$ is the order of G. Then α has the Rohklin property if and only if there is a projection $e \in A' \cap A^\infty$ such that $E^\infty(e) = \frac{1}{\#G} \cdot 1$, where E^∞ is the conditional expectation from A^∞ onto P^∞ induced by E.

Main result(Rokhlin:unital case)

Theorem (Osaka-Teruya: 2010)

Let $P \subset A$ be an inclusion of unital C*-algebras and $E\colon A \to P$ be a faithful conditional expectation of index finite. Suppose that E has the Rokhlin property and $\mathcal{D}$ is a separable unital self-absorbing C*-algebra.

1. If A is $\mathcal{D}$-absorbing, then P is $\mathcal{D}$-absorbing.
2. If A is an inductive limit of weakly semiprojective C*-algebras and is strongly self-absorbing, then P is strongly self-absorbing.
3. If A is a UHF-algebra of infinite type, $\mathcal{O}_2$, $\mathcal{O}_\infty$, and $\mathcal{O}_\infty \otimes UHF$-algebra of infinite type, then $P \cong A$ and $C^*\langle A, e_P\rangle$ is stably isomorphic to A.

Corollary (Hirshberg-Winter: 2007)

Let A be a separable, unital, simple C*-algebra and α be an action of a finite group G on A. Suppose that α has the Rokhlin property. If A is $\mathcal{D}$-absorbing, then the crossed product algebra $A \rtimes_\alpha G$ is $\mathcal{D}$-absorbing.

Theorem (Toms-Winter: 2007)

Let A and $\mathcal{D}$ be sparable C*-algebras and suppose that $\mathcal{D}$ is unital and strongly self-absorbing. Then there is an isomorphism $\phi : A \to A \otimes D$ iff there is a unital *-homomorphism $\rho : \mathcal{D} \to M(A)_\infty \cap A'$.

Theorem (Toms-Winter: 2007)

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Lemma

Let $P \subset A$ be an inclusion of unital C*-algebras and E be a conditional expectation from A onto P with a finite index. If E has the Rokhlin property with a Rokhlin projection $e \in A_\infty$, then there is a unital linear map $\beta \colon A^\infty \to P^\infty$ such that for any $x \in A^\infty$ there exists the unique element y of P^∞ such that $xe = ye = \beta(x)e$ and $\beta(A' \cap A^\infty) \subset P' \cap P^\infty$. In particular, $\beta_{|A}$ is a unital injective $*$-homomorphism and $\beta(x) = x$ for all $x \in P$.

Example (Osaka-Teruya:2010)

There exists a symmetry β with the tracial Rokhlin property on the universal UHF-algebra $\mathcal{U}_\infty$ such that $\mathcal{U}_\infty \rtimes_\beta \mathbf{Z}/2\mathbf{Z}$ is not strongly self-absorbing.

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Example (Phillips: 2010)

There exists a strongly self-absorbing UHF-algebra D $(= \otimes_{n\in\mathbf{N}} M_{2r(n)+1}, r(n) = \frac{1}{2}(3^n - 1))$, a D-absorbing separable infinite dimensional simple C*-algebra C, and an action $\gamma\colon \mathbf{Z}_2 \to \mathrm{Aut}(C)$ with the *tracial* Rokhlin property, such that $C \rtimes_\gamma \mathbf{Z}_2$ is not D-absorbing.

Main result (Tracial Rokhlin:unital case)

Definition (Phillips: 2003)

Let α be the action of a finite group G on an infinite dimensional finite simple separable unital C*-algebra A. An α is said to have the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every nonzero positive $x \in A$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. With $e = \sum_{g\in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.

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The flip action on the irrational rotation algebra A_θ has the tracial Rokhlin property.

Definition

Let $P \subset A$ be an inclusion of unital C*-algebras and $E\colon A \to P$ be a conditional expectation of index finite. A conditional expectation E is said to have the *tracial Rokhlin property* if for any nonzero positive $z \in A^\infty$ there exists a projection $e \in A' \cap A^\infty$ satisfying

$$(\mathrm{Index} E) E^\infty(e) = g$$

is a projection and $1 - g$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A^∞ generated by z, and a map $A \ni x \mapsto xe$ is injective. We call e a Rokhlin projection.

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Lemma

Let $P \subset A$ be an inclusion of unital C*-algebras and $E\colon A \to P$ be a conditional expectation of index finite type. Suppose that E has the tracial Rokhlin property, then A has the Property (SP) or E has the Rokhlin property.

Lemma

Let $P \subset A$ be an inclusion of unital C*-algebras and $E\colon A \to P$ be a conditional expectation of index finite type. Suppose that E has the tracial Rokhlin property, then A has the Property (SP) or E has the Rokhlin property.

Proposition

Let G be a finite group, α an action of G on an infinite dimensional finite simple separable unital C*-algebra A, and E the canonical conditional expectation from A onto the fixed point algebra $P = A^\alpha$ defined by

$$E(x) = \frac{1}{|G|}\sum_{g\in G} \alpha_g(x) \quad \text{for } x \in A,$$

where $|G|$ is the order of G. Then α has the tracial Rokhlin property if and only if E has the tracial Rokhlin property.

Lemma

Let $A \supset P$ be an inclusion of unital C*-algebras and E a conditional expectation from A onto P with index finite type. Suppose that A is simple. If E has the tracial Rokhlin property with a Rokhlin projection $e \in A_\infty$ and a projection $g = (\mathrm{Index} E)E^\infty(e)$, then there is a unital linear map $\beta\colon A^\infty \to P^\infty g$ such that for any $x \in A^\infty$ there exists the unique element y of P^∞ such that $xe = ye = \beta(x)e$ and $\beta(A' \cap A^\infty) \subset P' \cap P^\infty g$. In particular, $\beta_{|_A}$ is a unital injective *-homomorphism and $\beta(x) = xg$ for all $x \in P$.

Lemma

Let $P \subset A$ be an inclusion of unital C*-algebras with index finite type and $E\colon A \to P$ has the tracial Rokhlin property. Suppose that projections $p, q \in P^\infty$ satisfy $ep = pe$ and $q \preceq ep$ in A^∞, where e is the Rokhlin projection for E. Then $q \preceq p$ in P^∞.

Lemma

Let $P \subset A$ be an inclusion of unital C*-algebras with index finite type and $E\colon A \to P$ has the tracial Rokhlin property. Suppose that projections $p, q \in P^\infty$ satisfy $ep = pe$ and $q \preceq ep$ in A^∞, where e is the Rokhlin projection for E. Then $q \preceq p$ in P^∞.

Corollary

Let $P \subset A$ be an inclusion of unital C*-algebras and E a conditional expectation from A onto P with index finite type. Suppose that A is an infinite dimensional simple C*-algebra with tracial topological rank zero (resp. less than or equal to one) and E has the tracial Rokhlin property. Then P has tracial rank zero (resp. less than or equal to one).

Theorem

Let $P \subset A$ be an inclusion of unital C*-algebra and E be a conditional expectation from A onto P with index finite type. Suppose that A is simple, separable, nuclear, $\mathcal{Z}$-absorbing and E has the tracial Rohklin property. P is $\mathcal{Z}$-absorbing.

Theorem

Let $P \subset A$ be an inclusion of unital C*-algebra and E be a conditional expectation from A onto P with index finite type. Suppose that A is simple, separable, nuclear, $\mathcal{Z}$-absorbing and E has the tracial Rohklin property. P is $\mathcal{Z}$-absorbing.

Corollary

Let A be an infinite dimensional simple separable unital C*-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G with the tracial Rokhlin property. Suppose that A is $\mathcal{Z}$-absorbing. Then we have

1. (Hirshberg- Orovitz:2013) The fixed point algebra A^α and the crossed product $A \rtimes_\alpha G$ are $\mathcal{Z}$-absorbing.
2. For any subgroup H of G the fixed point algebra A^H is $\mathcal{Z}$-absorbing.

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Main result (Rokhlin: nonunital case)

Definition

Let $P \subset A$ be an inclusion of separable C*-algebras and $E\colon A \to P$ be a conditional expectation of index finite in the sense of Izumi. A conditional expectation E is said to have the *Rokhlin property* if there exists a projection $e \in A' \cap A^{\infty}$ satisfying

$$(\mathrm{Index}_p E) E^{\infty}(e) = f$$

is a projection and $fa = a$ $(\forall a \in A)$ and a map $A \ni x \mapsto xe$ is injective, where $\mathrm{Index}_p E = \sup\{\lambda > 0\colon \frac{1}{\lambda}E - Id \text{ is positive}\}$. We call e a Rokhlin projection.

Definition (cf. Santiago:2014)

Let α be the action of a finite group G on a unital an infinite dimensional, separable, C*-algebra A. An α is said to have the *Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. With $e = \sum_{g \in G} e_g$, the projection $\|ea - a\| < \varepsilon$ for all a in F.

Definition (cf. Santiago:2014)

Let α be the action of a finite group G on a unital an infinite dimensional, separable, C*-algebra A. An α is said to have the *Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. With $e = \sum_{g \in G} e_g$, the projection $\|ea - a\| < \varepsilon$ for all a in F.

As in the case of the Rokhlin property in the sense of Izumi we have the following characterization.

Proposition

Let α be an action of a finite group G on a unital C^*-algebra A and E the canonical conditional expectation from A onto the fixed point algebra $P = A^\alpha$ defined by

$$E(x) = \frac{1}{|G|} \sum_{g \in G} \alpha_g(x), \quad x \in A,$$

where $|G|$ is the order of G. Then α has the Rohlin property if and only if there is a projection $e \in A' \cap A^\infty$ and a projection $f \in P^\infty$ such that $E^\infty(e) = \frac{1}{|G|} \cdot f$ and $fa = a$ for any $a \in A$, where E^∞ is the conditional expectation from A^∞ onto P^∞ induced by E.

Theorem

Let $P \subset A$ be an inclusion of separable C*-algebras and E be a conditional expectation from A onto P with $\mathrm{Index}_p E < \infty$. Suppose that $\mathcal{D}$ is a separable unital self-absorbing C*-algebra and E has the Rokhlin property. Then if A is $\mathcal{D}$-absorbing, P is $\mathcal{D}$-absorbing.

Theorem

Let $P \subset A$ be an inclusion of separable C*-algebras and E be a conditional expectation from A onto P with $\mathrm{Index}_p E < \infty$. Suppose that $\mathcal{D}$ is a separable unital self-absorbing C*-algebra and E has the Rokhlin property. Then if A is $\mathcal{D}$-absorbing, P is $\mathcal{D}$-absorbing.

Corollary (cf. Santiago:2014)

Let A be a separable, simple C*-algebra and α be an action of a finite group G on A. Suppose that α has the Rokhlin property in the sense of Santiago. Then if A is $\mathcal{D}$-absorbing, the crossed product algebra $A \rtimes_\alpha G$ is $\mathcal{D}$-absorbing.