Strong algebras

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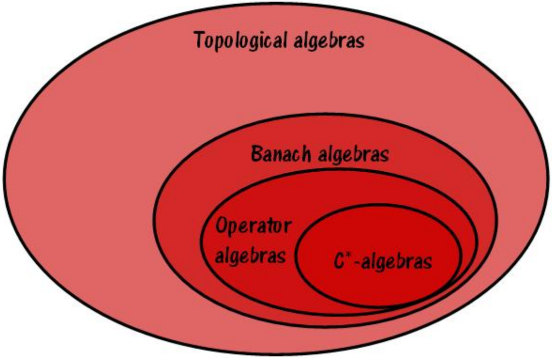
Some topological algebras



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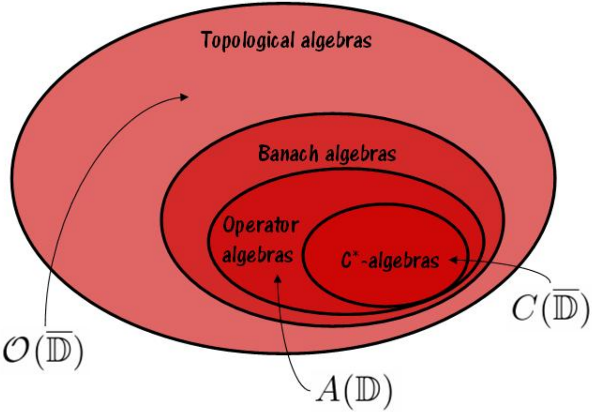

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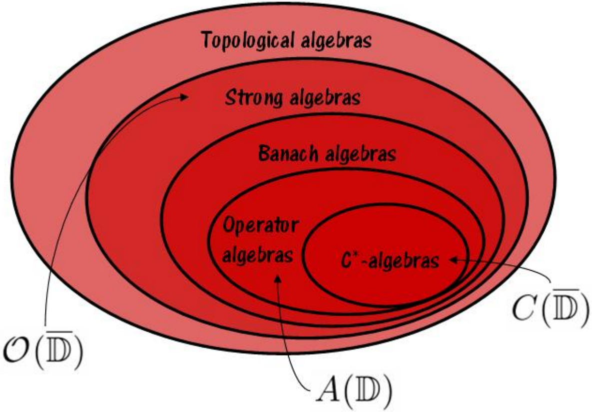


Strong algebras

Algebras which are inductive limits of Banach spaces

An inductive limit of Banach spaces

Definition

Let $\{X_\alpha : \alpha \in A\}$ be a family of subspaces of a vector space X, directed under inclusion, satisfying $X = \bigcup_\alpha X_\alpha$, such that on each X_α, a norm $\|\cdot\|_\alpha$ is given, and whenever $\alpha \leq \beta$, the topology induced by $\|\cdot\|_\beta$ on X_α is coarser than the topology induced by $\|\cdot\|_\alpha$.
Then X, topologized with the inductive limit topology is called **the inductive limit of the normed spaces** $\{X_\alpha : \alpha \in A\}$.

The inductive limit topology on X is the finest **locally convex** topology such that $X_\alpha \hookrightarrow X$ are continuous.

A strong algebra

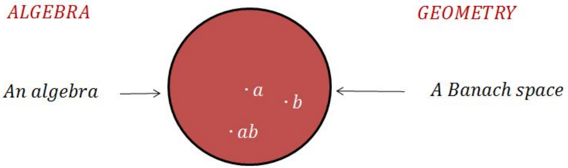
Definition (Alpay & S, 2013)

Let $\{X_\alpha : \alpha \in A\}$ *be a family of Banach spaces directed under inclusion, and let* $\mathcal{A} = \bigcup X_\alpha$ *be its inductive limit. We call* $\mathcal{A}$ *a* **strong algebra** *if it is an algebra satisfying the property that for any* $\alpha \in A$ *there exists* $h(\alpha) \in A$ *such that for any* $\beta \geq h(\alpha)$ *there is a positive constant* $A_{\beta,\alpha}$ *for which*

$$\|ab\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta, \quad \text{and} \quad \|ba\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta.$$

(in particular, $ab, ba \in X_\beta$*) for every* $a \in X_\alpha$ *and* $b \in X_\beta$*.*

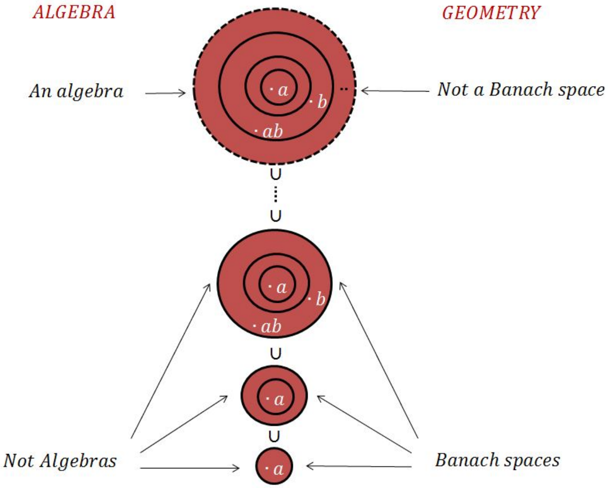
A Banach algebra



The relation between *ALGEBRA* and *GEOMETRY*

$$\|ab\| \leq \|a\| \, \|b\|$$

A strong algebra



The relation between *ALGEBRA* and *GEOMETRY*

$$\|ab\|_n \leq A_{n,m}\|a\|_m\|b\|_n \quad \text{and} \quad \|ba\|_n \leq A_{n,m}\|a\|_m\|b\|_n \quad \forall n \geq h(m)$$

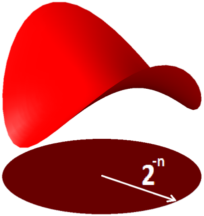
Examples

Example 1. Let X_0 be a Banach algebra. Then if the "family of Banach-spaces" is $\{X_0\}$, we obtain that for $A_{0,0} = 1$, $\mathcal{A} = X_0$ is a strong algebra; i.e. **every Banach algebra is also a strong algebra**.

Examples

Example 1. Let X_0 be a Banach algebra. Then if the "family of Banach-spaces" is $\{X_0\}$, we obtain that for $A_{0,0}=1$, $\mathcal{A}=X_0$ is a strong algebra; i.e. **every Banach algebra is also a strong algebra**.

Example 2. Let X_n be the Hardy space $H^2(2^{-n}\mathbb{D})$ $(n\in\mathbb{N})$, i.e. X_n is the space of holomorphic functions on $2^{-n}\mathbb{D}$ for which $\|f\|_n^2=\sup_{0<r<1}\frac{1}{2\pi}\int_{|z|=r\cdot 2^{-n}}|f(z)|^2\,dz<\infty$. We can show that



$$\|fg\|_n=\|gf\|_n\leq\frac{1}{\sqrt{1-4^{-(n-m)}}}\|f\|_m\|g\|_n$$

for every $n>m$ (we will prove a more general result in the sequel). In this case $\mathcal{A}=\bigcup X_n$ is **the algebra of germs of holomorphic functions at the origin**, which is, as we can see, a strong algebra.

The significance of the inequalities $\|ab\|_\beta \le A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta$ and $\|ba\|_\beta \le A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta$

Continuity

$a \mapsto ab$ and $a \mapsto ba$ are continuous. Thus we obtain a topological algebra (i.e. locally convex topological vector space with separately continuous multiplication).

Proposition (Alpay & S, 2013)

A strong algebra is **bornological**, *i.e. every balanced convex subset which absorbs every bounded set is a neighborhood of 0.*

$\Longrightarrow$

Theorem (Alpay & S, 2013)

If a set is bounded in $\mathcal{A}$ iff it is bounded in some of the X_α, *then the product is* **jointly continuous**.

The significance of the inequalities $\|ab\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta$ and $\|ba\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta$

Power series

If $f(z) = \sum_{n=0}^{\infty} f_n z^n$ is a power series converges in $|z| < R$, then for every $a \in \mathcal{A}$ with $A_{\beta,\alpha}\|a\|_\alpha < R$ (for some $\beta \geq h(\alpha)$),

$$\sum |f_n|\|a^n\|_\beta \leq \|1\|_\beta \sum |f_n|(A_{\beta,\alpha}\|a\|_\alpha)^n.$$

Thus, $f(a) \in \mathcal{A}$

In particular, this implies that if $A_{\beta,\alpha}\|a\|_\alpha < 1$ (for some $\beta \geq h(\alpha)$) then $1 - a$ is invertible.

The significance of the inequalities $\|ab\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta$ and $\|ba\|_\beta \leq A_{\beta,\alpha}\|a\|_\alpha\|b\|_\beta$

Boundedness of the spectrum

It holds that

$$\sigma(a) \subseteq \{z \in \mathbb{C} \ : \ |z| \leq \inf_{\beta \geq h(\alpha)} A_{\beta,\alpha}\|a\|_\alpha\}.$$

In each of the following cases, of inductive limit of a sequence of Banach spaces (X_n), any bounded set of $\bigcup X_n$ is bounded in some of the X_n.

The inductive limit is a Banach space

The inductive limit is a dual of reflexive Fréchet space

The maps $X_n \to X_{n+1}$ are compact

The topology of X_n induced by X_m is the initial topology of X_n

Thus, in case a strong algebra $\mathcal{A}$ is of one of these forms, then in particular the multiplication is jointly continuous.

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Thus, in case a strong algebra $\mathcal{A}$ is of one of these forms, then in particular the multiplication is jointly continuous.

Theorem

In case where the topology on $\mathcal{A}$ is the finest ~~locally convex~~ topology such that the mappings $X_\alpha \hookrightarrow \mathcal{A}$ are continuous, the set of invertible elements $GL(\mathcal{A})$ is open, and $(\cdot)^{-1} : GL(\mathcal{A}) \to GL(\mathcal{A})$ is continuous.

In the sequel we will see many examples of strong algebras where the set of indices A is $\mathbb{N}$, and the maps $X_n \to X_{n+1}$ are compact. Thus, we may apply the theorem to each of these SAs.

Wiener theorem

Let $\mathcal{U}$ be the space of periodic functions

$$f(z) = \sum_{n\in\mathbb{Z}} f_n z^n$$

on $\mathbb{T}$ to $\mathcal{A}$, such that $\|f\|_\alpha = \sum_{n\in\mathbb{Z}} \|f_n\|_\alpha < \infty$ for some α. Note that, for all $\beta \geq h(\alpha)$,

$$\|fg\|_\beta \leq \sum_n \sum_m \|f_m g_{n-m}\|_\beta \leq A_{\beta,\alpha} \|f\|_\alpha \|g\|_\beta;$$

so $\mathcal{U}$ **is also a strong algebra** (with the same $A_{\beta,\alpha}$).

Theorem (Wiener, Annals of Mathematics, 1932 (for $\mathbb{C}$); Bochner & Phillips, Annals of Mathematics, 1942 (for all Banach algebras), Alpay & S, 2013 (for all strong algebras))

f is left/right/both-sided invertible iff $f(z)$ is left/right/both-sided invertible for every $z \in \mathbb{T}$.

Strong convolution algebras

Strong algebras associated to a locally compact group

$L_2(G, \mu)$ is usually not an algebra

Let G be a locally compact topological group with a Haar measure μ. The convolution of two measurable functions f, g is defined by

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\mu(y).$$

It is well known that $L_1(G, \mu)$ is a Banach algebra with the convolution product, while $L_2(G, \mu)$ is usually not closed under convolution. More precisely,

Theorem (N.W. Rickert (1969))

For any locally compact group G, $L_2(G, \mu)$ is closed under convolution if and only if G is compact.

In case G is compact it holds that, $\|f * g\| \leq \sqrt{\mu(G)}\|f\|\|g\|$. Thus, $L_2(G, \mu)$ is a Banach algebra.

So what can be done if G is not compact, but we still want an "Hilbert environment"?

Strong convolution algebras

Let G be a locally compact topological group with a left Haar measure μ and let

$$S \subseteq G$$

be a Borel sub-semi-group. Let (μ_p) be a sequence of measures on G such that

$$\mu \gg \mu_1 \gg \mu_2 \gg \cdots .$$

Question: When is $\varinjlim L_2(S, \mu_p)$ a strong algebra?

Strong convolution algebras

Theorem (Alpay & S, JFA, 2013)

If for any $x, y \in S$ *and for any* $p \in \mathbb{N}$,

$$\frac{d\mu_p}{d\mu}(xy) \leq \frac{d\mu_p}{d\mu}(x)\frac{d\mu_p}{d\mu}(y),$$

then for every $f \in L_2(S, \mu_p)$ *and* $g \in L_2(S, \mu_q)$ *such that* $q \geq p$,

$$\|f * g\|_q \leq \left(\int_S \frac{d\mu_q}{d\mu_p} d\mu\right)^{\frac{1}{2}} \|f\|_p \|g\|_q \text{ and } \|g * f\|_q \leq \left(\int_S \frac{d\mu_q}{d\mu_p} d\widetilde{\mu}\right)^{\frac{1}{2}} \|f\|_p \|g\|_q,$$

where $\widetilde{\mu}$ *is the right Haar measure. In particular, if for any* p *there exists* $q \geq p$ *such that* $\int_S \frac{d\mu_q}{d\mu_p} d\mu < \infty$ *and* $\int_S \frac{d\mu_q}{d\mu_p} d\widetilde{\mu} < \infty$, *then* $\bigcup L_2(S, \mu_p)$ **is a strong algebra**.

We call such an algebra **a strong convolution algebra** (SCA).

Strong convolution algebras: the discrete case

Theorem (Alpay & S, JFA, 2013)

In the discrete case it holds that:

- *the sufficient condition* $\frac{d\mu_p}{d\mu}(xy) \leq \frac{d\mu_p}{d\mu}(x)\frac{d\mu_p}{d\mu}(y)$ *is also necessary.*
- *a SCA is nuclear*.*
- *the tensor product (with respect to the* π *or* ϵ *topology) of two SCA's is again SCA.*

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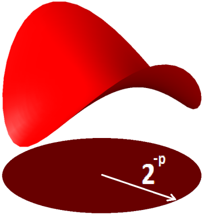
* A locally convex vector space X is said to be nuclear if to every continuous seminorm p on X there is another continuous seminorm on X, $q \geq p$, such that the canonical mapping $X_q \to X_p$ is nuclear, where X_r denotes the completion of the normed space $X/\ker r$.

The algebra of germs of holomorphic functions at the origin

The previous example of the strong algebra $\mathcal{O}_0$, namely the algebra of germs of holomorphic functions at the origin, is actually an example of SCA. We recall the details.

Let $H^2(2^{-p}\mathbb{D})$ be the Hardy space on the disk with radius 2^{-p}, i.e. X_p is the space of holomorphic functions on $2^{-p}\mathbb{D}$ for which $\|f\|_p^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_{|z|=r\cdot 2^{-p}} |f(z)|^2\, dz < \infty$.



If we write f in terms of power series $f = \sum_{n=0}^{\infty} f_n z^n$, we obtain that $\|f\|_p^2 = \sum_{n=0}^{\infty} |f_n|^2 2^{-2np}$ and that the pointwise multiplication is a convolution of the coefficients, i.e.

$$\mathcal{O}_0 = \bigcup H^2(2^{-p}\mathbb{D}) = \bigcup \ell_2(\mathbb{N}, 2^{-2np}).$$

But these measures are (sub-)multiplicative and $\sum 2^{-2n(q-p)} = (1-4^{-(q-p)})^{-1} < \infty$ for every $q > p$.

The algebra of germs of holomorphic functions at the origin

By the last theorem,

Theorem (Alpay & S, JFA, 2013)

$\mathcal{O}_0$ is a SCA, with an inequality

$$\|fg\|_q = \|gf\|_q \leq \frac{1}{\sqrt{1-4^{-(q-p)}}}\|f\|_p\|g\|_q.$$

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Proposition (Alpay & S, JFA, 2013)

An SCA associated to a discrete group is nuclear.

$\Longrightarrow$

Corollary

$\mathcal{O}_0$ is a nuclear space.

Replacing $\{0\}$ by any compact set K and the Hardy spaces $H_2(2^{-p}\mathbb{D})$ by appropriate Smirnov spaces $E_2(U_p)$ ((U_p) is a decreasing sequence "nice" open neighborhoods of K) yields

Theorem (Alpay & S)

$\mathcal{O}(K) = \varinjlim E_2(U_p)$ *is a strong algebra with an inequality*

$$\|fg\|_q \leq \sqrt{\frac{|\partial U_p|}{2\pi d(\partial U_p, \partial U_q)}}\|f\|_p\|g\|_q,$$

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Proposition

$E_2(U_p) \hookrightarrow E_2(U_{p+1})$ *are compact. As a result, the topology of* $\mathcal{O}(K)$ *is the finest topology such that the embeddings* $E_2(U_p) \hookrightarrow \mathcal{O}(K)$ *are continuous.*

$\Longrightarrow$

Corollary

The multiplication is jointly continuous, $GL(\mathcal{O}(K))$ *is open and* $f \mapsto f^{-1}$ *is continuous.*

Inductive limit of Fock spaces

Definition

Let H is a Hilbert space, then

$$\mathcal{F}(H) = \mathbb{C} \oplus H \oplus H^{\otimes 2} \oplus \cdots$$

is called the (full) Fock space associated to H.

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Now, suppose that one has a sequence of measures (weights) μ_p on $\mathbb{N}$, and consider the Fock spaces $\mathcal{F}(\ell_2(\mathbb{N}, \mu_p))$ with the product $\otimes$.

These of course are not algebras.

However, the inductive limit $\varinjlim \mathcal{F}(\ell_2(\mathbb{N}, \mu_p))$ can be a strong algebra.

Inductive limit of Fock spaces

Theorem (D. Alpay & S, SPA, 2013)

$\varinjlim \mathcal{F}(\ell_2(\mathbb{N},\mu_p))$ *with the multiplication* $\otimes$ *is a strong algebra if and only if* $\varinjlim \ell_2(\mathbb{N},\mu_p)$ *is nuclear.*

Inductive limit of Fock spaces

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$\varinjlim \mathcal{F}(\ell_2(\mathbb{N}, \mu_p))$ with the multiplication $\otimes$ is a strong algebra if and only if $\varinjlim \ell_2(\mathbb{N}, \mu_p)$ is nuclear.

In particular, if $\varinjlim \ell_2(\mathbb{N}, \mu_p)$ is nuclear, then $\varinjlim \mathcal{F}(\ell_2(\mathbb{N}, \mu_p))$ is closed under $\otimes$.

Applications to non-commutative stochastic analysis

A non-commutative algebra of stochastic distributions

Stochastic Gelfand triples

	commutative	non-commutative
Kondratiev space of stochastic test functions	$\mathcal{S}_1 = \varprojlim \mathcal{F}^s(\ell^2(\mathbb{N},(2n)^p))$	
White noise space	$\mathcal{W} = \mathcal{F}^s(\ell^2(\mathbb{N}))$	$\widetilde{\mathcal{W}} = \mathcal{F}(\ell^2(\mathbb{N}))$
Kondratiev space of stochastic distributions	$\mathcal{S}_{-1} = \varinjlim \mathcal{F}^s(\ell^2(\mathbb{N},(2n)^{-p}))$	

A non-commutative algebra of stochastic distributions

18 years ago Våge showed $\mathcal{S}_{-1}$ is a strong algebra.

Theorem (Våge, 1996)

In the space $\mathcal{S}_{-1} = \varinjlim \mathcal{F}^s(\ell^2(\mathbb{N}, (2n)^{-p}))$ *it holds that,*

$$\|g \otimes_s f\|_q = \|f \otimes_s g\|_q \leq A_{q,p}\|f\|_p\|g\|_q$$

for any $q \geq p + 2$*, where* $A_{q,p}^2 = \sum_{\alpha \in \ell}(2\mathbb{N})^{-\alpha(q-p)}$ *is finite due to Zhang (1992).*

This gave rise to many results in stochastic PDEs, stochastic linear systems, and stochastic control theory.

A non-commutative algebra of stochastic distributions

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A non-commutative algebra of stochastic distributions

Theorem (Alpay & S, SPA, 13)

$\varinjlim \mathcal{F}(\ell_2(\mathbb{N}, \mu_p))$ with the product $\otimes$ is a strong algebra if and only if $\varinjlim \ell_2(\mathbb{N}, \mu_p)$ is nuclear.

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Proposition

The Schwartz space $\mathscr{S}'$ of tempered distribution is nuclear and $\varinjlim \ell_2(\mathbb{N},(2n)^{-p}) \cong \mathscr{S}'$

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$$\Downarrow$$

Corollary (Alpay & S, SPA, 13)

$\varinjlim \mathcal{F}(\ell_2(\mathbb{N},(2n)^{-p}))$ *is a strong algebra. More precisely,*

$$\|f\otimes g\|_q \le A_{q,p}\|f\|_p\|g\|_q \text{ and } \|g\otimes f\|_q \le A_{q,p}\|f\|_p\|g\|_q$$

for any $q \ge p+2$*, where* $A_{q,p}^2 = \frac{1}{1-2^{-(q-p)}\zeta(q-p)}$.

A non-commutative algebra of stochastic distributions

This gives rise to the development of derivatives of free stochastic processes, e.g. the derivative of the free Brownian motion, namely the "free white noise" (Alpay-Jorgensen-S, SPA, 2014).

For further reading...

For further reading

- D. Alpay, P. Jorgensen and G. Salomon. On free stochastic processes and their derivatives. Stochastic Processes and their Applications, 2014.
- D. Alpay and G. Salomon. On algebras which are inductive limit of Banach algebras. preprint on arXiv, 2013.
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- D. Alpay and G. Salomon. A new family of $\mathbb{C}$-algebras with applications in linear systems. Infinite Dimensional Analysis, Quantum Probability and Related Topics, 2012.

Thank you!