The Bergman kernel and the Bergman metric

Gadadhar Misra

Indian Institute of Science
Bangalore

Recent Advances in Operator Theory and Operator Algebras
Indian Statistical Institute, Bangalore
December 15, 2014
kernel functions

Let $\mathcal{D}$ be a domain in $\mathbb{C}^d$, $V$ be a normed linear space and $K : \mathcal{D} \times \mathcal{D} \rightarrow V$ be a function, which is holomorphic in the first variable and anti-holomorphic in the second.

For two functions of the form $K(\cdot, w_i)\zeta_i$ in $V$ ($i = 1, 2$), define their inner product by the reproducing property, that is,

$$\langle K(\cdot, w_1)\zeta_1, K(\cdot, w_2)\zeta_2 \rangle = \langle K(w_2, w_1)\zeta_1, \zeta_2 \rangle.$$

This extends to an inner product on the linear span of the vectors

$$\mathcal{H} \zeta_0 = \{ \sum_{i=1}^{n} K(\cdot, w_i)\zeta_i | \zeta_1, \ldots, \zeta_n \in V; w_1, \ldots, w_n \in \mathcal{D} \text{ and } n \in \mathbb{N} \}$$

if and only if $K$ is positive definite in the sense that

$$\sum_{j,k=1}^{n} \langle K(z_j, z_k)\zeta_k, \zeta_j \rangle = \sum_{k=1}^{n} \langle K(\cdot, z_k)\zeta_k, \sum_{j=1}^{n} K(\cdot, z_j)\zeta_j \rangle = \sum_{k=1}^{n} \langle K(\cdot, z_k)\zeta_k \|^2 > 0.$$
**Gram matrix**

The completion \( \mathcal{H} \) of the linear space \( \mathcal{H}_0 \) is a Hilbert space with respect to the inner product induced by \( K \), or equivalently,

\[
\langle f, K(\cdot, w)\zeta \rangle_{\mathcal{H}} = \langle f(w), \zeta \rangle_V, \; w \in \mathcal{D}, \; \zeta \in V.
\]

Let \( G : \mathcal{D} \times \mathcal{D} \to V \) be the Grammian \( G(z, w) = \left( \langle u_j(w), u_k(z) \rangle \right)_{j,k} \) of a set of \( r(= \dim V) \) anti-holomorphic functions \( u_\ell : \mathcal{D} \to \mathcal{H}, \; 1 \leq \ell \leq r \), taking values in some Hilbert space \( \mathcal{H} \). We have

\[
\sum_{p,q=1}^{n} \langle G(z_p, z_q)\# \zeta_q, \zeta_p \rangle_V = \sum_{j,k=1}^{r} \sum_{pq=1}^{n} G(z_p, z_q)_{j,k} \zeta_q(j)\overline{\zeta_p(k)}
\]

\[
= \sum_{j,k=1}^{r} \left( \sum_{pq=1}^{n} \langle u_j(z_q), u_k(z_p) \rangle \zeta_q(j)\overline{\zeta_p(k)} \right)
\]

\[
= \left\| \sum_{jk} \zeta_q(j)u_q(z_q) \right\|^2 > 0.
\]

We therefore conclude that \( G(z, w)\# \) defines a positive definite kernel on \( \mathcal{D} \).
Let \( \{e_\ell : \mathcal{D} \overset{\text{hol}}{\to} V, \ell \in \mathbb{N}\} \) be an orthonormal basis in the Hilbert space \( \mathcal{H} \). Given \( \zeta \in V \), let \( \zeta^\# \) be the function \( \eta \to \langle \eta, \zeta \rangle_V \). Thus \( \zeta^\# \) defines an element in \( V^* \). Assume that \( f \to f(w), w \in \mathcal{D} \) is uniformly locally bounded. Then the sum \( \sum_\ell e_\ell(z)e_\ell(w)^\# \), is convergent on compact subsets of \( \mathcal{D} \). It also has the reproducing property:

\[
\langle f(\cdot), \sum_\ell e_\ell(\cdot)e_\ell(w)^\# \zeta \rangle = \langle f(\cdot), \sum_\ell e_\ell(\cdot)\langle \zeta, e_\ell(w) \rangle \rangle \\
= \sum_\ell \langle e_\ell(w), \zeta \rangle \langle f(\cdot), e_\ell(\cdot) \rangle \\
= \langle f(w), \zeta \rangle, \ z \in V.
\]

Since \( K \) is uniquely determined by the reproducing property, we have

\[
K(z, w) = \sum_\ell e_\ell(z)e_\ell(w)^\#.
\]
For $\zeta \in V$, let $\zeta^\dagger$ be the linear map $\xi \mapsto \langle \xi, \zeta \rangle_V$. For any domain $D$ in $V$, the function $K : D \times D \rightarrow \text{Hom}(V, V)$ defined by the formula $K(z, w) = zw^\dagger$ is positive definite, whereas $K(z, w)\dagger$ is not!

For the Bergman space $A^2(D^m)$, of the polydisc $D^m$, the orthonormal basis is $\{\sqrt{\prod_{i=1}^m (n_i + 1)}z_I : I = (i_1, \ldots, i_m)\}$. Clearly, we have

$$B_{D^m}(z, w) = \sum_{|I|=0}^{\infty} \left( \prod_{i=1}^m (n_i + 1) \right) z_I^* w_I = \prod_{i=1}^m (1 - z_i w_i)^{-2}.$$ 

Similarly, for the Bergman space of the ball $A^2(B^m)$, the orthonormal basis is $\{\sqrt{(-m-1)^{|I|}}z_I : I = (i_1, \ldots, i_m)\}$. Again, it follows that

$$B_{B^m}(z, w) = \sum_{|I|=0}^{\infty} \left( \begin{array}{c} -m - 1 \\ \ell \end{array} \right) \left( \sum_{|I|=\ell} \left( \begin{array}{c} |I| \\ I \end{array} \right) z_I^* w_I \right) = (1 - \langle z, w \rangle)^{-m-1}.$$
For $\zeta \in V$, let $\zeta^\dagger$ be the linear map $\xi \rightarrow \langle \xi, \zeta \rangle_V$. For any domain $D$ in $V$, the function $K : D \times D \rightarrow \text{Hom}(V, V)$ defined by the formula $K(z, w) = zw^\#$ is positive definite, whereas $K(z, w)^\#$ is not!

For the Bergman space $A^2(D^m)$, of the polydisc $D^m$, the orthonormal basis is $\{ \sqrt{\prod_{i=1}^m (n_i + 1)} z^I : I = (i_1, \ldots, i_m) \}$. Clearly, we have

$$B_{D^m}(z, w) = \sum_{|I|=0}^{\infty} \left( \prod_{i=1}^m (n_i + 1) \right) z^I \bar{w}^I = \prod_{i=1}^m (1 - z_i \bar{w}_i)^{-2}. $$

Similarly, for the Bergman space of the ball $A^2(B^m)$, the orthonormal basis is $\{ \sqrt{(-m-1)/|I|} z^I : I = (i_1, \ldots, i_m) \}$. Again, it follows that

$$B_{B^m}(z, w) = \sum_{|I|=0}^{\infty} \left( \frac{-m-1}{|I|} \right) \left( \sum_{|I|=\ell} \left( \frac{|I|}{|I|} \right) z^I \bar{w}^I \right) = (1 - \langle z, w \rangle)^{-m-1}. $$
For $\zeta \in V$, let $\zeta^\dagger$ be the linear map $\xi \to \langle \xi, \zeta \rangle_V$. For any domain $\mathcal{D}$ in $V$, the function $K : \mathcal{D} \times \mathcal{D} \to \text{Hom}(V, V)$ defined by the formula $K(z, w) = zw^\#$ is positive definite, whereas $K(z, w)^\#$ is not!

For the Bergman space $\mathbb{A}^2(\mathbb{D}^m)$, of the polydisc $\mathbb{D}^m$, the orthonormal basis is $\{\sqrt{\prod_{i=1}^m (n_i + 1)} z^I : I = (i_1, \ldots, i_m)\}$. Clearly, we have

$$B_{\mathbb{D}^m}(z, w) = \sum_{|I|=0}^{\infty} \left( \prod_{i=1}^m (n_i + 1) \right) z^I \bar{w}^I = \prod_{i=1}^m (1 - z_i \bar{w}_i)^{-2}.$$ 

Similarly, for the Bergman space of the ball $\mathbb{A}^2(\mathbb{B}^m)$, the orthonormal basis is $\{\sqrt{(-m-1)^{|I|}} \binom{|I|}{\ell} z^I : I = (i_1, \ldots, i_m)\}$. Again, it follows that

$$B_{\mathbb{B}^m}(z, w) = \sum_{|I|=0}^{\infty} \binom{-m-1}{\ell} \left( \sum_{|I|=\ell} \binom{|I|}{I} z^I \bar{w}^I \right) = (1 - \langle z, w \rangle)^{-m-1}.$$
new from old

Let $W$ be a second finite dimensional inner product space and $T : \mathcal{H} \to \text{Hol}(\mathcal{D}, W)$ be a linear map for which the evaluation at $z \in \mathcal{D}$, namely, $f \to (Tf)(z)$, $f \in \mathcal{H}$, is continuous. Transplant the inner product from $\mathcal{H}/\ker T$ to the linear space $T\mathcal{H}$. In consequence, $T(z)K(z, w)T^\#_w : W \to W$ is the reproducing kernel of $T\mathcal{H}$:

$$TK(z, w)\zeta := (T(z)K_w\zeta)(z) = \sum_\ell \langle \zeta, e_\ell(w) \rangle (Te_\ell)(z).$$

Linearity in $\zeta$ implies that $TK(z, w)$ is in $\text{Hom}(V, T\mathcal{H})$. We have

$$T(z)K(z, w) = \sum_\ell (Te_\ell(z))e_\ell(w)^\#$$

and

$$K(z, w)T^\# := (T(w)K(w, z))^\# = \sum_\ell e_\ell(z)(Te_\ell(w))^\#$$

(For fixed $w$, $\{Te_\ell(w)^\#\zeta\}$ is in $\ell^2$ for all $\zeta$. ) Applying $T$ to this we have

$$TK(z, w)T^\# = \sum_\ell (Te_\ell)(z)(Te_\ell(w)^\#).$$
Suppose \( \mathcal{H} \subseteq \text{Hol}(\mathcal{D}, V) \) is a Hilbert space possessing a reproducing kernel \( K \) and \( T : \mathcal{H} \rightarrow \text{Hol}(\mathcal{D}, W) \) is a linear map such that \( f \rightarrow (Tf)(z), f \in \mathcal{H}, \) is continuous. Let \( \mathcal{H}' \subseteq \text{Hol}(\mathcal{D}, W) \) be another Hilbert space with reproducing kernel \( K' : \mathcal{D} \times \mathcal{D} \rightarrow \text{Hom}(W, W). \)

**Lemma**

If \( TK(z, w)T^\# \prec CK'(z, w), \) then the image of \( T \) is contained in \( \mathcal{H}' \) and as an operator from \( \mathcal{H} \) to \( \mathcal{H}' \), it is bounded by \( C. \)

**Proof.** Without loss of generality, may assume \( C = 1. \) If \( \mathcal{H}_i, i = 1, 2 \) are two Hilbert spaces with reproducing kernels \( K_i, i = 1, 2, \) then their sum is the reproducing kernel of the Hilbert space \( \{ g | g = f_1 + f_2 \text{ for some } f_1 \in \mathcal{H}_1 \text{ and } f_2 \in \mathcal{H}_2 \} \) equipped with the norm \( \|g\|^2 = \inf \{ \|f_1\|^2 + \|f_2\|^2 | g = f_1 + f_2 \}. \)
Suppose $\mathcal{H} \subseteq \text{Hol}(D, V)$ is a Hilbert space possessing a reproducing kernel $K$ and $T : \mathcal{H} \to \text{Hol}(D, W)$ is a linear map such that $f \to (Tf)(z), f \in \mathcal{H}$, is continuous. Let $\mathcal{H}' \subseteq \text{Hol}(D, W)$ be another Hilbert space with reproducing kernel $K' : D \times D \to \text{Hom}(W, W)$.

Lemma
If $TK(z, w)T^\# \prec CK'(z, w)$, then the image of $T$ is contained in $\mathcal{H}'$ and as an operator from $\mathcal{H}$ to $\mathcal{H}'$, it is bounded by $C$.

Proof. Without loss of generality, may assume $C = 1$. If $\mathcal{H}_i, i = 1, 2$ are two Hilbert spaces with reproducing kernels $K_i, i = 1, 2$, then their sum is the reproducing kernel of the Hilbert space

$$\{g | g = f_1 + f_2 \text{ for some } f_1 \in \mathcal{H}_1 \text{ and } f_2 \in \mathcal{H}_2 \}$$
equipped with the norm $\|g\|^2 = \inf\{\|f_1\|^2 + \|f_2\|^2 | g = f_1 + f_2 \}$. 
Apply this with $\mathcal{H}_1 := T\mathcal{H}$, $K_1 := TKT^\dagger$. Set $\mathcal{H}_2$ to be the Hilbert space corresponding to the kernel function $K_2 := K' - K_1$, which is positive definite by assumption. For $f$ in $\mathcal{H}$, write $f = f_1 + f_2$, where $f_1 = Tf$ and $f_2 = 0$. Then we have

$$\|Tf\|^2_{\mathcal{H}'} \leq \|Tf\|^2_{\mathcal{H}_1} = \|Tf\|^2_{T\mathcal{H}} \leq \|f\|^2_{\mathcal{H}}.$$
Any bi-holomorphic map \( \varphi : \mathbb{D} \rightarrow \tilde{\mathbb{D}} \) induces a unitary operator \( U_\varphi : \mathbb{A}^2(\tilde{\mathbb{D}}) \rightarrow \mathbb{A}^2(\mathbb{D}) \) defined by the formula

\[
(U_\varphi f)(z) = (J(\varphi, z) (f \circ \varphi)(z), f \in \mathbb{A}^2(\tilde{\mathbb{D}}), z \in \mathbb{D}.
\]

This is an immediate consequence of the change of variable formula for the volume measure on \( \mathbb{C}^n \).

Consequently, if \( \{\tilde{e}_n\}_{n \geq 0} \) is any orthonormal basis for \( \mathbb{A}^2(\tilde{\mathbb{D}}) \), then \( \{e_n\}_{n \geq 0} \), where \( \tilde{e}_n = J(\varphi, \cdot)(\tilde{e}_n \circ \varphi) \) is an orthonormal basis for the Bergman space \( \mathbb{A}^2(\tilde{\mathbb{D}}) \).
quasi-invariance of $B$

Any bi-holomorphic map $\varphi : \mathcal{D} \to \tilde{\mathcal{D}}$ induces a unitary operator $U_\varphi : \mathbb{A}^2(\tilde{\mathcal{D}}) \to \mathbb{A}^2(\mathcal{D})$ defined by the formula

$$(U_\varphi f)(z) = (J(\varphi, z)(f \circ \varphi)(z), f \in \mathbb{A}^2(\tilde{\mathcal{D}}), z \in \mathcal{D}.$$ 

This is an immediate consequence of the change of variable formula for the volume measure on $\mathbb{C}^n$.

Consequently, if $\{\tilde{e}_n\}_{n \geq 0}$ is any orthonormal basis for $\mathbb{A}^2(\tilde{\mathcal{D}})$, then $\{e_n\}_{n \geq 0}$, where $\tilde{e}_n = J(\varphi, \cdot)(\tilde{e}_n \circ \varphi)$ is an orthonormal basis for the Bergman space $\mathbb{A}^2(\tilde{\mathcal{D}})$. 
quasi-invariance of $B$

Expressing the Bergman kernel $B_\mathcal{D}$ of the domains $\mathcal{D}$ as the infinite sum $\sum_{n=0}^{\infty} e_n(z)e_n(w)$ using the orthonormal basis in $A^2(\mathcal{D})$, we see that the Bergman Kernel $B$ is quasi-invariant, that is, If $\varphi: \mathcal{D} \to \tilde{\mathcal{D}}$ is holomorphic then we have the transformation rule

$$J(\varphi, z)B_{\tilde{\mathcal{D}}} (\varphi(z), \varphi(w))J(\varphi, w) = B_{\mathcal{D}}(z, w),$$

where $J(\varphi, w)$ is the Jacobian determinant of the map $\varphi$ at $w$.

If $\mathcal{D}$ admits a transitive group of bi-holomorphic automorphisms, then this transformation rule gives an effective way of computing the Bergman kernel. Thus

$$B_{\mathcal{D}}(z, z) = |J(\varphi_z, z)|^2 B_{\mathcal{D}}(0, 0), \quad z \in \mathcal{D},$$

where $\varphi_z$ is the automorphism of $\mathcal{D}$ with the property $\varphi_z(z) = 0$. 
quasi-invariance of $B$

Expressing the Bergman kernel $B_{\mathcal{D}}$ of the domains $\mathcal{D}$ as the infinite sum $\sum_{n=0}^{\infty} e_n(z)e_n(w)$ using the orthonormal basis in $\mathbb{A}^2(\mathcal{D})$, we see that the Bergman Kernel $B$ is quasi-invariant, that is, if $\varphi : \mathcal{D} \to \mathcal{\tilde{D}}$ is holomorphic then we have the transformation rule

$$J(\varphi, z)B_{\mathcal{\tilde{D}}}(\varphi(z), \varphi(w))|J(\varphi, w)| = B_{\mathcal{D}}(z, w),$$

where $J(\varphi, w)$ is the Jacobian determinant of the map $\varphi$ at $w$.

If $\mathcal{D}$ admits a transitive group of bi-holomorphic automorphisms, then this transformation rule gives an effective way of computing the Bergman kernel. Thus

$$B_{\mathcal{D}}(z, z) = |J(\varphi_z, z)|^2B_{\mathcal{D}}(0, 0), \ z \in \mathcal{D},$$

where $\varphi_z$ is the automorphism of $\mathcal{D}$ with the property $\varphi_z(z) = 0$. 
Consider the special case, where $\phi: \mathcal{D} \to \mathcal{D}$ is an automorphism. Clearly, in this case, $U_{\phi}$ is unitary on $A^2(\mathcal{D})$ for all $\phi \in \text{Aut}(\mathcal{D})$.

The map $J: \text{Aut}(\mathcal{D}) \times \mathcal{D} \to \mathbb{C}$ satisfies the cocycle property, namely

$$J(\psi \phi, z) = J(\phi, \psi(z))J(\psi, z), \quad \phi, \psi \in \text{Aut}(\mathcal{D}), \ z \in \mathcal{D}.$$ 

This makes the map $\phi \to U_{\phi}$ a homomorphism.

Thus we have a unitary representation of the Lie group $\text{Aut}(\mathcal{D})$ on $A^2(\mathcal{D})$. 

Consider the special case, where \( \varphi : \mathcal{D} \to \mathcal{D} \) is an automorphism. Clearly, in this case, \( U_\varphi \) is unitary on \( \mathbb{A}^2(\mathcal{D}) \) for all \( \varphi \in \text{Aut}(\mathcal{D}) \). The map \( J : \text{Aut}(\mathcal{D}) \times \mathcal{D} \to \mathbb{C} \) satisfies the cocycle property, namely

\[
J(\psi \varphi, z) = J(\varphi, \psi(z))J(\psi, z), \ \varphi, \psi \in \text{Aut}(\mathcal{D}), \ z \in \mathcal{D}.
\]

This makes the map \( \varphi \to U_\varphi \) a homomorphism.

Thus we have a unitary representation of the Lie group \( \text{Aut}(\mathcal{D}) \) on \( \mathbb{A}^2(\mathcal{D}) \).
Consider the special case, where \( \varphi : \mathcal{D} \rightarrow \mathcal{D} \) is an automorphism. Clearly, in this case, \( U_\varphi \) is unitary on \( \mathbb{A}^2(\mathcal{D}) \) for all \( \varphi \in \text{Aut}(\mathcal{D}) \). The map \( J : \text{Aut}(\mathcal{D}) \times \mathcal{D} \rightarrow \mathbb{C} \) satisfies the cocycle property, namely

\[
J(\psi \varphi, z) = J(\varphi, \psi(z))J(\psi, z), \quad \varphi, \psi \in \text{Aut}(\mathcal{D}), \ z \in \mathcal{D}.
\]

This makes the map \( \varphi \rightarrow U_\varphi \) a homomorphism. Thus we have a unitary representation of the Lie group \( \text{Aut}(\mathcal{D}) \) on \( \mathbb{A}^2(\mathcal{D}) \).
more representations

Exploit the quasi-invariance of the Bergman kernel to construct unitary representations of the automorphism group $\text{Aut}(\mathcal{D})$. Let $B^\lambda(z, w)$ be the polarization of the function $B(w, w)^\lambda$, $w \in \mathcal{D}$, $\lambda > 0$.

Now, as before,

$$J_\varphi(z)^\lambda B^\lambda(\varphi(z), \varphi(w))\overline{J_\varphi(w)^\lambda} = B^\lambda(z, w), \quad \varphi \in \text{Aut}(\mathcal{D}), \quad z, w \in \mathcal{D}.$$

Let $\mathcal{O}(\mathcal{D})$ be the ring of holomorphic functions on $\mathcal{D}$. Define

$$U^{(\lambda)} : \text{Aut}(\mathcal{D}) \to \text{End}(\mathcal{O}(\mathcal{D}))$$

by the formula

$$(U_\varphi^{(\lambda)}f)(z) = (J_\varphi^{-1}(z))^\lambda(f \circ \varphi^{-1})(z)$$

and note that $\varphi \mapsto U_\varphi$ is a homomorphism.

When is it unitarizable?
more representations

Exploit the quasi-invariance of the Bergman kernel to construct unitary representations of the automorphism group $\text{Aut}(\mathcal{D})$. Let $B^\lambda(z, w)$ be the polarization of the function $B(w, w)\lambda$, $w \in \mathcal{D}$, $\lambda > 0$.

Now, as before,

$$J_\varphi(z)^\lambda B^\lambda(\varphi(z), \varphi(w))\overline{J_\varphi(w)^\lambda} = B^\lambda(z, w), \quad \varphi \in \text{Aut}(\mathcal{D}), \quad z, w \in \mathcal{D}.$$  

Let $\mathcal{O}(\mathcal{D})$ be the ring of holomorphic functions on $\mathcal{D}$. Define

$$U^{(\lambda)} : \text{Aut}(\mathcal{D}) \rightarrow \text{End}(\mathcal{O}(\mathcal{D}))$$

by the formula

$$(U^{(\lambda)}_\varphi f)(z) = (J_{\varphi^{-1}}(z))^\lambda (f \circ \varphi^{-1})(z)$$

and note that $\varphi \mapsto U_\varphi$ is a homomorphism.

When is it unitarizable?
more representations

Exploit the quasi-invariance of the Bergman kernel to construct unitary representations of the automorphism group $\text{Aut}(\mathcal{D})$. Let $B^\lambda(z, w)$ be the polarization of the function $B(w, w)^\lambda$, $w \in \mathcal{D}$, $\lambda > 0$.

Now, as before,

$$J_\varphi(z)^\lambda B^\lambda(\varphi(z), \varphi(w))\overline{J_\varphi(w)^\lambda} = B^\lambda(z, w), \, \varphi \in \text{Aut}(\mathcal{D}), \, z, w \in \mathcal{D}.$$

Let $\mathcal{O}(\mathcal{D})$ be the ring of holomorphic functions on $\mathcal{D}$. Define

$$U^{(\lambda)} : \text{Aut}(\mathcal{D}) \to \text{End}(\mathcal{O}(\mathcal{D}))$$

by the formula

$$(U_\varphi^{(\lambda)}f)(z) = (J_{\varphi^{-1}})(z)^\lambda (f \circ \varphi^{-1})(z)$$

and note that $\varphi \mapsto U_\varphi$ is a homomorphism.

When is it unitarizable?
more representations

Exploit the quasi-invariance of the Bergman kernel to construct unitary representations of the automorphism group $\text{Aut}(\mathcal{D})$. Let $B^\lambda(z, w)$ be the polarization of the function $B(w, w)^\lambda$, $w \in \mathcal{D}$, $\lambda > 0$.

Now, as before,

$$J_\varphi(z)^\lambda B^\lambda(\varphi(z), \varphi(w))J_\varphi(w)^\lambda = B^\lambda(z, w), \varphi \in \text{Aut}(\mathcal{D}), z, w \in \mathcal{D}.$$ 

Let $\mathcal{O}(\mathcal{D})$ be the ring of holomorphic functions on $\mathcal{D}$. Define

$$U^{(\lambda)} : \text{Aut}(\mathcal{D}) \to \text{End}(\mathcal{O}(\mathcal{D}))$$

by the formula

$$(U_\varphi^{(\lambda)} f)(z) = (J_{\varphi^{-1}}(z))^\lambda (f \circ \varphi^{-1})(z)$$

and note that $\varphi \mapsto U_\varphi$ is a homomorphism.

When is it unitarizable?
new kernels?

Let \( K \) be a complex valued positive definite kernel on \( \mathbb{D} \). For \( w \) in \( \mathbb{D} \), and \( p \) in the set \( \{1, \ldots, d\} \), let \( e_p : \Omega \to \mathcal{H} \) be the antiholomorphic function:

\[
e_p(w) := K_w(\cdot) \otimes \frac{\partial}{\partial \bar{w}_p} K_w(\cdot) - \frac{\partial}{\partial \bar{w}_p} K_w(\cdot) \otimes K_w(\cdot).
\]

Setting \( G(z, w)_{p,q} = \langle e_p(w), e_q(z) \rangle \), we have

\[
\frac{1}{2} G(z, w)_{p,q}^\# = K(z, w) \frac{\partial^2}{\partial z_q \partial \bar{w}_p} K(z, w) - \frac{\partial}{\partial \bar{w}_p} K(z, w) \frac{\partial}{\partial z_q} K(z, w).
\]

The curvature \( K \) of the metric \( K \) is given by the \((1, 1)\) - form

\[
\sum \frac{\partial^2}{\partial w_q \partial \bar{w}_p} \log K(w, w) dw_q \wedge d\bar{w}_p.
\]

Set

\[
\mathcal{K}_K(z, w) := \left( \frac{\partial^2}{\partial z_q \partial \bar{w}_p} \log K(z, w) \right)_{q,p}.
\]

We note that \( K(z, w)^2 \mathcal{K}(z, w) = \frac{1}{2} G(z, w)^\# \). Hence \( K(z, w)^2 \mathcal{K}(z, w) \) defines a positive definite kernel on \( \mathbb{D} \) taking values in \( \text{Hom}(V, V) \).
transformation rule

Let $\varphi : \mathcal{D} \to \mathcal{D}$ be a holomorphic map. Applying the change of variable formula twice to the function $\log K(\varphi(z), \varphi(w))$, we have

$$
\left( \frac{\partial^2}{\partial z_i \partial \bar{w}_j} \log K(\varphi(z), \varphi(w)) \right)_{ij} = \left( \frac{\partial \varphi^\ell}{\partial z_i} \right)_{i\ell} \left( \frac{\partial^2}{\partial z_\ell \partial \bar{w}_k} \log K(\varphi(z), \varphi(w)) \right)_{\ell k} \left( \frac{\partial \bar{\varphi}_k}{\partial \bar{z}_j} \right)_{kj}.
$$

Now, we set $K(w, w) = B_\mathcal{D}(w, w)$, the Bergman kernel of $\mathcal{D}$, which transforms according to the rule:

$$
\det_C D\varphi(w) B_\mathcal{D}(\varphi(w), \varphi(w)) \overline{\det_C D\varphi(w)} = B_\mathcal{D}(w, w),
$$

Thus $K_{B_\mathcal{D} \circ (\varphi, \varphi)}(w, w)$ equals $K_{B_\mathcal{D}}(w, w)$. Hence we conclude that $K := K_{B_\mathcal{D}}$ is invariant under the automorphisms $\varphi$ of $\mathcal{D}$ in the sense that

$$
D\varphi(w)^\# K(\varphi(w), \varphi(w)) \overline{D\varphi(w)} = K(w, w), \ w \in \mathcal{D}.
$$
*rewrite the transformation rule*

Or equivalently,

\[
\mathcal{K}(\varphi(z), \varphi(w)) = D\varphi(z)^{-1} \mathcal{K}(z, w) D\varphi(z)^{-1}
\]

\[
= D\varphi(z)^{-1} \mathcal{K}(z, w) (D\varphi(w)^{-1})^*
\]

\[
= m_0(\varphi, z) \mathcal{K}(z, w) m_0(\varphi, w)^*,
\]

where \( m_0(\varphi, z) = D\varphi(z)^{-1} \) and multiplying both sides by \( K^2 \), we have

\[
K(\varphi(z), \varphi(w))^2 \mathcal{K}(\varphi(z), \varphi(w)) = m_2(\varphi, z) K(z, w)^2 \mathcal{K}(z, w) m_2(\varphi, w)^*,
\]

where \( m_2(\varphi, z) = (\det_C D\varphi(w)^2 D\varphi(z)^\dagger)^{-1} \) is a multiplier. Of course, we now have that

(i) \( K^{2+\lambda}(z, w) \mathcal{K}(z, w) \), \( \lambda > 0 \), is a positive definite kernel and

(ii) it transforms according with \( m_\lambda(\varphi, z) = (\det_C D\varphi(z)^2+\lambda D\varphi(z)^\dagger)^{-1} \) in place of \( m_2(\varphi, z) \).
**rewrite the transformation rule**

Or equivalently,

\[
\mathcal{K} (\varphi(z), \varphi(w)) = D\varphi(z)^\#^{-1} \mathcal{K}(z, w) D\varphi(z)^{-1} \\
= D\varphi(z)^\#^{-1} \mathcal{K}(z, w) (D\varphi(w)^\#^{-1})^* \\
= m_0(\varphi, z) \mathcal{K}(z, w) m_0(\varphi, w)^*,
\]

where \( m_0(\varphi, z) = D\varphi(z)^\#^{-1} \) and multiplying both sides by \( \mathcal{K}^2 \), we have

\[
\mathcal{K}(\varphi(z), \varphi(w))^2 \mathcal{K}(\varphi(z), \varphi(w)) = m_2(\varphi, z) \mathcal{K}(z, w)^2 \mathcal{K}(z, w) m_2(\varphi, w)^*,
\]

where \( m_2(\varphi, z) = (\det_C D\varphi(w)^2 D\varphi(z)^\#)^{-1} \) is a multiplier. Of course, we now have that

(i) \( \mathcal{K}^{2+\lambda}(z, w) \mathcal{K}(z, w) \), \( \lambda > 0 \), is a positive definite kernel and

(ii) it transforms according with \( m_\lambda(\varphi, z) = (\det_C D\varphi(z)^{2+\lambda} D\varphi(z)^\dagger)^{-1} \) in place of \( m_2(\varphi, z) \).
Or equivalently,

\[
K(\varphi(z), \varphi(w)) = D\varphi(z)^\#^{-1}K(z, w)D\varphi(z)^{-1}
\]
\[
= D\varphi(z)^\#^{-1}K(z, w)(D\varphi(w)^\#^{-1})^*
\]
\[
= m_0(\varphi, z)K(z, w)m_0(\varphi, w)^*,
\]

where \( m_0(\varphi, z) = D\varphi(z)^\#^{-1} \) and multiplying both sides by \( K^2 \), we have

\[
K(\varphi(z), \varphi(w))^2K(\varphi(z), \varphi(w)) = m_2(\varphi, z)K(z, w)^2K(z, w)m_2(\varphi, w)^*,
\]

where \( m_2(\varphi, z) = (\det \mathbb{C} D\varphi(w)^2D\varphi(z)^\#)^{-1} \) is a multiplier. Of course, we now have that

(i) \( K^{2+\lambda}(z, w)K(z, w), \lambda > 0, \) is a positive definite kernel and

(ii) it transforms according with \( m_\lambda(\varphi, z) = (\det \mathbb{C} D\varphi(z)^{2+\lambda}D\varphi(z)^\dagger)^{-1} \) in place of \( m_2(\varphi, z) \).
Thank you!