Talk 1: Definitions and Examples

Elias Katsoulis

The semicrossed product $C_0(X) \times_\sigma \mathbb{Z}^+$

It was introduced by Arveson (1967), Arveson and Josephson (1969) and formalized by Peters (1984).

- $X \subseteq \mathbb{C}$ is locally compact Hausdorff space.

- $C_0(X)$ are the continuous functions on X vanishing at infinity.

- $\sigma : X \to X$ a proper continuous map

Given $x \in X$ and $f \in C_0(X)$, we define

$$\pi_x(f) = \begin{pmatrix} f(x) & 0 & 0 & \dots & \dots \\ 0 & f(\sigma(x)) & 0 & \dots & \dots \\ 0 & 0 & f(\sigma^{(2)}(x)) & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$$S_x = \begin{pmatrix} 0 & 0 & 0 & \dots & \dots \\ 1 & 0 & 0 & \dots & \dots \\ 0 & 1 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Formally for each $x \in X$, $f \in C_0(X)$ define
$\pi_x(f)\xi = (f(x)\xi_0, (f \circ \sigma)(x)\xi_1, (f \circ \sigma^{(2)})(x)\xi_2, \dots)$.
and S_x is the forward shift

$$S\xi = (0, \xi_0, \xi_1, \xi_2, \dots).$$

DEFINITION. The semicrossed product $C_0(X) \times_\sigma \mathbb{Z}^+$ is defined as the norm closed operator algebra acting on $\oplus_{x\in X}\mathcal{H}_x$ and generated by the operators

$$\pi(f) \equiv \oplus_{x\in X}\pi_x(f), \quad f \in C_0(X), \text{ and}$$

$$S\pi(f), \quad f \in C_0(X),$$

where

$$S \equiv \oplus_{x\in X} S_x$$

.

Note the covariance relation

$$\pi(f)S = S\pi(f \circ \sigma), \quad f \in C_0(X)$$

The classification problem for semicrossed products.

Classify the semicrossed products $C_0(X) \times_\sigma \mathbb{Z}^+$ as algebras.

A sufficient condition: Assume that σ_1 and σ_2 are topologicaly conjugate, i.e., there exists a homeomorphism

$$\gamma : X_1 \to X_2$$

so that

$$\gamma \circ \sigma_1 = \sigma_2 \circ \gamma.$$

Then the semicrossed products $C_0(X_1) \times_{\sigma_1} \mathbb{Z}^+$ and $C_0(X_2) \times_{\sigma_2} \mathbb{Z}^+$ are isomorphic as algebras.

Necessity:

- Arveson and Josephson (1969). X_i compact, σ_i no fixed points, plus some extra conditions

- Peters (1985). X_i compact, σ_i no fixed points.

- Hadwin and Hoover (1988). X_i compact, the set

  $$\{x \in X_i \mid \sigma_1(x) \neq x, \sigma_1^{(2)}(x) = \sigma_1(x)\}$$

  has empty interior.

- Power (1992). X_i locally compact, σ_i homeomorphisms

THEOREM 1. (Davidson and Katsoulis, 2008) Let X_i be a locally compact Hausdorff space and let σ_i a proper continuous map on X_i, for $i = 1, 2$. Then the dynamical systems (X_1, σ_1) and (X_2, σ_2) are conjugate if and only if the semicrossed products $C_0(X_1) \times_{\sigma_1} \mathbb{Z}^+$ and $C_0(X_2) \times_{\sigma_2} \mathbb{Z}^+$ are isomorphic as algebras.

The tensor algebra $\mathcal{T}_{\mathcal{G}}^{+}$

- $\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1, r, s)$ a countable directed graph

- $\mathcal{G}_\infty$ the (finite) path space of $\mathcal{G}$

The path space consists of all vertices

$$v \in \mathcal{G}^0$$

and finite paths

$$p = e_k e_{k-1} \dots e_1$$

where the $e_i \in \mathcal{G}^1$ are edges satisfying $s(e_i) = r(e_{i-1})$, $i = 1, 2, \dots, k$, $k \in \mathbb{N}$.

Let $\{\xi_p\}_{p\in\mathcal{G}_\infty}$ denote the usual orthonormal basis of the Fock space $\mathcal{H}_\mathcal{G} \equiv l^2(\mathcal{G}_\infty)$, where ξ_p is the characteristic function of $\{p\}$. The left creation operator L_q, $q \in \mathcal{G}_\infty$, is defined by

$$L_q\xi_p = \begin{cases} \xi_{qp} & \text{if } s(q) = r(p) \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION. The norm closed algebra generated by $\{L_p \mid p \in \mathcal{G}_\infty\}$, denoted as $\mathcal{T}^+_\mathcal{G}$, is the tensor tensor algebra of the graph $\mathcal{G}$. Its weak closure, denoted as $\mathcal{L}_\mathcal{G}$, is the free semigroupoid algebra of $\mathcal{G}$.

THEOREM 2. (Katsoulis and Kribs, 2004) Let $\mathcal{G}_1$, $\mathcal{G}_2$ be directed graphs with no sinks. Then the tensor algebras $\mathcal{T}^+_{\mathcal{G}_1}$ and $\mathcal{T}^+_{\mathcal{G}_2}$ are isomorphic as algebras if and only if $\mathcal{G}_1$ are $\mathcal{G}_2$ are isomorphic as graphs.

DEFINITION. Let $\mathcal{G}$ be a finite undirected graph with no loop edges or multiple edges between any two of its vertices. A vertex-deleted subgraph of $\mathcal{G}$ is a subgraph formed by deleting exactly one vertex from $\mathcal{G}$ and its incidence edges.

DEFINITION For a graph $\mathcal{G}$, the deck of $\mathcal{G}$, denoted as $D(\mathcal{G})$, is the multiset of all vertex-deleted subgraphs of $\mathcal{G}$. Each graph in $D(\mathcal{G})$ is called a *card*. Two graphs that have the same deck are said to be *hypomorphic* or *reconstructions* of each other. With these definitions at hand, the famous

CONJECTURE (Kelly and Ulam) Any two hypomorphic graphs on at least three vertices have to be isomorphic.

A finite directed graph $\mathcal{G}$ will belong to the subclass $\mathfrak{G}_0$ of all directed graphs if $\mathcal{G}$ comes from a finite undirected graph by replacing each edge with two directed edges with opposite directions. The concepts of a card, a deck and hypomorphism transfer to graphs in $\mathfrak{G}$ and the Reconstruction Conjecture can be stated as

Reconstruction Conjecture (Kelly and Ulam). Any two hypomorphic graphs in $\mathfrak{G}_0$ on at least three vertices are necessarily isomorphic.

DEFINITION. If $\mathcal{G} \in \mathfrak{G}$, then a vertex-deleted subalgebra of $\mathcal{T}_{\mathcal{G}}^{+}$ is formed by deleting from $\mathcal{G}$ exactly one vertex and its incidence edges and then taking the subalgebra of $\mathcal{T}_{\mathcal{G}}^{+}$ formed by the partial isometries and projections corresponding to the remaining edges and vertices respectively.

DEFINITION For a tensor algebra $\mathcal{T}_{\mathcal{G}}^{+}$, the deck of $\mathcal{T}_{\mathcal{G}}^{+}$, denoted as $D(\mathcal{T}_{\mathcal{G}}^{+})$, is the multiset of all vertex-deleted subalgebras of $\mathcal{T}_{\mathcal{G}}^{+}$. Each graph in $D(\mathcal{T}_{\mathcal{G}}^{+})$ is called a *card*. Two tensor algebras that have the same deck are said to be *hypomorphic* or *reconstructions* of each other.

In an ongoing collaboration with Gunther Cornelissen at Utreht we have the following

THEOREM 3. If $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{G}_0$, then the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are hypomorphic if and only $\mathcal{T}^+_{\mathcal{G}_1}$ and $\mathcal{T}^+_{\mathcal{G}_2}$ are hypomorphic as operator algebras.

Therefore the reconstruction conjecture admits the following equivalent form

COROLLARY 4. The reconstruction conjecture in graph theory is equivalent to the assertion that hypomorphic tensor algebras of graphs in $\mathfrak{G}_0$ are necessarily isomorphic as algebras.

The formal definition of the semicrossed product $C_0(X) \times_\sigma \mathbb{Z}^+$

$(\mathcal{A}, \sigma)$ a C*-dynamical system, i.e.,

- A is a C*-algebra
- $\sigma : \mathcal{A} \to \mathcal{A}$ non-degenerate $*$-endomorphism.

DEFINITION. An isometric covariant representation (π, V) of the C*-dynamical system $(\mathcal{A}, \sigma)$ consists of a $*$-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and an isometry $V \in \mathcal{B}(\mathcal{H})$ so that

$$\pi(A)V = V\pi(\sigma(A)), \quad \forall A \in \mathcal{A}.$$

Similar definitions for unitary or contractive covariant representations.

DEFINITION Let $(\mathcal{A}, \sigma)$ be a C*-dynamical system. The algebra $\mathcal{A} \times_\sigma \mathbb{Z}^+$ is the universal operator algebra associated with "all" covariant representations of $(\mathcal{A}, \sigma)$, i.e., the universal algebra generated by a copy of $\mathcal{A}$ and an isometry V satisfying the covariant relations.

Note that each covariant representation (π, V) provides a contractive representation $\pi \times V$ of $\mathcal{A} \times_\sigma \mathbb{Z}^+$.

Similar definition for the universal operator algebra $\mathcal{A} \times_\sigma^{un} \mathbb{Z}^+$ (resp. $\mathcal{A} \times_\sigma^{con} \mathbb{Z}^+$) associated with "all" unitary (resp. contractive) covariant representations.

THEOREM 5. (Peters 1985) The representation $\oplus_{x \in X} \pi_x \times S_x$ of $C_0(X) \times_\sigma \mathbb{Z}^+$ is isometric.

THEOREM 6. (Peters) The algebras $\mathcal{A} \times_\sigma^{un} \mathbb{Z}^+$, $\mathcal{A} \times_\sigma \mathbb{Z}^+$ (and $\mathcal{A} \times_\sigma^{un} \mathbb{Z}^+$ in the injective case) are isometrically isomorphic.

Note that the last Theorem is not valid for $C*$-algebraic crossed products.

The formal definition of the tensor algebra $\mathcal{T}_{\mathcal{G}}^{+}$

$\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1, r, s)$, a countable directed graph

DEFINITION. A family of partial isometries $\{L_e\}_{e\in\mathcal{G}^{(1)}}$ and projections $\{L_p\}_{p\in\mathcal{G}^{(0)}}$ is said to obey the Cuntz-Krieger-Toeplitz relations associated with $\mathcal{G}$ if and only if they satisfy

$$(\dagger)\begin{cases} (1) \quad L_pL_p = 0 & \forall\, p, q \in \mathcal{G}^{(0)},\ p \neq q \\ (2) \quad L_e^*L_f = 0 & \forall\, e, f \in \mathcal{G}^{(1)},\ e \neq f \\ (3) \quad L_e^*L_e = L_{s(e)} & \forall\, e \in \mathcal{G}^{(1)} \\ (4) \quad L_eL_e^* \leq L_{r(e)} & \forall\, e \in \mathcal{G}^{(1)} \\ (5) \quad \sum_{r(e)=p} L_eL_e^* \leq L_p & \forall\, p \in \mathcal{G}^{(0)} \end{cases}$$

DEFINITION. The tensor algebra $\mathcal{T}_{\mathcal{G}}^{+}$ of a graph $\mathcal{G}$ is the universal operator algebra for all families of partial isometries $\{L_e\}_{e\in\mathcal{G}^{(1)}}$ and projections $\{L_p\}_{p\in\mathcal{G}^{(0)}}$ which obey the Cuntz-Krieger-Toeplitz relations associated with $\mathcal{G}$.

THEOREM 7 (Fowler, Muhly and Reaburn, 2001). The representation of $\mathcal{T}_{\mathcal{G}}^{+}$ on the Fock space $\mathcal{H}_G$ is isometric.

A is a C*-algebra.

An *inner-product right A-module* is a linear space X which is a right A-module together with a map

$$(\cdot, \cdot) \mapsto \langle \cdot, \cdot \rangle : X \times X \to A$$

such that

$$\begin{aligned} \langle \xi, \lambda y + \eta \rangle &= \lambda \langle x, y \rangle + \langle x, \eta \rangle \\ \langle \xi, \eta a \rangle &= \langle \xi, \eta \rangle a \\ \langle \eta, \xi \rangle &= \langle \xi, \eta \rangle^* \\ \langle \xi, \xi \rangle &\geq 0; \text{ if } \langle \xi, \xi \rangle = 0 \text{ then } \xi = 0. \end{aligned}$$

For $\xi \in X$ we write $\|\xi\|_X := \|\langle \xi, \xi \rangle\|_A$ and one can deduce that $\|\cdot\|_X$ is actually a norm. X equipped with that norm will be called *Hilbert A-module* if it is complete and will be denoted as X_A.

For a Hilbert A-module X we define the set $L(X)$ of the *adjointable maps* that consists of all maps $s : X \to X$ for which there is a map $s^* : X \to X$ such that

$$\langle s\xi, \eta \rangle = \langle \xi, s^*\eta \rangle \,, \qquad\qquad (\xi, \eta \in X).$$

DEFINITION. A C*-correspondence (X, A, φ) consists of a Hilbert A-module (X, A) and a left action

$$\varphi : A \longrightarrow L(X).$$

If φ is injective then the C*-correspondence (X, A, φ) is said to be injective.

A (Toeplitz) representation (π, t) of X into a C*-algebra B, is a pair of a $*$-homomorphism $\pi\colon A \to B$ and a linear map $t\colon X \to B$, such that

1. $\pi(a)t(\xi) = t(\varphi_X(a)(\xi))$,
2. $t(\xi)^*t(\eta) = \pi(\langle \xi, \eta \rangle_X)$,

for $a \in A$ and $\xi, \eta \in X$. An easy application of the C*-identity shows that $t(\xi)\pi(a) = t(\xi a)$ is also valid. A representation (π, t) is said to be *injective* iff π is injective; in that case t is an isometry.

DEFINITION. The *tensor algebra* $\mathcal{T}_X^+$ of a C*-correspondence (X, A, φ) is the norm-closed algebra generated by all elements of the form $\pi_\infty(a), t_\infty(\xi)$, $a \in A$, $\xi \in X$, where (π_∞, t_∞) denotes the universal Toeplitz representation of (X, A, φ).

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Talk 2: C*-correspondences

Elias Katsoulis

A is a C*-algebra.

An *inner-product right A-module* is a linear space X which is a right A-module together with a map

$$(\cdot,\cdot) \mapsto \langle \cdot,\cdot \rangle : X \times X \to A$$

such that

$$\begin{aligned} \langle \xi, \lambda y + \eta \rangle &= \lambda \langle x, y \rangle + \langle x, \eta \rangle \\ \langle \xi, \eta a \rangle &= \langle \xi, \eta \rangle a \\ \langle \eta, \xi \rangle &= \langle \xi, \eta \rangle^* \\ \langle \xi, \xi \rangle &\geq 0; \text{ if } \langle \xi, \xi \rangle = 0 \text{ then } \xi = 0. \end{aligned}$$

For $\xi \in X$ we write $\|\xi\|_X^2 := \|\langle \xi, \xi \rangle\|_A$ and one can deduce that $\|\cdot\|_X$ is actually a norm. X equipped with that norm will be called *Hilbert A-module* if it is complete and will be denoted as X_A.

For a Hilbert A-module X we define the set $L(X)$ of the *adjointable maps* that consists of all maps $s : X \to X$ for which there is a map $s^* : X \to X$ such that

$$\langle s\xi, \eta \rangle = \langle \xi, s^*\eta \rangle \,, \qquad\qquad (\xi, \eta \in X).$$

The compact operators $K(X) \subseteq L(X)$ is the closed subalgebra of $L(X)$ generated by the "rank one" operators

$$\theta_{\xi,\eta}(z) := \xi \, \langle \eta, z \rangle \,, \quad \xi, \eta z \in X$$

DEFINITION. A C*-correspondence (X, A, φ) consists of a Hilbert A-module (X, A) and a left action

$$\varphi : A \longrightarrow L(X).$$

If φ is injective then the C*-correspondence (X, A, φ) is said to be injective.

Representations of C*-correspondences

A (Toeplitz) representation (π, t) of X into a C*-algebra B, is a pair of a $*$-homomorphism $\pi\colon A \to B$ and a linear map $t\colon X \to B$, such that

1. $\pi(a)t(\xi) = t(\varphi_X(a)(\xi))$,

2. $t(\xi)^* t(\eta) = \pi(\langle \xi, \eta \rangle_X)$,

for $a \in A$ and $\xi, \eta \in X$. An easy application of the C*-identity shows that

3. $t(\xi)\pi(a) = t(\xi a)$

is also valid. A representation (π, t) is said to be *injective* iff π is injective; in that case t is an isometry.

DEFINITION. The *Toeplitz-Cuntz-Pimsner* C*-algebra $\mathcal{T}_X$ of a C*-correspondence (X, A, φ) is the C*-algebra generated by all elements of the form $\pi_\infty(a), t_\infty(\xi)$, $a \in A$, $\xi \in \mathcal{X}$, where (π_∞, t_∞) denotes the universal Toeplitz representation of (X, A, φ).

DEFINITION. The *tensor algebra* $\mathcal{T}_X^+$ is the norm-closed subalgebra of $\mathcal{T}_X$ generated by all elements of the form $\pi_\infty(a), t_\infty(\xi)$, $a \in A$, $\xi \in \mathcal{X}$, where (π_∞, t_∞) denotes the universal Toeplitz representation of (X, A, φ).

THEOREM 1. Let (π, t) be a representation of a C*-correspondence (X, A, φ). Then there exists a map

$$\psi_t \colon K(X) \longrightarrow \mathrm{C}^*(\pi, t)$$

so that $\psi_t(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*$, for all $\xi, \eta \in X$.

DEFINITION. A representation (π, t) of a C*-correspondence (X, A, φ) is said to be a <u>covariant representation</u> iff

$$\pi(a) = \psi_t(\varphi(a)), \qquad \text{for all } a \in J_X,$$

where $J_X = \varphi^{-1}(K(X)) \cap (\ker \varphi)^{\perp}$.

DEFINITION. The *Cuntz-Pimsner* C*-algebra $\mathcal{O}_X$ of a C*-correspondence (X, A, φ) is the C*-algebra generated by all elements of the form $\overline{\pi}_\infty(a), \bar{t}_\infty(\xi)$, $a \in A$, $\xi \in X$, where $(\overline{\pi}_\infty, \bar{t}_\infty)$ denotes the universal covariant representation of (X, A, φ).

In order to show that Toeplitz representations do exist, we introduce the interior tensor product of C*-correspondences.

The *interior* or *stabilized tensor product*, denoted by $X \otimes X$ or simply by $X^{\otimes 2}$, is the quotient of the vector space tensor product $X \otimes_{\mathrm{alg}} X$ by the subspace generated by the elements of the form

$$\xi a \otimes \eta - \xi \otimes \varphi(a)\eta, \quad \xi, \eta \in X, a \in A.$$

It becomes a pre-Hilbert B-module when equipped with

$$(\xi \otimes \eta)a := \xi \otimes (\eta a),$$
$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle := \langle y_1, \varphi(\langle \xi_1, \xi_2 \rangle)\eta_2 \rangle$$

For $s \in L(X)$ we define $s \otimes \mathrm{id}_X \in L(X \otimes X)$ as the mapping

$$\xi \otimes y \mapsto s(\xi) \otimes y.$$

Hence $X \otimes Y$ becomes a C*-correspondence by defining $\varphi_{X \otimes X}(a) := \varphi_X(a) \otimes \mathrm{id}_X$.

The *Fock space* $\mathcal{F}_X$ over the correspondence X is the interior direct sum of the $X^{\otimes n}$ with the structure of a direct sum of C*-correspondences over A,

$$\mathcal{F}_X = A \oplus X \oplus X^{\otimes 2} \oplus \dots .$$

Given $\xi \in X$, the (left) creation operator $t_\infty(\xi) \in \mathcal{L}(\mathcal{F}_X)$ is defined as

$$t_\infty(\xi)(a, \zeta_1, \zeta_2, \dots) = (0, \xi a, \xi \otimes \zeta_1, \xi \otimes \zeta_2, \dots).$$

For any $a \in A$, we define

$$\pi_\infty(a) = L_a \oplus \varphi(a) \oplus (\oplus_{n=1}^{\infty} \varphi(a) \otimes \mathrm{id}_n)$$

.

It is easy to verify that (π_∞, t_∞) is a representation of X which is called the *Fock representation* of X.

The gauge-invariance uniqueness Theorems

DEFINITION. A representation (π, t) of X is said to admit a gauge action if for each $z \in \mathbb{T}$ there exists a $*$-homomorphism

$$\beta_z \colon \mathrm{C}^*(\pi, t) \to \mathrm{C}^*(\pi, t)$$

such that $\beta_z(\pi(a)) = \pi(a)$ and $\beta_z(t(\xi)) = zt(\xi)$, for all $a \in A$ and $\xi \in X$.

THEOREM 2. (Katsura 2004). Let (X, A, φ) be a C*-correspondence and (π, t) a covariant representation that admits a gauge action and is faithful on A. Then the integrated representation $\pi \times t$ is faithful on $\mathcal{O}_X$.

THEOREM 3. (Katsura 2004) Let (X, A, φ) be a C*-correspondence and (π, t) a representation that admits a gauge action and satisfies

$$I'(\pi, t) \equiv \{a \in A \mid \pi(a) \in \psi_t(K(X))\} = 0$$

Then the integrated representation $\pi \times t$ is faithful on $\mathcal{T}_X$.

Talk 3: Adding tails to a C*-correspondence

Elias Katsoulis

Loosely speaking, we say that an injective C*-correspondence (Y, B, ψ) arises from (X, A, φ) by adding a tail iff

- $X \subseteq Y$ and $A \subseteq B$, with

$$\psi(a)\xi = \varphi(a)\xi, \quad a \in A, \xi \in X$$

- a covariant representation of (Y, B, ψ) restricts to a covariant representation of (X, A, φ)

- $\mathcal{O}_X$ is a full corner of $\mathcal{O}_Y$

The origins are in the theory of graph C*-algebras

Let $\mathcal{G}$ be a connected, directed graph with a distinguished sink $p_0 \in \mathcal{G}^0$ and no sources. We assume that $\mathcal{G}$ is contractible at p_0. So there exists a unique infinite path $w_0 = e_1 e_2 e_3 \dots$ ending at p_0, i.e. $r(w_0) = p_0$. Let $p_n \equiv s(e_n)$, $n \geq 1$.

Let $(A_p)_{p\in\mathcal{G}^0}$ be a family of C*-algebras parameterized by the vertices of $\mathcal{G}$ so that $A_{p_0} = A$. For each $e \in \mathcal{G}^1$, we now consider a full, right Hilbert $A_{s(e)}$ - module X_e and a $*$-homomorphism

$$\varphi_e\colon A_{r(e)} \longrightarrow \mathcal{L}(X_e)$$

satisfying the following requirements.

- If $e \neq e_1$, φ_e is injective and maps onto $\mathcal{K}(X_e)$.

- $\mathcal{K}(X_{e_1}) \subseteq \varphi_{e_1}(A)$ and

$$J_X \subseteq \ker\varphi_{e_1} \subseteq (\ker\varphi_X)^{\perp}. \tag{1}$$

- The maps φ_X and φ_{e_1} satisfy the *linking condition*

$$\varphi_{e_1}^{-1}(\mathcal{K}(X_{e_1})) \subseteq \varphi_X^{-1}(\mathcal{K}(X)) \tag{2}$$

Let

$$T_0 = c_0(\,(A_p)_{p\in\mathcal{G}^0_-}),$$

where $\mathcal{G}^0_- \equiv \mathcal{G}^0\backslash\{p_0\}$.

Let T_1 be the completion of $c_{00}((X_e)_{e\in\mathcal{G}^1})$ with respect to the inner product

$$\langle u, v\rangle (p) = \sum_{s(e)=p} \langle u_e, v_e\rangle\,, \quad p \in \mathcal{G}^0_-.$$

Equip now T_1 with a right T_0 - action, so that

$$(ux)_e = u_e x_{s(e)}, \quad e \in \mathcal{G}^1, x \in T_0.$$

The pair (T_0, T_1) is the <u>tail</u> for (X, A, φ).

To the C*-correspondence (X, A, φ) and the data

$$\tau \equiv \Big(\mathcal{G}, (X_e)_{e\in\mathcal{G}^1}, (A_p)_{p\in\mathcal{G}^0}, (\varphi_e)_{e\in\mathcal{G}^1}\Big),$$

we now associate

$$\begin{aligned} A_\tau &\equiv A \oplus T_0 \\ X_\tau &\equiv X \oplus T_1 \end{aligned} \tag{3}$$

and we view X_τ as a A_τ-Hilbert module.

We define a left A_τ-action $\varphi_\tau : A_\tau \to \mathcal{L}(X_\tau)$ on X_τ by setting

$$\varphi_\tau(a, x)(\xi, u) = (\varphi_X(a)\xi, v),$$

where

$$v_e = \begin{cases} \varphi_{e_1}(a)(u_{e_1}), & \text{if } e = e_1 \\ \varphi_e(x_{r(e)})u_e, & \text{otherwise} \end{cases}$$

for $a \in A$, $\xi \in X$, $x \in T_0$ and $u \in T_1$.

THEOREM 1. (Kakariadis and Katsoulis, 2012). Let (X, A, φ) be a non-injective C*- correspondence and let X_τ be the graph C*-correspondence over A_τ defined above. Then X_τ is an injective C*-correspondence and the Cuntz-Pimsner algebra $\mathcal{O}_X$ is a full corner of $\mathcal{O}_{X_\tau}$.

Furthermore, if (π, t) is a covariant representation of X_τ , then its restriction on X produces a covariant representation of (X, A, φ) .

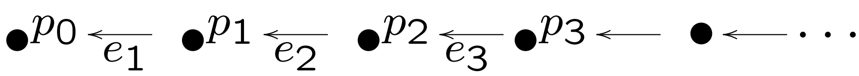
The Muhly-Tomforde tail

Given a (non-injective) correspondence (X, A, φ_X), Muhly and Tomforde construct the tail that results from the previous construction, with respect to data

$$\tau = \left(\mathcal{G}, (X_e)_{e\in\mathcal{G}^{(1)}}, (A_p)_{p\in\mathcal{G}^{(0)}}, (\varphi_e)_{e\in\mathcal{G}^{(1)}}\right)$$

defined as follows.

The graph $\mathcal{G}$ is illustrated in the figure below.



$A_p = X_e = \ker \varphi_X$, for all $p \in \mathcal{G}^{(0)}_{-}$ and $e \in \mathcal{G}^{(1)}$. Finally,

$$\varphi_e(a)u_e = au_e, \quad e \in \mathcal{G}^{(1)}, u_e \in X_e, a \in A_{r(e)}$$

Our tail for (A, A, α)

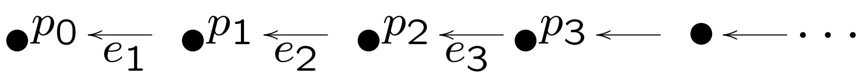
Given a (non-injective) correspondence (X, A, φ_X), we construct the tail that results from the previous construction, with respect to data

$$\tau = \Big(\mathcal{G}, (X_e)_{e\in\mathcal{G}^{(1)}}, (A_p)_{p\in\mathcal{G}^{(0)}}, (\varphi_e)_{e\in\mathcal{G}^{(1)}}\Big)$$

defined as follows.

Let $\theta : A \to M(\ker \varphi_X)$.

The graph $\mathcal{G}$ is once again



but $A_p = X_e = \theta(A)$, for all $p \in \mathcal{G}^{(0)}_-$ and $e \in \mathcal{G}^{(1)}$. Finally,

$$\varphi_e(a)u_e = \theta(a)u_e, \quad e \in \mathcal{G}^{(1)}, u_e \in X_e, a \in A_{r(e)}$$

A first application of adding tails.

THEOREM 2. Let (A, α) a C*-dynamical system and X_α the pertinent correspondence. Then the Cuntz-Pimsner C*-algebra O_{X_α} is strongly Morita equivalent to a crossed product C*-algebra.

A more interesting application for the correspondence coming from a multivariable C*-dynamical system $(A, \alpha_1, \alpha_2, \ldots, \alpha_n)$

Another application.

PROPOSITION 3 (Katsoulis and Kribs, 2006). If (X, A, φ) is an injective correspondence, then

$$\mathrm{alg}(\overline{\pi}_\infty, \overline{t}_\infty)/K(F_X) \simeq \mathrm{alg}(\overline{\pi}_\infty, \overline{t}_\infty)$$

COROLLARY 4. If (X, A, φ) is an injective correspondence, then $\mathcal{T}_X^+$ embedds isometrically and canonically in $\mathcal{O}_X$.

By using tails

THEOREM 5 (Katsoulis and Kribs 2006). If (X, A, φ) is any C*-correspondence, then $\mathcal{T}_X^+$ embedds isometrically and canonically in $\mathcal{O}_X$.

Kakariadis and Katsoulis, Contributions to C^*-correspondences..., Trans. Amer Math Soc. 364 (2012) 6605–6630.

Katsoulis and Kribs, Tensor algebras of C^*-correspondences and their C^*-envelopes, JFA 234 (2006) 226–233.

Muhly and Tomforde, Adding tails to C^*-correspondences, Documenta Mathematica 9 (2004) 79-106

Talk 4: The C*-envelope of an operator algebra

Elias Katsoulis

- all operator spaces $\mathcal{S}$ satisfy $1 \in \mathcal{S} \subseteq \mathrm{C}^*(\mathcal{S})$.

- all completely contractive maps between operator spaces preserve the unit.

THEOREM 1. (Arveson 1969). A (unital) completely contractive map $\varphi : \mathcal{S} \to B(\mathcal{H})$ admits a completely contractive (unital) extension

$$\tilde{\varphi} : \mathrm{C}^*(\mathcal{S}) \longrightarrow B(\mathcal{H}).$$

DEFINITION. A completely contractive (cc) map $\varphi : \mathcal{S} \to B(\mathcal{H})$ is said to have the <u>unique extension property</u> iff any completely contractive extension

$$\tilde{\varphi} : \mathrm{C}^*(\mathcal{S}) \longrightarrow B(\mathcal{H})$$

is multiplicative.

DEFINITION. If $\varphi_i : \mathcal{S} \to B(\mathcal{H}_i)$, $i = 1, 2$, are cc maps then φ_2 is said to be a dilation of φ_1 (denoted as $\varphi_2 \geq \varphi_1$) if $\mathcal{H}_2 \supseteq \mathcal{H}_1$ and

$$c_{\mathcal{H}_1}(\varphi_2(s)) \equiv P_{\mathcal{H}_1}\varphi_2(s)\mid_{\mathcal{H}_1} = \varphi_1(s), \quad \forall s \in \mathcal{S}.$$

DEFINITION. A completely contractive (cc) map $\varphi : \mathcal{S} \to B(\mathcal{H})$ is said to be <u>maximal</u> if it has no non-trivial dilations: $\varphi' \geq \varphi \implies \varphi' = \varphi \oplus \psi$ for some cc map ψ.

THEOREM 2. (Muhly and Solel 1998). A cc map $\varphi : \mathcal{S} \to B(\mathcal{H})$ is maximal iff it has the unique extension property.

THEOREM 3.(Dritschel and McCullough 2005). Every cc map $\varphi : \mathcal{S} \to B(\mathcal{H})$ can be dilated to a maximal cc map $\varphi' : \mathcal{S} \to B(\mathcal{H}')$

PROPOSITION 4. (Arveson 1969). Let $\mathcal{S}, \mathcal{T}$ be operator spaces and

$$\alpha : \mathcal{S} \longrightarrow \mathcal{T}$$

be a completely isometric (unital) map. If

$$\varphi : \mathcal{T} \to B(\mathcal{H})$$

is maximal then

$$\varphi \circ \alpha : \mathcal{S} \to B(\mathcal{H})$$

is also maximal.

COROLLARY 5. If $\mathcal{A}, \mathcal{B}$ are unital operator algebras and

$$\alpha : \mathcal{A} \longrightarrow \mathcal{B}$$

is a complete isometry, then α is multiplicative.

THEOREM 6.(Hamana 1979). Let $\varphi : \mathcal{S} \to B(\mathcal{H})$ be a completely isometric maximal map. If $\mathcal{J} \subseteq \mathrm{C}^*(\mathcal{S})$ is an ideal so that the quotient map

$$q : \mathrm{C}^*(\mathcal{S}) \longrightarrow \mathrm{C}^*(\mathcal{S})/\mathcal{J}$$

is faithful on $\mathcal{S}$, then

$$\mathcal{J} \subseteq \ker \tilde{\varphi}$$

where $\tilde{\varphi}$ is the unique cc extension of φ to $\mathrm{C}^*(\mathcal{S})$. (The ideal $\ker \tilde{\varphi}$ is said to be the <u>Shilov ideal</u> of $\mathcal{S} \subseteq \mathrm{C}^*(\mathcal{S})$.)

THEOREM 7. (Hamana, 1979). Let $\mathcal{S}$ be a unital operator space. Then there exists a C*-algebra $\mathrm{C}^*_{\mathrm{env}}(\mathcal{S})$ (= the C*-envelope of $\mathcal{S}$) and a complete unital isometry

$$\theta\colon \mathcal{S} \longrightarrow \mathrm{C}^*_{\mathrm{env}}(\mathcal{S})$$

so that for any other completely isometric unital embedding

$$\varphi\colon \mathcal{S} \longrightarrow \mathcal{C} = \mathrm{C}^*(\varphi(\mathcal{S}))$$

we have $*$:homomorphism $\pi\colon \mathcal{C} \to \mathrm{C}^*_{\mathrm{env}}(\mathcal{S})$ so that $\pi \circ \varphi = \theta$.

The C*-envelope of an arbitrary operator algebra

If $\mathcal{A} \subseteq B(\mathcal{H})$ is a non-degenerately acting operator algebra with $I_{\mathcal{H}} \notin \mathcal{A}$, then $\mathcal{A}_1$ will denote its unitization.

THEOREM 8. (Meyer, 2001). Let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a completely contractive homomorphism between operator algebras. Then its unitization $\varphi_1\colon \mathcal{A}_1 \to \mathcal{B}_1$ is also completely contractive.

This allows us to consider the category of operator algebras with morphisms the completely contractive homomorphisms.

In that category

COROLLARY 9. Every cc homomorphism $\varphi : \mathcal{A} \to B(\mathcal{H})$ of an operator algebra $\mathcal{A}$ can be dilated to a maximal cc homomorphism. $\varphi' : \mathcal{S} \to B(\mathcal{H}')$

The C^*-envelope of a (non degenerately acting) operator algebra $\mathcal{A}$ is the C^*-algebra generated by $\mathcal{A}$ inside $C^*_{env}(\mathcal{A}_1)$.

THEOREM 10. (Katsoulis and Kribs, 2006). If $(X, A, \varphi)$is a C*-correspondence, then

$$\mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_X^+) \simeq \mathcal{O}_X$$

THEOREM 11. (Davidson and Katsoulis) Let $A, B \in B(\mathcal{H})$ contractions satisfying

$$AB = B\varphi(A)$$

for some finte Blashke product (e.g., $AB = BA^2$). Then there exist unitary operators $V, W \in B(\mathcal{K})$, with $\mathcal{K} \supset \mathcal{H}$, so that

(i) $VW = W\varphi(V)$, and

(ii) $P_{\mathcal{H}} W^m V^n |_{\mathcal{H}} = A^m B^n$, $m, n \in \mathbb{Z}$.

Talk 5: Dynamics and classification of operator algebras

Elias Katsoulis

Gunther Cornelissen, Matilde Marcolli, Quantum Statistical. Mechanics, L-series and Anabelian Geometry, arXiv:1009.0736.

A complex number a is called *algebraic* if there exists a nonzero polynomial $p(X) \in \mathbb{Q}[X]$ such that $p(a) = 0$. The polynomial is unique if we require that it be irreducible and monic. We say that a is an algebraic integer if the unique irreducible, monic polynomial which it satisfies has integer coefficients. We know that the set $\overline{\mathbb{Q}}$ of all algebraic numbers is a field, and the algebraic integers form a ring. For an algebraic number a, the set K of all $f(a)$, with $f(X) \in \mathbb{Q}[X]$ is a field, called an algebraic number field. If all the roots of the polynomial $p(X)$ are in K, then K is called Galois over $\mathbb{Q}$.

QUESTION. Which invariants of a number field characterize it up to isomorphism?

The absolute Galois group of a number field K is the group $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ consisting of all automorphisms σ of $\overline{\mathbb{Q}}$ such that $\sigma(a) = a$ for all $a \in K$. Let $f(X) \in K[X]$ be irreducible, and let Z_f be the set of its roots. The group of permutations of Z_f is a finite group, which is given the discrete topology. Then G_K acts on Z_f. We put a topology on G_K, so that the homomorphism of G_K to the group of permutations of Z_f is continuous for every such $f(X)$. Then G_K is a topological group; it is compact and totally disconnected.

THEOREM. [Uchida, 1976] Number fields E and F are isomorphic as fields if and only if G_E and G_F are isomorphic as topological groups.

The absolute Galois group is not well understood at all (it is considered an anabelian object). What we do understand well are abelian

Galois groups. For a number field K we denote by K^{ab} the maximal abelian extension of K. This is the maximal extension which is Galois (i.e., any irreducible polynomial which has a root in K^{ab} has all its roots in it), and such that the Galois group of K^{ab} over K is abelian. For example, the theorem of Kronecker and Weber says that $\mathbb{Q}^{\text{ab}}$ is the field generated by all the numbers $\exp(\frac{2\pi i}{n})$, i.e., by all roots of unity. Unfortunately,

EXAMPLE. The abelianized Galois groups of $\mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-3})$ are isomorphic.

THEOREM. [Cornelissen and Marcolli, to appear] Let E and F be number fields. Then, E and F are isomorphic if and only if there exists an isomorphism of topological groups

$$\psi\colon G_E^{\mathrm{ab}} \to G_F^{\mathrm{ab}}$$

such that for every character χ of G_F^{ab} we have $L_{F,\chi} = L_{E,\psi\circ\chi}$, where $L_{F,\chi}$ denotes the L-function associated with ψ .

Cornelissen and Marcolli make essential use of the work of Davidson and Katsoulis on multivariable dynamics. At the epicenter of this interaction between number theory and non-selfadjoint operator algebras lies the concept of piecewise conjugacy and the fact that piecewise conjugacy is an invariant for isomorphisms between certain operator algebras associated with multivariable dynamical systems.

(X, σ) a topological dynamical system, i.e.,

- X locally compact Hausdorff space
- $\sigma : X \to X$ proper continuous map.

Similarly

(A, α) a C*-dynamical system, i.e.,

- A is a C*-algebra
- $\sigma : A \to A$ non-degenerate $*$-endomorphism.

Multivariably...

(X, σ) is a multivariable dynamical system:
X locally compact Hausdorff
$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$, where $\sigma_i : X \to X$, $1 \leq i \leq n$, are continuous (proper) maps.

and a similar definition for a multivariable C*-dynamical system (A, α).

We want an operator algebra $\mathcal{A}$ that encodes (X, σ):

$\mathcal{A}$ contains $C_0(X)$ and $S_1, \dots, S_n$ satisfying covariance relations:

$$fS_i = S_i(f \circ \sigma_i)$$

for $1 \leq i \leq n$ and $f \in C_0(X)$

$\mathbb{F}_n^+$ is the free semigroup on n letters.
For $w \in \mathbb{F}_n^+$, say $w = i_k \dots i_1$, write $S_w = S_{i_k} \dots S_{i_1}$.

The covariance algebra is

$$\mathcal{A}_0 = \Big\{ \sum_{\text{finite}} S_w f_w : f_w \in C_0(X) \Big\}.$$

This is an algebra since:

$$(S_v)(fS_w g) = S_{vw}(f \circ \sigma_w) g$$

where $\sigma_w = \sigma_{i_k} \circ \dots \circ \sigma_{i_1}$.

We need a norm condition in order to complete $\mathcal{A}_0$.
Given the choices:

(1) Contractive: $\|S_i\| \leq 1$ for $1 \leq i \leq n$

(2) Row Contractive: $\left\| \begin{bmatrix} S_1 & S_2 & \dots & S_n \end{bmatrix} \right\| \leq 1$.

we get:

Completing $\mathcal{A}_0$ using (1) yields the semicrossed product $C_0(X) \times_\sigma \mathbb{F}_n^+$.

Completing $\mathcal{A}_0$ using (2) yields the tensor algebra $\mathcal{T}_+(X, \sigma)$.

Piecewise conjugate multisystems

Two multivariable dynamical systems (X, σ) and (Y, τ) are said to be *conjugate* if there exists a homeomorphism γ of X onto Y and a permutation $\alpha \in S_n$ so that $\tau_i = \gamma \sigma_{\alpha(i)} \gamma^{-1}$ for $1 \leq i \leq n$.

DEFINITION. We say that two multivariable dynamical systems (X, σ) and (Y, τ) are *piecewise conjugate* if there is a homeomorphism γ of X onto Y and an open cover $\{\mathcal{U}_\alpha : \alpha \in S_n\}$ of X so that for each $\alpha \in S_n$,

$$\gamma^{-1} \tau_i \gamma|_{\mathcal{U}_\alpha} = \sigma_{\alpha(i)}|_{\mathcal{U}_\alpha}.$$

The difference in the two concepts of conjugacy lies on the fact that the permutations depend on the particular open set. As we shall see, a single permutation generally will not suffice.

PROPOSITION. Let (X, σ) and (Y, τ) be piecewise conjugate multivariable dynamical systems. Assume that X is connected and that

$$E := \{x \in X : \sigma_i(x) = \sigma_j(x), \text{for some } i \neq j\}$$

has empty interior. Then (X, σ) and (Y, τ) are conjugate.

For $n = 2$, we can be more definitive.

PROPOSITION. Let X be connected and let $\sigma = (\sigma_1, \sigma_2)$; and let E as above. Then piecewise conjugacy coincides with conjugacy if and only if $\overline{X \backslash E}$ is connected.

The multivariable classification problem.

THEOREM. Let (X,σ) and (Y,τ) be two multivariable dynamical systems. If $\mathcal{T}_+(X,\sigma)$ and $\mathcal{T}_+(Y,\tau)$ or $\mathrm{C}_0(X) \times_\sigma \mathbb{F}_n^+$ and $\mathrm{C}_0(Y) \times_\tau \mathbb{F}_n^+$ are isomorphic as algebras, then the dynamical systems (X,σ) and (Y,τ) are piecewise conjugate.

For the tensor algebras, sufficiency holds in the following cases:

(i) X has covering dimension 0 or 1

(ii) σ consists of no more than 3 maps. ($n \leq 3$.)

For instance:

THEOREM. Suppose that X is a compact subset of $\mathbb{R}$. Then for two multivariable dynamical systems (X, σ) and (Y, τ), the following are equivalent:

1. (X, σ) and (Y, τ) are piecewise topologically conjugate.

2. $\mathcal{T}_+(X, \sigma)$ and $\mathcal{T}_+(Y, \tau)$ are isomorphic.

3. $\mathcal{T}_+(X, \sigma)$ and $\mathcal{T}_+(Y, \tau)$ are completely isometrically isomorphic.

The analysis of the $n = 3$ case is the most demanding and required non-trivial topological information about the Lie group $SU(3)$. The conjectured converse reduces to a question about the unitary group $U(n)$.

CONJECTURE. Let Π_n be the $n!$-simplex with vertices indexed by S_n. Then there should be a continuous function u of Π_n into $U(n)$ so that:

1. each vertex is taken to the corresponding permutation matrix,

2. for every pair of partitions (A, B) of the form

   $$\{1, \ldots, n\} = A_1 \dot{\cup} \ldots \dot{\cup} A_m = B_1 \dot{\cup} \ldots \dot{\cup} B_m,$$

   where $|A_s| = |B_s|$, $1 \leq s \leq m$, let

   $$\mathcal{P}(A, B) = \{\alpha \in S_n : \alpha(A_s) = B_s, 1 \leq s \leq m\}.$$

   If $x = \sum_{\alpha \in \mathcal{P}(A,B)} x_\alpha \alpha$, then the non-zero matrix coefficients of $u_{ij}(x)$ are supported on $\bigcup_{s=1}^m B_s \times A_s$. We call this the *block decomposition condition*.

We have established this conjecture for $n = 2$ and 3 and Chris Ramsey the cases $n = 4, 5$.

With Ken Davidson we considered only classical dynamical systems (dynamical systems over commutative C*-algebras) and our notion of piecewise conjugacy applies exclusively to such systems. Motivated by the interaction between number theory and non-selfadjoint operator algebras, one wonders whether a useful analogue of piecewise conjugacy can be developed for multivariable systems over arbitrary C*-algebras. The goal here is to obtain a natural notion of piecewise conjugacy that generalizes that of Davidson and Katsoulis from the commutative case while remaining an invariant for isomorphisms between non-selfadjoint operator algebras associated with such systems.

DEFINITION. Let A be a unital C*-algebra and let $P(A)$ be its pure state space equipped with the w^*-topology. The *Fell spectrum* $\hat{A}$ of A is the space of unitary equivalence classes of non-zero irreducible representations of A. (The usual unitary equivalence of representations will be denoted as $\sim$.) The GNS construction provides a surjection $P(A) \to \hat{A}$ and $\hat{A}$ is given the quotient topology.

Let A be a unital C*-algebra A and $\alpha = (a_1, \alpha_2, \ldots, \alpha_n)$ be a multivariable system consisting of unital $*$-epimorphisms. Any such system (A, α) induces a multivariable dynamical system $(\hat{A}, \hat{\alpha})$ over its Fell spectrum $\hat{A}$.

DEFINITION. Two multivariable systems (A, α) and $(B, \vec{e})$ are said to be *piecewise conjugate on their Fell spectra* if the induced systems $(\hat{A}, \hat{\alpha})$ and $(\hat{B}, \hat{\vec{e}})$ are piecewise conjugate, in the sense of the definition above.

We have the following result with Kakariadis.

THEOREM. Let (A, α) and $(B, \vec{e})$ be multivariable dynamical systems consisting of $*$-epimorphisms. Assume that either $\mathcal{T}_+(A, \alpha)$ and $\mathcal{T}_+(B, \vec{e})$ or $A \times_\alpha \mathbb{F}_{n_\alpha}^+$ and $B \times_\beta \mathbb{F}_{n_\beta}^+$ are isometrically isomorphic. Then the multivariable systems (A, α) and $(B, \vec{e})$ are piecewise conjugate over their Fell spectra.

PROBLEM. Is there an analogous result for the Jacobson spectrum?

In particular this implies that when the associated operator algebras are isomorphic then both (A, α) and $(B, \vec{e})$ have the same number of $*$-*epimorphisms*. (We call this property invariance of the dimension). In the commutative case, the invariance of the dimension holds for systems consisting of arbitrary endomorphisms. Is it true here?

THEOREM. There exist multivariable systems (A, α_1, α_2) and $(B, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ consisting of $*$-monomorphisms for which $\mathcal{T}_+(A, \alpha_1, \alpha_2)$ and $\mathcal{T}_+(B, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ are isometrically isomorphic.

PROBLEM. [Invariance of dimension for semi-crossed products] Let (A, α) and $(B, \vec{e})$ be multivariable dynamical systems consisting of $*$-endomorphisms. Prove or disprove: if $A \times_\alpha \mathbb{F}_{n_\alpha}^+$ and $B \times_\beta \mathbb{F}_{n_\beta}^+$ are isometrically isomorphic then $n_\alpha = n_{\vec{e}}$.

THEOREM. Let (A, α) and $(B, \vec{e})$ be two automorphic multivariable C*-dynamical systems and assume that A is primitive. Then the following are equivalent:

1. $A \times_\alpha \mathbb{F}^+_{n_\alpha}$ and $B \times_{\vec{e}} \mathbb{F}^+_{n_{\vec{e}}}$ are isometrically isomorphic.

2. $\mathcal{T}^+(A, \alpha)$ and $\mathcal{T}^+(B, \vec{e})$ are isometrically isomorphic.

3. (A, α) and $(B, \vec{e})$ are outer conjugate.

DEFINITION. We say that two multivariable C*-dynamical systems (A, α) and $(B, \vec{e})$ are *outer conjugate* if they have the same dimension and there are $*$-isomorphism $\gamma : A \to B$, unitary operators $U_i \in B$ and $\pi \in S_n$ so that

$$\gamma^{-1}\alpha_i\gamma(b) = U_i^*\vec{e}_{\pi(i)}(b)U_i.$$

for all $b \in B$ and i.

Assume now that (A, α) and $(B, \vec{e})$ are two multivariable dynamical systems such that $\mathcal{T}^+(A, \alpha)$ and $\mathcal{T}^+(B, \vec{e})$ (or $A \times_\alpha \mathbb{F}^+_{n_\alpha}$ and $B \times_{\vec{e}} \mathbb{F}^+_{n_{\vec{e}}}$) are isometrically isomorphic via a mapping α. Since α is isometric, it follows that $\alpha|_A$ is a $*$-monomorphism that maps A onto B (This is the only point where we use that α is isometric.) We will be denoting $\alpha|_A$ by α as well.

Let S_i, $i = 1, \ldots, n_\alpha$, (resp. T_i, $i = 1, 2, \ldots, n_{\vec{e}}$) be the generators in $\mathcal{T}^+(A, \alpha)$ (resp. $\mathcal{T}^+(B, \vec{e})$) and let b_{ij} be the T_i-Fourier coefficient of $\alpha(s_j)$, i.e.,

$$\alpha(S_j) = b_{0j} + T_1 b_{1j} + T_2 b_{2j} + \cdots + T_n b_{nj} + Y,$$

where Y involves Fourier terms of order 2 or higher.

Since α is a homomorphism,

$$\alpha(a)\alpha(S_j) = \alpha(aS_j) = \alpha(S_j \alpha_j(a)) = \alpha(S_j)\alpha\alpha_j(a),$$

for all $a \in A$. Hence, $\vec{e}_i \alpha(a) b_{ij} = b_{ij} \alpha\alpha_j(a)$, $a \in A$, and so

$$\vec{e}_i(b) b_{ij} = b_{ij} \alpha\alpha_j \alpha^{-1}(b) = b_{ij} \tilde{\alpha}_j(b),$$

for all $b \in B$.

From the intertwining equation

$$\vec{e}_i(b)b_{ij} = b_{ij}\tilde{\alpha}_j(b), b \in B \qquad (*)$$

we obtain.

- Since A is primitive, $b_{i,j}$ is either zero or invertible!

- If $b_{ij} \neq 0$ then $\vec{e}_i \sim \tilde{\alpha}_j$.

Therefore each equivalence class $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is equivalent to exactly one class $\{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_m\}$.

Need to show that $m = n$. Bwoc let $m < n$.

Start with an "arbitrary" n-tuple $(y_1, y_2, \ldots, y_n)$.

From the equation

$$T_1 y_1 + T_2 y_2 + \cdots + T_n y_n = \lim_e \alpha(x_e),$$

where x_e are non-commutative polynomials in $S_1, S_2, \ldots, S_m$ and remembering that

$$\alpha(S_j) = b_{0j} + T_1 b_{1j} + T_2 b_{2j} + \cdots + T_n b_{nj} + Y,$$

we obtain

$$
\begin{aligned}
y_1 &= \lim_e b_{11} x_e^1 + b_{12} x_e^2 + \cdots + b_{1m} x_e^m, \\
y_2 &= \lim_e b_{21} x_e^1 + b_{22} x_e^2 + \cdots + b_{2m} x_e^m, \\
&\vdots \\
y_n &= \lim_e b_{n1} x_e^1 + b_{n2} x_e^2 + \cdots + b_{nm} x_e^m.
\end{aligned}
$$

Perform Gaussian elimination to reduce this system to

$$
\begin{aligned}
\bar{y}_2 &= \lim_e \bar{b}_{22} x_e^2 + \bar{b}_{23} x_e^3 + \cdots + \bar{b}_{2m} x_e^m, \\
\bar{y}_3 &= \lim_e \bar{b}_{32} x_e^2 + \bar{b}_{33} x_e^3 + \cdots + \bar{b}_{3m} x_e^m, \\
&\vdots \\
\bar{y}_n &= \lim_e \bar{b}_{n2} x_e^2 + \bar{b}_{n3} x_e^3 + \cdots + \bar{b}_{nm} x_e^m,
\end{aligned}
$$

We continue this short of "Gaussian elimination" and we arrive at a system that contains one column and at least two non-trivial rows of the form

$$w_1 = \lim_e d_1 x_e^m$$
$$w_2 = \lim_e d_2 x_e^m,$$

where the data (w_1, w_2) is arbitrary. Therefore d_1, d_2 are non-zero, hence invertible. By letting $w_1 = 1$ we obtain that $\lim_e x_e^m = d_1^{-1}$. Therefore, if we let $w_2 = 0$, then we get that $0 = d_2 d_1^{-1}$, which is a contradiction.

THEOREM. Let (A, α) and $(B, \vec{e})$ be multivariable dynamical systems consisting of $*$-epimorphisms The tensor algebras $\mathcal{T}_+(A, \alpha)$ and $\mathcal{T}_+(B, \vec{e})$ are isometrically isomorphic if and only if the correspondences $((A, \alpha)$ and $(B, \vec{e})$ are unitarily equivalent.

In light of the above result we ask

PROBLEM. Let (A, α) and $(B, \vec{e})$ be multivariable dynamical systems consisting of $*$-monomorphis If the tensor algebras $\mathcal{T}_+(A, \alpha)$ and $\mathcal{T}_+(B, \vec{e})$ are isometrically isomorphic does it follow that the correspondences $((A, \alpha)$ and $(B, \vec{e})$ are unitarily equivalent.

Talk 6: Local maps and representation theory

Elias Katsoulis