Tensor algebras and subproduct systems arising from stochastic matrices

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OAOT 2014 at ISI, Bangalore
We will discuss some of the results of the following paper:


**General Problem**

We can encode several objects into operator algebras, especially using subproduct systems of W*-correspondences.

How much information can we recover from the algebras?
**Definition**

Let $M$ be a von Neumann algebra. A right $M$-module $E$ is called a **Hilbert $W^*$-module** if it is endowed with a map $\langle \cdot, \cdot \rangle : E \times E \to M$ such that for all $\xi, \eta, \eta' \in E$ and $m \in M$,

- it is $M$-linear in the second variable
- $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$
- $\langle \xi, \xi \rangle \geq 0$ and $\langle \xi, \xi \rangle = 0 \iff \xi = 0$
- $E$ is complete with respect to the norm $\|\xi\|_E = \|\langle \xi, \xi \rangle^{1/2}\|_M$
- it is self-dual, i.e. for every bounded $M$-linear functional $f : E \to M$ there exists $\eta_f \in E$ such that $f(\xi) = \langle \eta_f, \xi \rangle$

The set $\mathcal{L}(E)$ of adjointable $M$-linear operators on $E$ is also a $W^*$-algebra. We say that $E$ is a **$W^*$-correspondence** when in addition $E$ is has a left multiplication by $M$ given by a normal homomorphism $M \to \mathcal{L}(E)$. 

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**Correspondences and subproduct systems**

**Basic framework**
Examples

- **Hilbert spaces** are $W^*$-correspondences over $M = \mathbb{C}$

- **Finite graph correspondences**: Given a graph $G = (G_0, G_1)$ with $d$ vertices, we define $M \subseteq M_d(\mathbb{C})$ to be the set of diagonal matrices, and $E_G = \{ A \in M_d(\mathbb{C}) | A_{ij} = 0 \text{ if } (i, j) \notin G_1 \}$. The left & right actions are given by usual multiplication, and inner product is $\langle A, B \rangle = \text{Diag}(A^*B)$.

- Let $M \subseteq B(H)$ be a vN algebra and let $\theta : M \to M$ be a unital normal completely positive map. Let $\pi : M \to B(M \otimes_\theta H)$ be the minimal Stinespring dilation of $\theta$. The **Arveson-Stinespring $W^*$-correspondence of $\theta$** is the correspondence over $M'$ given by

$$\text{Arv}(\theta) = \{ T \in B(H, M \otimes_\theta H) | \pi(x)T = Tx, \forall x \}$$

with operations as follows: for every $T, S \in \text{Arv}(\theta), a \in M'$,

$$T \cdot a = T \circ a, \quad a \cdot T = (I \otimes a) \circ T, \quad \langle T, S \rangle = T^*S.$$
Definition (Shalit-Solel ’09, Bhat-Mukherjee ’10)

Let $M$ be a vN algebra, let $X = (X_n)_{n \in \mathbb{N}}$ be a family of $W^*$-correspondences over $M$, and let $U = (U_{m,n} : X_m \otimes X_n \to X_{m+n})$ be a family of bounded $M$-linear maps. We say that $X$ is a subproduct system over $M$ if for all $m, n, p \in \mathbb{N}$,

1. $X_0 = M$
2. $U_{m,n}$ is co-isometric
3. The family $U$ “behaves like multiplication”: $U_{m,0}$ and $U_{0,n}$ are the right/left multiplications and

$$U_{m+n,p}(U_{m,n} \otimes I_p) = U_{m,n+p}(I_m \otimes U_{n,p})$$

When $U_{m,n}$ is unitary for all $m, n$ we say that $X$ is a product system.

– Bhat-Mukherjee ’10: case $M = \mathbb{C}$, under the name inclusion systems.
– Product systems of Hilbert spaces were first defined by Arveson, when studying semigroups of endomorphisms of $B(H)$. 
Examples

- (Product systems $\mathcal{P}^E$) Given a W*-correspondence $E$ over $M$, define $\mathcal{P}^E_0 = M$, $\mathcal{P}^E_n = E^\otimes n$ and let $U_{m,m} : E^\otimes m \otimes E^\otimes n \to E^\otimes m+n$ be the canonical unitary embodying associativity.

- (Standard Finite-dimensional Hilbert space fibers) Suppose that $X = (X_n)_{n \in \mathbb{N}}$ is a family of fin. dim. Hilbert spaces such that $X_{m+n} \subseteq X_m \otimes X_n$ (standard)

Let $U_{m,n} : X_m \otimes X_n \to X_{m+n}$ be the projection. Then $X$ is a subproduct system.

Theorem (Muhly-Solel '02, Solel-Shalit '09)

Let $M$ be a vN algebra. Suppose that $\theta : M \to M$ is a unital normal CP map, and let $X_n = \text{Arv}(\theta^n)$. Then there is a canonical family of multiplication maps $U = (U_{m,n})$ for which $X$ is a subproduct system.
Given a subproduct system \((X, U)\), we define the Fock W*-correspondence

\[ \mathcal{F}_X = \bigoplus_{n=0}^{\infty} X_n \]

Define for every \(\xi \in X_m\) the shift operator \(S^{(m)}_{\xi} \in \mathcal{L}(\mathcal{F}_X)\)

\[ S^{(m)}_{\xi} \psi = U_{m,n}(\xi \otimes \psi), \quad \psi \in X_n \]

We shall consider several natural operator algebras associated to \((X, U)\).

- **Tensor algebra**: \(\mathcal{T}_+(X) = \overline{\text{Alg}}_\| \cdot \| \{ S^{(m)}_{\xi} | \forall \xi \in X_m, \forall m \} \) (not self-adjoint)
- **Toeplitz algebra**: \(\mathcal{T}(X) = C^* (\mathcal{T}_+(X))\)
- **Cuntz-Pimsner algebra**: \(\mathcal{O}(X) = \mathcal{T}(X)/\mathcal{J}(X)\)

Viselter '12 suggested the following ideal for subproduct systems: let \(Q_n\) denote the orthogonal projection onto the \(n^{th}\) summand of Fock module:

\[ \mathcal{J}(X) = \{ T \in \mathcal{T}(X) : \lim_{n \to \infty} \| TQ_n \| = 0 \} \].
Example (Product system $\mathcal{P}^C$)

Let $E = M = C$, and let $X = \mathcal{P}^C$ be the associated product system.

- We have $\mathcal{F}_X = \bigoplus_{n \in \mathbb{N}} C \cong \ell^2(\mathbb{N})$ and $\mathcal{T}_+(\mathcal{P}^C)$ is closed algebra generated by the unilateral shift. Hence,
  - $\mathcal{T}_+(\mathcal{P}^C) = \mathbb{A}(\mathbb{D})$,
  - $\mathcal{T}(\mathcal{P}^C)$ is the original Toeplitz algebra,
  - $\mathcal{O}(\mathcal{P}^C) = C(\mathbb{T})$.

Theorem (Viselter ’12)

If $E$ is a correspondence and its associated product system $X_E$ is faithful, then $\mathcal{O}(\mathcal{P}^E) = \mathcal{O}(E)$.

So the algebras for subproduct systems generalize the case of single correspondences. (via the associated product system).
Q: How much does the tensor algebra remember of the original structure?

**Theorem**

Let $G$ and $G'$ be countable directed graphs.

- **Solel '04**: $T_+\left(\mathcal{P}^E_G\right)$ and $T_+\left(\mathcal{P}^E_{G'}\right)$ are isometrically isomorphic if and only if $G$ and $G'$ are isomorphic as directed graphs.

- **Kribs-Katsoulis '04**: $T_+\left(\mathcal{P}^E_G\right)$ and $T_+\left(\mathcal{P}^E_{G'}\right)$ are boundedly isomorphic if and only if $G$ and $G'$ are isomorphic as directed graphs. Furthermore, if $G, G'$ have no sinks or sources, algebraic isomorphisms are bounded.

**Theorem (Davidson-Ramsey-Shalit '11)**

Let $X$ and $Y$ be standard subproduct systems with fin. dim. Hilbert space fibers. Then $T_+(X)$ is isometrically isomorphic to $T_+(Y)$ if and only if $X$ and $Y$ are unitarily isomorphic.

Similar results for multivariable dyn. systems (Davidson-Katsoulis '11), C*-dynamical systems (Davidson-Kakariadis '12), and many more.
Stochastic matrices

Definition

Given a countable (possibly infinite) set $\Omega$, a stochastic matrix over $\Omega$ is a function $P : \Omega \times \Omega \rightarrow \mathbb{R}$ such that

- $P_{ij} \geq 0$ for all $i, j$
- $\sum_{j \in \Omega} P_{ij} = 1$

$\text{Arv}(P)$ and $\mathcal{T}_+(P)$ for $P$ stochastic

There is a 1-1 correspondence between unital normal CP maps of $\ell^\infty(\Omega)$ and stochastic matrices over $\Omega$ given by

$$\theta_P(f)(i) = \sum_{j \in \Omega} P_{ij} f(j)$$

Therefore, a stochastic matrix $P$ gives rise to

- A subproduct system $\text{Arv}(P) := \text{Arv}(\theta_P)$
- A tensor algebra $\mathcal{T}_+(P) := \mathcal{T}_+(\text{Arv}(P))$
Theorem (Dor-On-Markiewicz ’14)

Let $P$ be a stochastic matrix over a state space $\Omega$. Then up to isomorphism of subproduct systems we have

$$Arv(P)_n = \{[a_{ij}] : \forall (i, j), a_{ij} = 0 \text{ if } (P^n)_{ij} = 0, \sum_{j \in \Omega} |a_{ij}|^2 < \infty\}$$

where $\ell^\infty(\Omega)$ acts as multiplication by diagonals on the left and on the right, and inner product is $\langle A, B \rangle = \text{Diag}(A^*B)$ and subproduct maps are given by

$$[U_{m,n}(A \otimes B)]_{ij} = \sum_{k \in \Omega} \sqrt{\frac{(P^m)_{ik}(P^n)_{kj}}{(P^{m+n})_{ij}}} a_{ik} b_{kj}$$
Question

Suppose that $P, Q$ are stochastic matrices over $\Omega$, and $T_+(P) \simeq T_+(Q)$. What can we say about the associated relation between $P$ and $Q$? What is the suitable version of equivalence $\simeq$?

We have several natural isomorphism relations for tensor algebras of stochastic matrices:

- Algebraic isomorphism
- Bounded isomorphism
- Isometric isomorphism
- Completely isometric isomorphism
- Completely bounded isomorphism

However, the situation turns out to be much simpler for stochastic matrices.
**Theorem (Dor-On-M. ’14 – Automatic Continuity)**

Let $P$ and $Q$ be stochastic matrices over $\Omega$. If $\psi : T_+ (P) \to T_+ (Q)$ is algebraic isomorphism, then it is bounded.

**Remark:** Tensor algebras are not semi-simple in general (see Davidson-Katsoulis ’11), so not a consequence of general machinery. The proof uses an automatic continuity lemma due to Sinclair, which has become a stepping stone for many similar results in a variety of contexts.

**Theorem (Dor-On-M. ’14)**

Let $P$ and $Q$ be stochastic matrices over $\Omega$. TFAE:

1. There is an isometric isomorphism of $T_+ (P)$ onto $T_+ (Q)$.
2. There is a graded comp. isometric isomorphism $T_+ (P)$ onto $T_+ (Q)$.
3. $\text{Arv}(P)$ and $\text{Arv}(Q)$ are unitarily isomorphic up to change of base.

Furthermore, if $P$ and $Q$ are recurrent (i.e. $\sum_n (P^n)_{ii} = \infty$ for all $i$), those conditions hold if and only if $P$ and $Q$ are the same up to permutation of $\Omega$. 
Recall that a stochastic matrix $P$ is **essential** if for every $i$, $P^m_{ij} > 0$ for some $n$ implies that $\exists m$ such that $P^m_{ji} > 0$.

We also say that the **support of $P$** is the matrix $\text{supp}(P)$ given by

$$\text{supp}(P)_{ij} = \begin{cases} 1, & P_{ij} \neq 0 \\ 0, & P_{ij} = 0 \end{cases}$$

**Theorem (Dor-On-M. ’14)**

Let $P$ and $Q$ be finite stochastic matrices over $\Omega$. TFAE:

1. There is an algebraic isomorphism of $T_+(P)$ onto $T_+(Q)$.
2. There is a graded **comp. bounded** isomorphism $T_+(P)$ onto $T_+(Q)$.
3. Arv($P$) and Arv($Q$) are similar up to change of base

Furthermore, if $P$ and $Q$ are essential, those conditions hold if and only if $P$ and $Q$ have the same supports up to permutation of $\Omega$. 
So when $P$ and $Q$ are finite, there are only two types of isomorphism problems:

- isometric iso. classes = graded completely isometric iso. classes
  = completely isometric iso. classes
- algebraic iso. classes = bounded iso. classes
  = completely bounded iso. classes

**Example**

For every $r \in (0, \frac{1}{2}]$, let

$$P_r = \begin{bmatrix} r & 1 - r \\ r & 1 - r \end{bmatrix}$$

(it is an essential and recurrent matrix since $P_r^2 = P_r$).

Then $\mathcal{T}_+(P_r)$ and $\mathcal{T}_+(P_s)$ are:

- algebraically isomorphic for every $r, s$.
- only isometrically isomorphic for $r = s$. 
A word about the proof.

- In Davidson-Ramsey-Shalit ’12, they show that in the orbit of the action of canonical “Bogolyubov” transformations of the tensor algebra on the space of isomorphisms there are always graded isomorphisms.

- This does not work so easily in our case.

- We first notice that for so called reducing projections $p_j$ (onto the state $j \in \Omega$), the cut down $\mathcal{T}_+(p_j \operatorname{Arv}(P)p_j)$ is like a disk algebra. For such regular $j$, the Bogolyubov trick works with minor generalization.

- For singular $j$ there may be complicated interrelations. To get a large enough group of Bogolyubov transformations, we need to define an equivalence relation $R$ on $\Omega$ which allows the action of a torus $\mathbb{T}^\Omega/\sim$ as Bogolyubov transformations $\alpha_\Lambda$ for $\Lambda$ in the torus.

- Given $\varphi : \mathcal{T}_+(P) \to \mathcal{T}_+(Q)$ an isomorphism, we show that in the orbit of $(\Lambda, \Theta) \mapsto \varphi \circ \alpha_\Lambda \circ \varphi \circ \alpha_\Theta \circ \varphi$ there is a graded completely isometric/bounded iso as required.
Definition (Arveson ’69)
Let $\mathcal{B} \subseteq B(H)$ be a unital closed subalgebra and let $\mathcal{A} = C^*(\mathcal{B})$. We will say that a two-sided ideal $\mathcal{I} \subseteq \mathcal{A}$ is a boundary ideal for $\mathcal{B}$ if the quotient map $q : \mathcal{A} \rightarrow \mathcal{A}/\mathcal{I}$ is completely isometric on $\mathcal{B}$.

Theorem (Hamana)
Let $\mathcal{B} \subseteq B(H)$ be a unital closed subalgebra and let $\mathcal{A} = C^*(\mathcal{B})$. Then there exists a largest boundary ideal $S_B \subseteq \mathcal{A}$ for $\mathcal{B}$, called the Shilov ideal of $\mathcal{A}$ for $\mathcal{B}$.

Definition
Let $\mathcal{B} \subseteq B(H)$ be a unital closed subalgebra and let $\mathcal{A} = C^*(\mathcal{B})$. The C*-envelope of $\mathcal{B}$ is the C*-algebra $C^*_{\text{env}}(\mathcal{B}) = \mathcal{A}/S_B$. It is the unique smallest C*-algebra (up to isomorphism) generated by a completely isometric copy of $\mathcal{B}$.
Some Examples:

Theorem (Katsoulis and Kribs ’06)

If $E$ is a $C^*$-correspondence, then $C^*_{env}(\mathcal{T}_+(E)) = \mathcal{O}(E)$.

Theorem (Davidson, Ramsey and Shalit ’11)

If $X$ is a commutative subproduct system of fin. dim. Hilbert space fibers, then $C^*_{env}(\mathcal{T}_+(X)) = \mathcal{T}(X)$.

Theorem (Kakariadis and Shalit – personal communication)

If $X$ is a subproduct system of fin. dim. Hilbert space fibers arising from a subshift of finite type, then $C^*_{env}(\mathcal{T}_+(X))$ is either $\mathcal{T}(X)$ or $\mathcal{O}(X)$.

So far, this dichotomy gives some reassurance of the soundness of Viselter’s definition of Cuntz-Pimsner algebra for subproduct systems.
Let $k \in \Omega$ and let $P$ be an irreducible stochastic matrix with period

$$r = \gcd\{n : (P^n)_{11} > 0\}.$$

We say that $P$ has a stationary $k^{th}$ column if $P(e_k) = P^{r+1}(e_k)$. We denote by $\Omega_0$ the set of states corresponding to stationary columns.

**Theorem (Dor-On-M.)**

Let $P$ be an irreducible stochastic finite matrix.

- If no columns of $P$ are stationary, then $C^*_{env}(T_+(P)) = T(P)$.
- If all columns of $P$ are stationary, then $C^*_{env}(T_+(P)) = O(P)$.

**Example (Dor-On-M. – Dichotomy fails)**

$$P = \frac{1}{6} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

is a 1-periodic irreducible stochastic matrix for which the three algebras $C^*_{env}(T_+(P))$, $T(P)$ and $O(P)$ are all different.
Theorem (Dor-On-M.)

Let $P$ be a finite irreducible stochastic matrix. Let $X = Arv(P)$ and for each $k \in \Omega$, let $F_k = F_X \cdot p_k$. Then the Shilov ideal of $\mathcal{T}_+(P)$ is given by

$$\bigcap_{k \in \Omega_0} \{ T \in \mathcal{T}_+(P) : T \upharpoonright F_k \text{ is compact} \} \cap \bigcap_{k \notin \Omega_0} \{ T \in \mathcal{T}_+(P) : T \upharpoonright F_k = 0 \}$$

The idea of the proof is working with a faithful representation whose irreducible decomposition is computable, and each piece is a boundary representation. From there we obtain a faithful representation with the unique extension property, hence we get the kernel is the Shilov ideal (and the range is the C*-envelope).
Thank you!