Tensor algebras and subproduct systems arising from stochastic matrices

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We will discuss some of the results of the following paper:

Dor-On-M. '14 Adam Dor-On and Daniel Markiewicz,
"Operator algebras and subproduct systems arising from stochastic matrices",
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General Problem

We can encode several objects into operator algebras, especially using subproduct systems of W*-correspondences.

How much information can we recover from the algebras?

W*-modules and W*-correspondences

Definition

Let M be a von Neumann algebra. A right M-module E is called a Hilbert W*-module if it is endowed with a map $\langle \cdot, \cdot \rangle : E \times E \to M$ such that for all $\xi, \eta, \eta' \in E$ and $m \in M$,

- it is M-linear in the second variable
- $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$
- $\langle \xi, \xi \rangle \geq 0$ and $\langle \xi, \xi \rangle = 0 \iff \xi = 0$
- E is complete with respect to the norm $\|\xi\|_E = \| \langle \xi, \xi \rangle^{1/2} \|_M$
- it is self-dual, i.e. for every bounded M-linear functional $f : E \to M$ there exists $\eta_f \in E$ such that $f(\xi) = \langle \eta_f, \xi \rangle$

The set $\mathcal{L}(E)$ of adjointable M-linear operators on E is a also W*-algebra. We say that E is a W*-correspondence when in addition E is has a left multiplication by M given by a normal homomorphism $M \to \mathcal{L}(E)$.

Examples

- Hilbert spaces are W*-correspondences over $M = \mathbb{C}$
- Finite graph correspondences: Given a graph $G = (G_0, G_1)$ with d vertices, we define $M \subseteq M_d(\mathbb{C})$ to be the set of diagonal matrices, and $E_G = \{A \in M_d(\mathbb{C}) \mid A_{ij} = 0 \text{ if } (i,j) \notin G_1\}$. The left & right actions are given by usual multiplication, and inner product is $\langle A, B \rangle = \mathrm{Diag}(A^* B)$.
- Let $M \subseteq B(H)$ be a vN algebra and let $\theta : M \to M$ be a unital normal completely positive map. Let $\pi : M \to B(M \otimes_\theta H)$ be the minimal Stinespring dilation of θ. The Arveson-Stinespring W*-correspondence of θ is the correspondence over M' given by

$$\mathrm{Arv}(\theta) = \{T \in B(H, M \otimes_\theta H) \mid \pi(x)T = Tx, \quad \forall x\}$$

with operations as follows: for every $T, S \in \mathrm{Arv}(\theta)$, $a \in M'$,

$$T \cdot a = T \circ a, \qquad a \cdot T = (I \otimes a) \circ T, \qquad \langle T, S \rangle = T^* S.$$

Definition (Shalit-Solel '09, Bhat-Mukherjee '10)

Let M be a vN algebra, let $X = (X_n)_{n\in\mathbb{N}}$ be a family of W*-correspondences over M, and let $U = (U_{m,n} : X_m \otimes X_n \to X_{m+n})$ be a family of bounded M-linear maps. We say that X is a subproduct system over M if for all $m, n, p \in \mathbb{N}$,

1. $X_0 = M$
2. $U_{m,n}$ is co-isometric
3. The family U "behaves like multiplication": $U_{m,0}$ and $U_{0,n}$ are the right/left multiplications and

$$U_{m+n,p}(U_{m,n} \otimes I_p) = U_{m,n+p}(I_m \otimes U_{n,p})$$

When $U_{m,n}$ is unitary for all m, n we say that X is a product system.

– Bhat-Mukherjee '10: case $M = \mathbb{C}$, under the name inclusion systems.
– Product systems of Hilbert spaces were first defined by Arveson, when studying semigroups of endomorphisms of $B(H)$.

Examples

- (Product systems $\mathscr{P}^E$) Given a W*-correspondence E over M, define $\mathscr{P}_0^E = M$, $\mathscr{P}_n^E = E^{\otimes^n}$ and let $U_{m,m} : E^{\otimes^m} \otimes E^{\otimes^n} \to E^{\otimes^{m+n}}$ be the canonical unitary embodying associativity.
- (Standard Finite-dimensional Hilbert space fibers) Suppose that $X = (X_n)_{n \in \mathbb{N}}$ is a family of fin. dim. Hilbert spaces such that

$$X_{m+n} \subseteq X_m \otimes X_n \qquad \text{(standard)}$$

  Let $U_{m,n} : X_m \otimes X_n \to X_{m+n}$ be the projection. Then X is a subproduct system.

Theorem (Muhly-Solel '02, Solel-Shalit '09)

Let M be a vN algebra. Suppose that $\theta : M \to M$ is a unital normal CP map, and let $X_n = \mathrm{Arv}(\theta^n)$. Then there is a canonical family of multiplication maps $U = (U_{m,n})$ for which X is a subproduct system.

Given a subproduct system (X, U), we define the Fock W*-correspondence

$$\mathcal{F}_X = \bigoplus_{n=0}^{\infty} X_n$$

Define for every $\xi \in X_m$ the shift operator $S_\xi^{(m)} \in \mathcal{L}(\mathcal{F}_X)$

$$S_\xi^{(m)}\psi = U_{m,n}(\xi \otimes \psi), \qquad \psi \in X_n$$

We shall consider several natural operator algebras associated to (X, U).

- Tensor algebra: $\mathcal{T}_+(X) = \overline{\mathrm{Alg}}^{\|\cdot\|}\{S_\xi^{(m)} \mid \forall \xi \in X_m, \forall m\}$ (not self-adjoint)
- Toeplitz algebra: $\mathcal{T}(X) = C^*(\mathcal{T}_+(X))$
- Cuntz-Pimsner algebra: $\mathcal{O}(X) = \mathcal{T}(X)/\mathcal{J}(X)$

Viselter '12 suggested the following ideal for subproduct systems: let Q_n denote the orthogonal projection onto the nth summand of Fock module:

$$\mathcal{J}(X) = \{T \in \mathcal{T}(X) : \lim_{n\to\infty} \|TQ_n\| = 0\}.$$

Example (Product system $\mathscr{P}^{\mathbb{C}}$)

Let $E = M = \mathbb{C}$, and let $X = \mathscr{P}^{\mathbb{C}}$ be the associated product system.

- We have $\mathcal{F}_X = \oplus_{n\in\mathbb{N}}\mathbb{C} \simeq \ell^2(\mathbb{N})$ and $\mathcal{T}_+(\mathscr{P}^{\mathbb{C}})$ is closed algebra generated by the unilateral shift. Hence,
- $\mathcal{T}_+(\mathscr{P}^{\mathbb{C}}) = \mathbb{A}(\mathbb{D})$,
- $\mathcal{T}(\mathscr{P}^{\mathbb{C}})$ is the original Toeplitz algebra,
- $\mathcal{O}(\mathscr{P}^{\mathbb{C}}) = C(\mathbb{T})$.

Theorem (Viselter '12)

If E is a correspondence and its associated product system X_E is faithful, then $\mathcal{O}(\mathscr{P}^E) = \mathcal{O}(E)$.

So the algebras for subproduct systems generalize the case of single correspondences. (via the associated product system).

Q: How much does the tensor algebra remember of the original structure?

Theorem

Let G and G' be countable directed graphs.

- *Solel '04: $\mathcal{T}_+(\mathscr{P}^{E_G})$ and $\mathcal{T}_+(\mathscr{P}^{E_{G'}})$ are isometrically isomorphic if and only if G and G' are isomorphic as directed graphs.*
- *Kribs-Katsoulis '04: $\mathcal{T}_+(\mathscr{P}^{E_G})$ and $\mathcal{T}_+(\mathscr{P}^{E_{G'}})$ are boundedly isomorphic if and only if G and G' are isomorphic as directed graphs. Furthermore, if G, G' have no sinks or sources, algebraic isomorphisms are bounded.*

Theorem (Davidson-Ramsey-Shalit '11)

Let X and Y be standard subproduct systems with fin. dim. Hilbert space fibers. Then $\mathcal{T}_+(X)$ is isometrically isomorphic to $\mathcal{T}_+(Y)$ if and only if X and Y are unitarily isomorphic.

Similar results for multivariable dyn. systems (Davidson-Katsoulis '11), C*-dynamical systems (Davidson-Kakariadis '12), and many more.

Definition

Given a countable (possibly infinite) set Ω, a stochastic matrix over Ω is a function $P : \Omega \times \Omega \to \mathbb{R}$ such that

- $P_{ij} \geq 0$ for all i, j
- $\sum_{j \in \Omega} P_{ij} = 1$

$\mathrm{Arv}(P)$ and $\mathcal{T}_+(P)$ for P stochastic

There is a 1-1 correspondence between unital normal CP maps of $\ell^\infty(\Omega)$ and stochastic matrices over Ω given by

$$\theta_P(f)(i) = \sum_{j \in \Omega} P_{ij} f(j)$$

Therefore, a stochastic matrix P gives rise to

- A subproduct system $\mathrm{Arv}(P) := \mathrm{Arv}(\theta_P)$
- A tensor algebra $\mathcal{T}_+(P) := \mathcal{T}_+(\mathrm{Arv}(P))$

Theorem (Dor-On-Markiewicz '14)

Let P be a stochastic matrix over a state space Ω. Then up to isomorphism of subproduct systems we have

$$Arv(P)_n = \{[a_{ij}] : \forall (i,j), a_{ij} = 0 \text{ if } (P^n)_{ij} = 0, \sum_{j \in \Omega} |a_{ij}|^2 < \infty\}$$

where $\ell^\infty(\Omega$ acts as multiplication by diagonals on the left and on the right, and inner product is $\langle A, B \rangle = Diag(A^* B)$ and subproduct maps are given by

$$[U_{m,n}(A \otimes B)]_{ij} = \sum_{k \in \Omega} \sqrt{\frac{(P^m)_{ik}(P^n)_{kj}}{(P^{m+n})_{ij}}} \; a_{ik} b_{kj}$$

Question

Suppose that P, Q are stochastic matrices over Ω, and $\mathcal{T}_+(P) \simeq \mathcal{T}_+(Q)$. What can we say about the associated relation between P and Q? What is the suitable version of equivalence $\simeq$?

We have several natural isomorphism relations for tensor algebras of stochastic matrices:

- Algebraic isomorphism
- Bounded isomorphism
- Isometric isomorphism
- Completely isometric isomorphism
- Completely bounded isomorphism

However, the situation turns out to be much simpler for stochastic matrices.

Theorem (Dor-On-M. '14 – Automatic Continuity)

Let P and Q be stochastic matrices over Ω. If $\psi : \mathcal{T}_+(P) \to \mathcal{T}_+(Q)$ is algebraic isomorphism, then it is bounded.

Remark: Tensor algebras are not semi-simple in general (see Davidson-Katsoulis '11), so not a consequence of general machinery.

The proof uses an automatic continuity lemma due to Sinclair, which has become a stepping stone for many similar results in a variety of contexts.

Theorem (Dor-On-M. '14)

Let P and Q be stochastic matrices over Ω. TFAE:

1. *There is an isometric isomorphism of $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.*
2. *there is a graded comp. isometric isomorphism $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.*
3. $\mathrm{Arv}(P)$ *and* $\mathrm{Arv}(Q)$ *are unitarily isomorphic up to change of base*

Furthermore, if P and Q are recurrent (i.e. $\sum_n (P^n)_{ii} = \infty$ for all i), those conditions hold if and only if P and Q are the same up to permutation of Ω.

Recall that a stochastic matrix P is essential if for every i, $P_{ij}^n > 0$ for some n implies that $\exists m$ such that $P_{ji}^m > 0$.

We also say that the support of P is the matrix $\mathrm{supp}(P)$ given by

$$\mathrm{supp}(P)_{ij} = \begin{cases} 1, & P_{ij} \neq 0 \\ 0, & P_{ij} = 0 \end{cases}$$

Theorem (Dor-On-M. '14)

Let P and Q be finite stochastic matrices over Ω. TFAE:

1. *There is an algebraic isomorphism of $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.*
2. *there is a graded comp. bounded isomorphism $\mathcal{T}_+(P)$ onto $\mathcal{T}_+(Q)$.*
3. *$\mathrm{Arv}(P)$ and $\mathrm{Arv}(Q)$ are similar up to change of base*

Furthermore, if P and Q are essential , those conditions hold if and only if P and Q have the same supports up to permutation of Ω.

So when P and Q are finite, there are only two types of isomorphism problems:

- isometric iso. classes $=$ graded completely isometric iso. classes $=$ completely isometric iso. classes
- algebraic iso. classes $=$ bounded iso. classes $=$ completely bounded iso. classes

Example

For every $r \in (0, \frac{1}{2}]$, let

$$P_r = \begin{bmatrix} r & 1-r \\ r & 1-r \end{bmatrix}$$

(it is an essential and recurrent matrix since $P_r^2 = P_r$).
Then $\mathcal{T}_+(P_r)$ and $\mathcal{T}_+(P_s)$ are:

- algebraically isomorphic for every r, s.
- only isometrically isomorphic for $r = s$.

A word about the proof.

- In Davidson-Ramsey-Shalit '12, they show that in the orbit of the action of canonical "Bogolyubov" transformations of the tensor algebra on the space of isomorphisms there are always graded isomorphisms.
- This does not work so easily in our case.
- We first notice that for so called *reducing* projections p_j (onto the state $j \in \Omega$), the cut down $\mathcal{T}_+(p_j \operatorname{Arv}(P) p_j)$ is like a disk algebra. For such regular j, the Bogolyubov trick works with minor generalization.
- For singular j there may be complicated interrelations. To get a large enough group of Bogolyubov transformations, we need to define an equivalence relation R on Ω which allows the action of a torus $\mathbb{T}^{\Omega/\sim}$ as Bogolyubov transformations α_Λ for Λ in the torus.
- Given $\varphi : \mathcal{T}_+(P) \to \mathcal{T}_+(Q)$ an isomorphism, we show that in the orbit of $(\Lambda, \Theta) \mapsto \varphi \circ \alpha_\Lambda \circ \varphi \circ \alpha_\Theta \circ \varphi$ there is a graded completely isometric/bounded iso as required.

Definition (Arveson '69)

Let $\mathcal{B} \subseteq B(H)$ be a unital closed subalgebra and let $\mathcal{A} = C^*(\mathcal{B})$. We will say that a two-sided ideal $\mathcal{I} \trianglelefteq \mathcal{A}$ is a boundary ideal for $\mathcal{B}$ if the quotient map $q : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is completely isometric on $\mathcal{B}$.

Theorem (Hamana)

Let $\mathcal{B} \subseteq B(H)$ be a unital closed subalgebra and let $\mathcal{A} = C^*(\mathcal{B})$. Then there exists a largest boundary ideal $\mathcal{S}_{\mathcal{B}} \trianglelefteq \mathcal{A}$ for $\mathcal{B}$, called the Shilov ideal of $\mathcal{A}$ for $\mathcal{B}$.

Definition

Let $\mathcal{B} \subseteq B(H)$ be a unital closed subalgebra and let $\mathcal{A} = C^*(\mathcal{B})$. The C*-envelope of $\mathcal{B}$ is the C*-algebra $C^*_{\mathrm{env}}(\mathcal{B}) = \mathcal{A}/\mathcal{S}_{\mathcal{B}}$. It is the unique smallest C*-algebra (up to isomorphism) generated by a completely isometric copy of $\mathcal{B}$.

Some Examples:

Theorem (Katsoulis and Kribs '06)

If E is a C-correspondence, then $C^*_{\mathrm{env}}(\mathcal{T}_+(E)) = \mathcal{O}(E)$.*

Theorem (Davidson, Ramsey and Shalit '11)

If X is a commutative subproduct system of fin. dim. Hilbert space fibers, then $C^*_{\mathrm{env}}(\mathcal{T}_+(X)) = \mathcal{T}(X)$.

Theorem (Kakariadis and Shalit – personal communication)

If X is a subproduct system of fin. dim. Hilbert space fibers arising from a subshift of finite type, then $C^*_{\mathrm{env}}(\mathcal{T}_+(X))$ is either $\mathcal{T}(X)$ or $\mathcal{O}(X)$.

So far, this dichotomy gives some reassurance of the soundness of Viselter's definition of Cuntz-Pimsner algebra for subproduct systems.

Let $k \in \Omega$ and let P be an irreducible stochastic matrix with period

$$r = \gcd\{n : (P^n)_{11} > 0\}.$$

We say that P has a stationary k^{th} column if $P(e_k) = P^{r+1}(e_k)$. We denote by Ω_0 the set of states corresponding to stationary columns.

Theorem (Dor-On-M.)

Let P be an irreducible stochastic finite matrix.

- *If no columns of P are stationary, then $C^*_{\text{env}}(\mathcal{T}_+(P)) = \mathcal{T}(P)$.*
- *if all columns of P are stationary, then $C^*_{\text{env}}(\mathcal{T}_+(P)) = \mathcal{O}(P)$.*

Example (Dor-On-M. – Dichotomy fails)

$$P = \frac{1}{6}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

is a 1-periodic irreducible stochastic matrix for which the three algebras $C^*_{\text{env}}(\mathcal{T}_+(P))$, $\mathcal{T}(P)$ and $\mathcal{O}(P)$ are all different.

Theorem (Dor-On-M.)

Let P be a finite irreducible stochastic matrix. Let $X = Arv(P)$ and for each $k \in \Omega$, let $F_k = \mathcal{F}_X \cdot p_k$. Then the Shilov ideal of $\mathcal{T}_+(P)$ is given by

$$\bigcap_{k \in \Omega_0} \{T \in \mathcal{T}_+(P) : T \restriction_{F_k} \text{ is compact}\} \quad \cap \quad \bigcap_{k \notin \Omega_0} \{T \in \mathcal{T}_+(P) : T \restriction_{F_k} = 0\}$$

The idea of the proof is working with a faithful representation whose irreducible decomposition is computable, and each piece is a boundary representation. From there we obtain a faithful representation with the unique extension property, hence we get the kernel is the Shilov ideal (and the range is the C*-envelope).

Thank you!