Matricial Families and Weighted Shifts

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Introduction

We studied tensor operator algebras associated with correspondences E (over a C^*- or W^*-algebra M) (to be defined shortly) and their ultraweak closures: the **Hardy algebras**.
A useful property of these algebras is that one knows their representation theory. We showed that the representations can be parameterized by a **matricial family of sets** (TBDS) and the elements of the algebra (viewed as functions on the space of all representations) can be described by **matricial families of operator valued functions** (TBDS).
In fact, the matricial family of sets (parameterizing the representations) is $\{\overline{D}_\sigma\}_{\sigma \in Rep(M)}$ where $\overline{D}_\sigma$ is the **unit ball** of a certain space (TBDS) associated with σ.
Thus, these algebras can be studied as **algebras of matricial families of functions** defined on a family of unit balls.

Thus, these algebras can be studied as algebras of matricial families of functions defined on a family of **unit balls**.

The motivation for the current work: Replace "unit balls" by more general matricial families of sets.

Following works of Muller and of Popescu, we were led to replace the tensor algebra (which is generated by a family of shifts) with a "weighted tensor algebra" generated by a family of **weighted shifts**.

♣ In the following we assume that M is a W^*-algebra. But (almost) everything works for a C^*-algebra.

The unweighted case

I will first review the unweighted case.

We begin with the following setup:

- M - a W^*-algebra.
- E - a W^*-correspondence over M. This means that E is a **bimodule** over M which is endowed with an M-**valued inner product** (making it a right-Hilbert C^*-module that is self dual). The left action of M on E is given by a unital, normal, *-homomorphism φ of M into the (W^*-) algebra of all bounded adjointable operators $\mathcal{L}(E)$ on E.

Examples

- (Very basic Example) $M = \mathbb{C}$, $E = \mathbb{C}$.
- (Basic Example) $M = \mathbb{C}$, $E = \mathbb{C}^d$, $d > 1$.
- M- arbitrary , $\alpha : M \to M$ a normal unital, endomorphism. $E = M$ with right action by multiplication, left action by $\varphi = \alpha$ and inner product $\langle \xi, \eta \rangle := \xi^* \eta$. Denote it ${}_\alpha M$.

Note: If σ is a representation of M on H, $E \otimes_\sigma H$ is a Hilbert space with $\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2 \rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2 \rangle_E) h_2 \rangle_H$.

Similarly, given two correspondences E and F over M, we can form the (internal) tensor product $E \otimes F$ by setting

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle = \langle f_1, \varphi(\langle e_1, e_2 \rangle_E) f_2 \rangle_F$$

$$\varphi_{E \otimes F}(a)(e \otimes f)b = \varphi_E(a)e \otimes fb$$

and applying an appropriate completion.
In particular we get "tensor powers" $E^{\otimes k}$.

Also, given a sequence $\{E_k\}$ of correspondences over M, the direct sum $E_1 \oplus E_2 \oplus E_3 \oplus \cdots$ is also a correspondence (after an appropriate completion).

For a correspondence E over M we define the Fock correspondence

$$\mathcal{F}(E) := M \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \cdots$$

For every $a \in M$ define the operator $\varphi_\infty(a)$ on $\mathcal{F}(E)$ by

$$\varphi_\infty(a)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n$$

and $\varphi_\infty(a)b = ab$.
For $\xi \in E$, define the "unweighted shift" (or "creation") operator T_ξ by

$$T_\xi(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n.$$

and $T_\xi b = \xi b$. So that T_ξ maps $E^{\otimes k}$ into $E^{\otimes (k+1)}$.

Definition

(1) The norm-closed algebra generated by $\varphi_\infty(M)$ and $\{T_\xi : \xi \in E\}$ will be called the **tensor algebra** of E and denoted $\mathcal{T}_+(E)$.

(2) The ultra-weak closure of $\mathcal{T}_+(E)$ will be called the **Hardy algebra** of E and denoted $H^\infty(E)$.

Examples

1. If $M = E = \mathbb{C}$, $\mathcal{F}(E) = \ell^2$, $\mathcal{T}_+(E) = A(\mathbb{D})$ and $H^\infty(E) = H^\infty(\mathbb{D})$.
2. If $M = \mathbb{C}$ and $E = \mathbb{C}^d$ then $\mathcal{F}(E) = \ell^2(\mathbb{F}_d^+)$, $\mathcal{T}_+(E)$ is Popescu's $\mathcal{A}_d$ and $H^\infty(E)$ is F_d^∞ (Popescu) or $\mathcal{L}_d$ (Davidson-Pitts). These algebras are generated by d shifts.

Theorem

Every completely contractive representation of $\mathcal{T}_+(E)$ on H is given by a pair $(\sigma, \mathfrak{z})$ where

1. σ *is a normal representation of* M *on* $H = H_\sigma$. ($\sigma \in NRep(M)$)
2. $\mathfrak{z} : E \otimes_\sigma H \to H$ *is a contraction that satisfies*

$$\mathfrak{z}(\varphi(\cdot) \otimes I_H) = \sigma(\cdot)\mathfrak{z}.$$

We write $\sigma \times \mathfrak{z}$ *for the representation and we have* $(\sigma \times \mathfrak{z})(\varphi_\infty(a)) = \sigma(a)$ *and* $(\sigma \times \mathfrak{z})(T_\xi)h = \mathfrak{z}(\xi \otimes h)$ *for* $a \in M$, $\xi \in E$ *and* $h \in H$.

Write $\mathcal{I}(\varphi \otimes I, \sigma)$ for the intertwining space and D_σ for the open unit ball there. Thus the c.c. representations of the tensor algebra are parameterized by the family $\{\overline{D_\sigma}\}_{\sigma \in NRep(M)}$.

Examples

(1) $M = E = \mathbb{C}$. So $\mathcal{T}_+(E) = A(\mathbb{D})$, σ is the trivial representation on H, $E \otimes H = H$ and D_σ is the (open) unit ball in $B(H_\sigma)$.

(2) $M = \mathbb{C}$, $E = \mathbb{C}^d$. $\mathcal{T}_+(E) = \mathcal{A}_d$ (Popescu's algebra) and D_σ is the (open) unit ball in $B(\mathbb{C}^d \otimes H, H)$. Thus the c.c. representations are parameterized by row contractions $(T_1, \ldots, T_d)$.

(3) M general, $E =_\alpha M$ for an automorphism α.
$\mathcal{T}_+(E) =$ the analytic crossed product.
The intertwining space $\mathcal{I}(\varphi \otimes I, \sigma)$ can be identified with $\{\mathfrak{z} \in B(H) : \sigma(\alpha(T))\mathfrak{z} = \mathfrak{z}\sigma(T), T \in B(H)\}$ and the c.c. representations are $\sigma \times \mathfrak{z}$ where $\mathfrak{z}$ is a contraction there.

Weighted Shifts and algebras

Definition

A weight sequence is a sequence $Z = \{Z_k\}$ such that

- $Z_k \in \mathcal{L}(E^{\otimes k}) \cap \varphi_k(M)' = \mathcal{A}_k$.
- Z_k is invertible for all $k \geq 1$.
- $\sup_k ||Z_k|| < \infty$.

Notation: We write

$$Z^{(m)} = Z_m(I_E \otimes Z_{m-1}) \cdots (I_{E^{\otimes(m-1)}} \otimes Z_1)$$

for "powers".

For $\xi \in E$, define the "Z-weighted shift" operator $W_\xi \in \mathcal{L}(\mathcal{F}(E))$ by

$$W_\xi(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = Z_{n+1}(\xi \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n).$$

and $W_\xi b = Z_1(\xi b)$.

Definition

(1) The norm-closed algebra generated by $\varphi_\infty(M)$ and $\{W_\xi : \xi \in E\}$ will be called the **Z-tensor algebra** of E and denoted $\mathcal{T}_+(E, Z)$.

(2) The ultra-weak closure of $\mathcal{T}_+(E, Z)$ will be called the **Z-Hardy algebra** of E and denoted $H^\infty(E, Z)$.

Motivation for the definition of the domains

What are the c.c. representations of $\mathcal{T}_+(E,Z)$?

Start with an easier question: Let $\mathcal{T}_{0+}(E,Z)$ be the **algebra** generated by $\varphi_\infty(M)$ and $\{W_\xi : \xi \in E\}$. What are its representations?

They are determined by the images of the generators and a simple calculation shows that each representation ρ (on H) is associated with a pair $(\sigma, \mathfrak{z})$ where σ is a representation of M (and we shall assume it is normal) and $\mathfrak{z} \in \mathcal{I}(\varphi \otimes I, \sigma)$ such that

- $\rho(\varphi_\infty(a)) = \sigma(a)$, $a \in M$
- $\rho(W_\xi)h = \mathfrak{z}(\xi \otimes h)$, $\xi \in E$, $h \in H$.

It will be convenient to write L_ξ for the operator $h \mapsto \xi \otimes h$ and then we have $\rho(W_\xi) = \mathfrak{z}L_\xi$.

Which pairs extend to the norm closure?

Simple examples of c.c. representations

(1) **Induced representations :** Fix a normal representation π of M on K and write $\pi^{\mathcal{F}(E)}$ for the representation of $\mathcal{T}_+(E,Z)$ on $\mathcal{F}(E)\otimes_\pi K$ defined by $\pi^{\mathcal{F}(E)}(X)=X\otimes I_K$. This is, in fact, a representation of $H^\infty(E,Z)$ and, if π is faithful, it is completely isometric.

(2) **Compressions of induced representations:** Let π, K be as in (1), H be a Hilbert space and $V : H \to \mathcal{F}(E)\otimes_\pi K$ be an isometry whose final space is coinvariant under the induced representation. Then $\rho(X)=V^*(X\otimes I_K)V$ is a c.c. representation of $\mathcal{T}_+(E,Z)$.

In the unweighted case, the representations arising as in (2) are "almost all" the c.c. representations of the tensor algebra . More precisely, they contain all the representations that are given by points in the **open** unit ball of $\mathcal{I}(\varphi\otimes I,\sigma)$. This raises the hope that,even in the weighted case, if we understand these representations, we will understand all the representations.

What are the representations that are compressions of induced representations?

In the case $M = E = \mathbb{C}$, this was studied by V. Muller (88). In his context the question is: What operators are compressions of a weighted shift (to a coinvariant subspace)? or, equivalently, what operators are parts of a weighted shift (i.e. are the restriction of a weighted shift to an invariant subspace)?

So, fix ρ given by $(\sigma, \mathfrak{z})$ (i.e $\rho(\varphi_\infty(a)) = \sigma(a)$ and $\rho(W_\xi) = \mathfrak{z} L_\xi$) and an isometry $V : H \to \mathcal{F}(E) \otimes K$ such that

- $V^*(W_\xi \otimes I_K)V = \mathfrak{z} L_\xi$
- $V^*(\varphi_\infty(a) \otimes I_K)V = \sigma(a)$.

We can write $Vh = (V_0 h, V_1 h, \ldots)$ where $V_m h \in E^{\otimes m} \otimes K$. Using this, the definition of W_ξ and the fact that the image of V is coinvariant, a simple computation shows that, for $m \geq 0$, $V^*_{m+1}(Z_{m+1} \otimes I_K) = \mathfrak{z}(I_K \otimes V^*_m)$. Hence

$$V^*_{m+1} = \mathfrak{z}(I_K \otimes V^*_m)(Z^{-1}_{m+1} \otimes I_K).$$

Applying this recursively we get

$$V^*_m = \mathfrak{z}^{(m)}((Z^{(m)})^{-1} \otimes V^*_0)$$

where

$$\mathfrak{z}^{(m)} = \mathfrak{z}(I_E \otimes \mathfrak{z}) \cdots (I_{E^{\otimes(m-1)}} \otimes \mathfrak{z}) : E^{\otimes m} \otimes H \to H.$$

Since V is an isometry, we have $I = \sum_{m=0}^{\infty} V_m^* V_m$. Thus, using $V_m^* = \mathfrak{z}^{(m)}((Z^{(m)})^{-1} \otimes V_0^*)$, we get

$$I = \sum_{m=0}^{\infty} \mathfrak{z}^{(m)}((Z^{(m)*} Z^{(m)})^{-1} \otimes V_0^* V_0)\mathfrak{z}^{(m)*}.$$

Write $R_m^2 = (Z^{(m)*} Z^{(m)})^{-1}$ (with $R_0 = I$) and consider the CP map

$$\Theta_{\mathfrak{z}}^R(T) = \sum_{m=0}^{\infty} \mathfrak{z}^{(m)}(R_m^2 \otimes T)\mathfrak{z}^{(m)*}$$

for $T \in \sigma(M)'$.
Then

$$\Theta_{\mathfrak{z}}^R(V_0^* V_0) = I.$$

One can now consider the set of all $\mathfrak{z}$ such that this equation holds for some contraction V_0. But it does not seem to be a tractable domain. So we now make further assumptions.

Suppose that $\Theta_{\mathfrak{z}}^{R}$ has an inverse of a similar form, i.e

$$(\Theta_{\mathfrak{z}}^{R})^{-1}(T) = \Theta_{\mathfrak{z}}^{Y}(T) := \sum_{m=0}^{\infty} \mathfrak{z}^{(m)}(Y_m \otimes T)\mathfrak{z}^{(m)*}$$

for some $Y = \{Y_m\}$ with $Y_m \in \mathcal{A}_m$. Then $\mathfrak{z}$ should satisfy

$$\Theta_{\mathfrak{z}}^{Y}(I) \geq 0.$$

This suggests considering the domain

$$\{\mathfrak{z} : \sum_{m=0}^{\infty} \mathfrak{z}^{(m)}(Y_m \otimes I)\mathfrak{z}^{(m)*} \geq 0\}.$$

But: **Can we find such Y?** and **Will this give us all the representations?**

Composing the two maps ($\Theta_{\mathfrak{z}}^{R}$ and $\Theta_{\mathfrak{z}}^{Y}$) and setting it equal to the identity suggest that $Y_0 = I$ and the equations

$$\sum_{k=0}^{m} Y_k \otimes R_{m-k}^2 = 0$$

hold for every $m > 0$.
These equations can be easily solved.
For $m = 1$: $R_1^2 + Y_1 = 0$. Thus $Y_1 = -R_1^2$.
For $m = 2$: $R_2^2 + Y_1 \otimes R_1^2 + Y_2 = 0$. Thus $Y_2 = -R_2^2 + R_1^2 \otimes R_1^2$, etc.

But, is the map $\Theta_{\mathfrak{z}}^{Y}$ well defined?
We don't know the answer in general (even for the scalar case: $M = E = \mathbb{C}$). Even if one imposes conditions that will ensure convergence, it is not clear that this domain will describe **all** the representations of the algebra.

In the scalar case Muller studied two situations where positive results can be obtained. Subsequent research has been successful in the following cases:

(1) When $Y_m \leq 0$ for $m > 0$.
We write $X_m = -Y_m \geq 0$ (and assume a convergence condition). The domain is then

$$\{\mathfrak{z} \in \mathcal{I}(\varphi \otimes I, \sigma) : \quad \sum_{k=1}^{\infty} \mathfrak{z}^{(k)} (X_k \otimes I_{H_\sigma}) \mathfrak{z}^{(k)*} \leq 1\}.$$

See: V. Muller, G.Popescu (Memoir), P.Muhly-B.S., J.Good.

(2) When Y_m can be derived from $\Theta^Y = (id - \Theta^X)^k$ (with X as in (1)): J. Agler, V. Muller, G. Popescu (JFA), I. Martziano.

From now on, I will discuss our work assuming (1).

Start with the domains

The domains

To define the domains, we consider now a sequence $X = \{X_k\}_{k=1}^{\infty}$ of operators satisfying

- $X_k \in \mathcal{L}(E^{\otimes k}) \cap \varphi_k(M)' = \mathcal{A}_k$.
- $X_k \geq 0$ for all $k \geq 1$ and X_1 is invertible.
- $\overline{lim} ||X_k||^{1/k} < \infty$.

Definition

A sequence $X = \{X_k\}_{k=1}^{\infty}$ satisfying (1)-(3) above is said to be admissible.

Associated to an admissible sequence X, we now set

$$\overline{D}_{X,\sigma} := \{\mathfrak{z} \in \mathcal{I}(\varphi \otimes I, \sigma) : \quad \sum_{k=1}^{\infty} \mathfrak{z}^{(k)}(X_k \otimes I_{H_\sigma})\mathfrak{z}^{(k)*} \leq 1\}$$

where $\mathfrak{z}^{(k)} = \mathfrak{z}(I_E \otimes \mathfrak{z}) \cdots (I_{E^{\otimes k}} \otimes \mathfrak{z}) : E^{\otimes k} \otimes H \to H$.

Examples

(1) If $X_1 = I_E$ and $X_k = 0$ for $k > 1$, $\overline{D}_{X,\sigma} = \overline{D}_\sigma$.

(2) If $E = M = \mathbb{C}$, σ is on H and $X_k = x_k \in \mathbb{C}$,
$\overline{D}_{X,\sigma} = \{T \in B(H) : \sum_k x_k T^k T^{*k} \leq I\}$.

(3) If $M = \mathbb{C}$, $E = \mathbb{C}^d$, σ is on H and X_k is the $d^k \times d^k$ matrix $(x_{\alpha,\beta})$ (where α, β are words of length k in $\{1, \dots, d\}$),

$$\overline{D}_{X,\sigma} = \{T = (T_1, \dots, T_d) : \sum_{\alpha,\beta} x_{\alpha,\beta} T_\alpha T_\beta^* \leq I\} \quad (1)$$

where $T_\alpha = T_{\alpha_1} \cdots T_{\alpha_k}$.

Theorem

- *Given an admissible sequence X, one can construct a weight sequence $Z = \{Z_k\}$ such that, writing $R_m^2 = (Z^{(m)*}Z^{(m)})^{-1}$ as above and setting $Y_0 = I$ and $Y_m = -X_m$ (for $m > 0$), we have that*
$$\sum_{k=0}^{m} Y_k \otimes R_{m-k}^2 = 0$$
for every $m > 0$.
- *A weight sequence Z associated to X is not unique but the algebras (tensor, Hardy) associated with two different weight sequences are unitarily equivalent.*
- *One can always choose Z (associated with a given admissible sequence X) such that either each Z_k is positive (for every k) or each $Z^{(k)}$ is positive.*

From now on we fix an admissible X and a weight sequence Z associated to it

Theorem

Every completely contractive representation ρ of $\mathcal{T}_+(E,Z)$ on H is given by a pair $(\sigma, \mathfrak{z})$ where

1. *σ is a normal representation of M on $H = H_\sigma$.*
   $(\sigma \in NRep(M))$
2. *$\mathfrak{z} \in \overline{D}_{X,\sigma}$.*

In fact, $\rho(\varphi_\infty(a)) = \sigma(a)$ and $\rho(W_\xi)h = \mathfrak{z}(\xi \otimes h) = \mathfrak{z}L_\xi$.
Conversely, every such pair gives rise to a c.c. representation.

Thus the c.c. representations of the Z-tensor algebra are parameterized by the family $\{\overline{D}_{X,\sigma}\}_{\sigma \in NRep(M)}$.

We write $\sigma \times \mathfrak{z}$ for the representation ρ above.

Dilations

Lemma

Given $\mathfrak{z} \in \overline{D}_{X,\sigma}$, the map defined by

$$\Phi_{\mathfrak{z}}(T) = \sum_{k=1}^{\infty} \mathfrak{z}^{(k)}(X_k \otimes T)\mathfrak{z}^{(k)*}$$

(where the convergence is in ultraweak operator topology) is a completely positive map on $\sigma(M)'$ and the sequence $\{\Phi_{\mathfrak{z}}^m(I)\}$ is decreasing. (Write $Q_{\mathfrak{z}}$ for its limit)

Recall that, if π is a normal representation of M on K, then the *associated induced representation* is $X \in \mathcal{T}_+(E,Z) \mapsto X \otimes I_K \in B(\mathcal{F}(E) \otimes_\pi K)$.

Definition

An element $\mathfrak{v} \in \overline{D}_{X,\tau}$ (and the associated representation) is said to be coisometric if $\sum_{k=1}^{\infty} \mathfrak{v}^{(k)}(X_k \otimes I_{\mathcal{U}})\mathfrak{v}^{(k)*} = I_{\mathcal{U}}$ (where τ is a normal representation of M on $\mathcal{U}$).

Theorem

Let σ be a normal representation of M on H and $\mathfrak{z} \in \overline{D}_{X,\sigma}$. Then the associated representation $\sigma \times \mathfrak{z}$ is a compression (in the sense defined above) of a representation that is the direct sum of an induced representation and a coisometric one.
If $Q_{\mathfrak{z}} = 0$, it is a compression of an induced representation.

Theorem

Under the following additional conditions

(a) $\mathcal{L}(E^{\otimes m}) = \mathcal{K}(E^{\otimes m})$ *for all* $m \geq 1$.

(b) *There is some* $\epsilon > 0$ *such that, for all* $k \geq 1$, $Z_k \geq \epsilon I_{E^{\otimes k}}$ *and*

(c) *There is some* a *such that* $(\overline{\lim}||X_k||^{1/k})I_E < aI_E \leq X_1$.

the representation $\sigma \times \mathfrak{z}$ *is a compression (in the sense defined above) of a representation that is the direct sum of an induced representation and a coisometric one where the coisometric representation is a* C^**-representation (i.e. it extends to a* C^**-representation of the* C^**-algebra* $\mathcal{T}(E,Z)$ *that is generated by* $\mathcal{T}_+(E,Z)$*).*

Note: The "induced parts" in the two theorems are isomorphic (although the constructions are different). The "coisometric parts" we obtain in the two constructions may differ.

The families of functions

Given $F \in \mathcal{T}_+(E, Z)$, we define a family $\{\widehat{F}_\sigma\}_{\sigma \in NRep(M)}$ of (operator valued) functions called the *Berezin transform of* F. Each function $\widehat{F}_\sigma$ is defined on $\overline{D}_{X,\sigma}$ and takes values in $B(H_\sigma)$:

$$\widehat{F}_\sigma(\mathfrak{z}) = (\sigma \times \mathfrak{z})(F).$$

Here $NRep(M)$ is the set of all normal representations of M. Note that the family of domains is a matricial family in the following sense.

Definition

A family of sets $\{\mathcal{U}(\sigma)\}_{\sigma \in NRep(M)}$, with $\mathcal{U}(\sigma) \subseteq \mathcal{I}(\varphi \otimes I, \sigma)$, satisfying $\mathcal{U}(\sigma) \oplus \mathcal{U}(\tau) \subseteq \mathcal{U}(\sigma \oplus \tau)$ is called a *matricial family* of sets (or an nc set).

We shall be interested here mainly with the following matricial families.

Examples

(1) For a given admissible sequence X, the families $\{D_{X,\sigma}\}_{\sigma \in NRep(M)}$ and $\{\overline{D_{X,\sigma}}\}_{\sigma \in NRep(M)}$ are matricial families.

(2) For $\sigma \in NRep(M)$, write $\mathcal{AC}(\sigma)$ for the set of all $\mathfrak{z} \in \overline{D_{X,\sigma}}$ such that the representation $\sigma \times \mathfrak{z}$ extends to an ultraweakly continuous representation of $H^\infty(E,Z)$. Then the family $\{\mathcal{AC}(\sigma)\}_{\sigma \in NRep(M)}$ is a matricial family.

Note: $D_{X,\sigma} \subseteq \mathcal{AC}(\sigma)$.

If $F \in H^\infty(E,Z)$, $\widehat{F}_\sigma$ is defined on $\mathcal{AC}(\sigma)$ (or $D_{X,\sigma}$).

Definition

Suppose $\{\mathcal{U}(\sigma)\}_{\sigma\in NRep(M)}$ is a matricial family of sets and suppose that for each $\sigma \in NRep(M)$, $f_\sigma : \mathcal{U}(\sigma) \to B(H_\sigma)$ is a function. We say that $f := \{f_\sigma\}_{\sigma\in NRep(M)}$ is a *matricial family of functions* (or an nc function) in case

$$Cf_\sigma(\mathfrak{z}) = f_\tau(\mathfrak{w})C \tag{2}$$

for every $\mathfrak{z} \in \mathcal{U}(\sigma)$, every $\mathfrak{w} \in \mathcal{U}(\tau)$ and every $C \in \mathcal{I}(\sigma\times\mathfrak{z}, \tau\times\mathfrak{w})$ (equivalently, $C \in \mathcal{I}(\sigma,\tau)$ and $C\mathfrak{z} = \mathfrak{w}(I_E \otimes C)$).

Theorem

For every $F \in \mathcal{T}_+(E,Z)$, the family $\{\widehat{F}_\sigma\}$ is a matricial family of functions on $\{\overline{D}_{X,\sigma}\}_\sigma$. Similarly, For $F \in H^\infty(E,Z)$, we get a matricial family of functions on $\{D_{X,\sigma}\}$ and on $\{\mathcal{AC}(\sigma)\}$.

Does the converse hold?

In the unweighted case, for $H^\infty(E)$ and $\{\mathcal{AC}(\sigma)\}$, the converse holds. Here, we have the following.

Theorem

If $f = \{f_\sigma\}_{\sigma \in NRep(M)}$ *is a matricial family of functions, with* f_σ *defined on* $\mathcal{AC}(\sigma)$ *and mapping to* $B(H_\sigma)$*, then there is an* $F \in H^\infty(E, Z)$ *such that* f *and the Berezin transform of* F *agree on* $D_{X,\sigma}$ *; i.e.,*

$$f_\sigma(\mathfrak{z}) = \widehat{F}_\sigma(\mathfrak{z})$$

for every σ *and every* $\mathfrak{z} \in D_{X,\sigma}$.

Note: (1) We don't know if equality holds for every $\mathfrak{z}$ in $\mathcal{AC}(\sigma)$. What is missing is a better understanding of the representations in $\mathcal{AC}(\sigma)$.
(2) The proof uses the identification of the commutant of $H^\infty(E, Z)$ (M-S) and the fact that this algebra has the double commutant property (G).

Thank You !