Star-generating vectors of Rudins quotient modules

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(joint work with A. Chattopadhyay and J. Sarkar)

Introduction

Let $T = (T_1, \ldots, T_n)$ be a commuting tuple of operators on a seperable Hilbert space H.

Definition

A non-empty subset S of H *is a T-generating set if*

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- The rank of the tuple T:

$$\text{rank } T = \inf \left\{ \#S : S \subseteq \mathrm{H}, [S]_T = \mathrm{H} \right\} \in \mathbb{N} \cup \{\infty\}.$$

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- For a proper quotient module $\mathcal{Q}$ of $H^2(\mathbb{D})$, co-rank $\mathcal{Q} = 1$.

Rudin's quotient module

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Let ϕ be a Blaschke product with $Z(\phi) = \{\alpha_n : n \in \mathbb{N}\}$ and $l_n = \text{ord}(b_{\alpha_n}, \phi)$ $(n \in \mathbb{N})$.

- $\mathcal{Q}_\phi = \bigvee_{\alpha_n \in Z(\phi)} \mathcal{Q}_{b_{\alpha_n}^{l_n}}$.

Different representations

Set $Z_k := Z(\phi_{1,k}) \times \cdots \times Z(\phi_{n,k})$ $(k \in \mathbb{Z})$ and $Z := \bigcup_{k \in \mathbb{Z}} Z_k$. For $(\alpha_1, \ldots, \alpha_n) \in Z_k$, let $(\alpha_i, k) = \mathrm{ord}(b_{\alpha_i}, \phi_{i,k})$, $i = 1, \ldots, n$.

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$$\begin{aligned}\mathcal{Q}_\Phi &= \bigvee_{k=-\infty}^{\infty} \mathcal{Q}_{\phi_{1,k}} \otimes \cdots \otimes \mathcal{Q}_{\phi_{n,k}} \\ &= \bigvee_{k=-\infty}^{\infty} \bigvee_{(\alpha_1,\ldots,\alpha_n)\in Z_k} \mathcal{Q}_{b_{\alpha_1}^{(\alpha_1,k)}} \otimes \cdots \otimes \mathcal{Q}_{b_{\alpha_n}^{(\alpha_n,k)}}.\end{aligned}$$

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Lower bound of co-rank

$$\mathcal{Q}_\Phi = \bigvee_{(\alpha_1,\ldots,\alpha_n)\in Z} \mathcal{Q}(\alpha_1,\ldots,\alpha_n)$$

For $(\beta_1,\ldots,\beta_n) \neq (\alpha_1,\ldots,\alpha_n) \in Z$, one can find Blaschke products ϕ_i' $(1 \leq i \leq n)$ such that

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- $\sup_{(\alpha_1,\dots,\alpha_n)\in Z} \text{co-rank } \mathcal{Q}(\alpha_1,\dots,\alpha_n) \leq \text{co-rank } \mathcal{Q}_\Phi$.

Minimal representation

$$\mathcal{Q}(\alpha_1, \dots, \alpha_n) = \bigvee_{k \in \mathcal{I}(\alpha_1, \dots, \alpha_n)} \mathcal{Q}_{b_{\alpha_1}^{(\alpha_1, k)}} \otimes \cdots \otimes \mathcal{Q}_{b_{\alpha_n}^{(\alpha_n, k)}}$$

Note that for $k_1, k_2 \in \mathbb{Z}$, if $(\alpha_i, k_1) \leq (\alpha_i, k_2)$ for all $i = 1, \dots, n$ then

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Proposition

For all $\mathcal{Q}(\alpha_1,\ldots,\alpha_n)$, there exists $\tilde{\mathcal{I}}(\alpha_1,\ldots,\alpha_n) \subseteq \mathcal{I}(\alpha_1,\ldots,\alpha_n)$ with finite minimal cardinality such that

$$\mathcal{Q}(\alpha_1,\ldots,\alpha_n) = \bigvee_{k\in\tilde{\mathcal{I}}(\alpha_1,\ldots,\alpha_n)} \mathcal{Q}_{b_{\alpha_1}^{(\alpha_1,k)}} \otimes \cdots \otimes \mathcal{Q}_{b_{\alpha_n}^{(\alpha_n,k)}}.$$

Co-rank of $\mathcal{Q}(\alpha_1, \ldots, \alpha_n)$

Theorem

Let $\bigvee_{k \in \tilde{\mathcal{I}}(\alpha_1,\ldots,\alpha_n)} \mathcal{Q}_{b_{\alpha_1}^{(\alpha_1,k)}} \otimes \cdots \otimes \mathcal{Q}_{b_{\alpha_n}^{(\alpha_n,k)}}$ *be a minimal representation of* $\mathcal{Q}(\alpha_1, \ldots, \alpha_n)$. *Then*

$$\textit{co-rank}\ \mathcal{Q}(\alpha_1, \ldots, \alpha_n) = \#\tilde{\mathcal{I}}(\alpha_1, \ldots, \alpha_n).$$

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$$\textit{co-rank } \mathcal{Q}(\alpha_1, \ldots, \alpha_n) = \#\tilde{\mathcal{I}}(\alpha_1, \ldots, \alpha_n).$$

Idea of the proof: Let $\#\tilde{\mathcal{I}}(\alpha_1, \ldots, \alpha_n) = r$.

- $f_{i,k}$ is a star-generator of $\mathcal{Q}_{b_{\alpha_i}^{(\alpha_i,k)}}$ for all i and k.

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- For any closed subspace $\mathcal{E}$ with $\mathcal{Q}(\alpha_1, \ldots, \alpha_n) \ominus \mathcal{E}$ is a quotient module,
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- $g_k = b_{\alpha_1}^{(\alpha_1,k)-1} M_z^* b_{\alpha_1} \otimes \cdots \otimes b_{\alpha_n}^{(\alpha_n,k)-1} M_z^* b_{\alpha_n}$ for all k.

co-rank of $\mathcal{Q}_\Phi$

$$\sup_{(\alpha_1,\ldots,\alpha_n)\in Z} \#\tilde{\mathcal{I}}(\alpha_1,\ldots,\alpha_n) = \sup_{(\alpha_1,\ldots,\alpha_n)\in Z} \text{co-rank } \mathcal{Q}(\alpha_1,\ldots,\alpha_n)$$

$$\leq \text{co-rank } \mathcal{Q}_\Phi.$$

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Theorem

Let $\mathcal{Q}_\Phi$ be the Rudin quotient module corresponding to Φ. Then

$$\textit{co-rank } \mathcal{Q}_\Phi = \sup_{(\alpha_1,\ldots,\alpha_n)\in Z} \textit{co-rank } \mathcal{Q}(\alpha_1,\ldots,\alpha_n) = \sup_{(\alpha_1,\ldots,\alpha_n)\in Z} \#\tilde{\mathcal{I}}(\alpha_1,\ldots,\alpha_n).$$

Examples

Example (1)

Let $\phi_1 = \{b_{\alpha_k}\}_{k=-\infty}^{\infty}$ *and* $\phi_2 = \{b_{\beta_k}\}_{k=-\infty}^{\infty}$ *with* $\sum_k (1-|\alpha_k|), \sum_k (1-|\beta_k|) < \infty$. *Then*

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- $Z = \{(\alpha_k, \beta_k) : k \in \mathbb{Z}\}$.

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Let $\phi_1 = \{b_{\alpha_k}\}_{k=-\infty}^{\infty}$ *and* $\phi_2 = \{b_{\beta_k}\}_{k=-\infty}^{\infty}$ *with* $\sum_k (1 - |\alpha_k|), \sum_k (1 - |\beta_k|) < \infty$. *Then*

$$\mathcal{Q}_\Phi = \bigvee_{k=-\infty}^{\infty} \mathcal{Q}_{b_{\alpha_k}} \otimes \mathcal{Q}_{b_{\beta_k}}.$$

- $Z = \{(\alpha_k, \beta_k) : k \in \mathbb{Z}\}$.
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Example (2)

Let $\phi_1 = \{z^k\}_{k=0}^{\infty}$ *and* $\phi_2 = \{\psi_k\}_{k=0}^{\infty}$, *where* $\psi_k = \prod_{i=k+1}^{\infty} b_{\alpha_i}^{i-k}$. *Here* α_i*'s are distinct. Then*

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References

A. Chattopadhyay, B. K. Das and J. Sarkar, *Star-generating vectors of Rudin's quotient modules*, J. Funct. Anal. **267** (2014) no. 11, 4341-4360.

K.J. Izuchi, K.H. Izuchi and Y. Izuchi, *Blaschke products and the rank of backward shift invariant subspaces over the bidisk*, J. Funct. Anal. **261** (2011), 1457-1468.

K.J. Izuchi, K.H. Izuchi and Y. Izuchi, *Ranks of invariant subspaces of the Hardy space over the bidisk*, J. Reine Angew. Math. **659** (2011), 101-139.