A class of Sub-Hardy Hilbert Spaces Associated with Weighted Shifts

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Joint work with Dinesh Singh
Outline of the talk

- History and Motivation
- Some Notations and Definitions
- Statement of our Main Result
- Analogue of Wold’s Decomposition
- Sketch of the proof of the main result
- Important consequences
History and Motivation

- **Arne Beurling (1949)** - Characterizes the closed subspaces of $H^2$ that are invariant under the action of $T_z$, the operator of multiplication with the coordinate function $z$.

- **Peter Lax (1959)** - Vector-valued generalization of Beurling’s work for shifts of finite multiplicity.

- **Paul Halmos (1961)** - Vector-valued generalization of Beurling’s work for shifts of infinite multiplicity.

- **Louis de Branges** - Not only extended Beurling’s theorem but also its vector-valued generalizations due to Lax and Halmos.

$H^2$- the class of analytic function on $\mathbb{D}$ whose Taylor coefficients are square summable.

(i) $H^2$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

for $f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$ in $H^2$.

(ii) $\{z^n\}_{n=0}^{\infty}$ forms an orthonormal basis for $H^2$. 

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Notations and Definitions

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  (ii) $\{z^n\}_{n=0}^{\infty}$ forms an orthonormal basis for $H^2$.

- $H^\infty$ - the class of bounded analytic functions on $\mathbb{D}$.
  
  (i) $H^\infty$ is a Banach algebra with $||\phi||_\infty = sup\{|\phi(z)| : z \in \mathbb{D}\}$.

  (ii) $H^\infty = \{\phi \in H^2 : \phi H^2 \subseteq H^2\}$. 

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Let \( \{\beta_n\} \) be a sequence of positive numbers.

\[
H^2(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} \alpha_n z^n : \sum_{n=0}^{\infty} |\alpha_n|^2 \beta_n^2 < \infty \right\}
\]

with the inner product

\[
\langle f, g \rangle = \sum_{n=0}^{\infty} \alpha_n \overline{\gamma_n} \beta_n^2
\]

for all \( f = \sum_{n=0}^{\infty} \alpha_n z^n \) and \( g = \sum_{n=0}^{\infty} \gamma_n z^n \) in \( H^2(\beta) \).

\( H^2(\beta) \) is a Hilbert space with respect to the above inner product space.

For \( \beta_n = 1 \) for all \( n \), \( H^2(\beta) = H^2 \).
Notations and Definition contd...

- $T \in \mathcal{B}(H)$ is called an **injective weighted shift** with weight sequence $\{w_n\}_{n=0}^{\infty}$ if

$$Te_n = w_n e_{n+1},$$

where $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis for $H$ and $\{w_n\}_{n=0}^{\infty}$ is a bounded sequence of positive numbers.

When $H = H^2$, $e_n = z^n$ and $w_n = 1$, we use $T_z$ to denote the injective weighted shift operator.

- $T \in \mathcal{B}(H)$ is said to **shift an orthogonal basis** $\{h_n\}$ of $H$ if $Th_n = h_{n+1}$ for each $n$. 
A. Beurling: If $M$ is a closed subspace of $H^2$ invariant under the action of $T_z$, then there is an inner function $b$ (i.e., $|b| = 1$ a.e. on $\mathbb{T}$) such that $M = bH^2$. 
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**de Branges:** Let \( M \) be a Hilbert space such that:

(i) \( M \) is contractively contained in \( H^2 \), that is, \( M \subseteq H^2 \) and \( \|f\|_2 \leq \|f\|_M \),

(ii) \( T_z(M) \subseteq M \) and \( T_z \) is an isometry on \( M \).

Then there exists a \( b \in H^\infty \) with \( \|b\|_\infty \leq 1 \) such that

\[
M = bH^2 \quad \text{and} \quad \|bf\|_M = \|f\|_2 \quad \forall f \in H^2
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**Singh and Singh:** Let $M$ be a Hilbert space such that:

(i) $M \subseteq H^2$,

(ii) $T_z(M) \subseteq M$ and $T_z$ acts isometrically on $M$.

Then there exists a $b \in H^\infty$ such that

$$M = bH^2 \quad \text{and} \quad \|bf\|_M = \|f\|_2 \quad \forall f \in H^2.$$
Can we weaken the hypotheses any further?
Our Theorem

Theorem (L. & Singh)

Let $M$ be a Hilbert space contained in $H^2$. Suppose the operator $T_z$, which denotes multiplication by $z$, is well defined on $M$, and satisfies:

(i) There exists a $\delta > 0$ such that $\delta \|f\|_M \leq \|T_z f\|_M \leq \|f\|_M$ for all $f \in M$.

(ii) For each $n \in \mathbb{N}$, $T_z^* T_z^{n+1}(M) \subseteq T_z(M)$ (the adjoint of $T_z$ is with respect to the inner product on $M$).
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Then $T_z$ acts as a weighted shift on $M$, and there exists a $b \in H^\infty$ such that

$$M = \overline{bH^2} \quad \text{(the closure is in the norm of } M)$$

and

$$\|bf\|_M \leq \|f\|_2 \quad \text{for all } f \in H^2.$$
Lemma (L. & Singh)

Let $T \in \mathcal{B}(H)$ be bounded below and $T^* T^{n+1}(H) \subseteq T(H)$ for all $n \in \mathbb{N}$. Let $N$ be the orthogonal complement of the range of $T$. Then:

(i) $H = \bigoplus_{n=0}^{\infty} T^n(N) \oplus \bigcap_{n=1}^{\infty} T^n(H)$.

(ii) The subspace $\bigcap_{n=1}^{\infty} T^n(H)$ is reducing for $T$ and $T$ restricted to it is an invertible operator.
Analogue of Wold’s decomposition

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Example

Take $H = H^2(\beta)$ and $T = T_z$ where

$$
\beta_n = \begin{cases} 
\frac{1}{2^{n/2}} & \text{if } n \text{ even}, \\
\frac{1}{2^{(n-1)/2}} & \text{if } n \text{ odd}.
\end{cases}
$$
Outline of the proof

- Using the lemma,

\[ M = \sum_{n=0}^{\infty} T_z^n(N) \oplus \bigcap_{n=1}^{\infty} T_z^n(M), \]

where \( N = M \ominus T_z(M) \).

- Since elements of \( M \) are analytic on \( \mathbb{D} \), \( \bigcap_{n=1}^{\infty} T_z^n(M) = \{0\} \).
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- \( \text{dim}(N) = 1. \)

- Take \( b \) a unit vector in \( N. \) Then \( \{bz^n\}_{n=0}^{\infty} \) is an orthogonal basis for \( M. \) Therefore, \( bH^2 \) is dense in \( M. \)
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- \( T_z \) shifts this orthogonal basis.
Theorem (A. Shields, 1974)

$T \in \mathcal{B}(H)$ is an injective weighted shift if and only if $T$ shifts an orthogonal basis of $H$. 
Theorem

Let $M$ be a Hilbert space contained in $H^2$. Suppose the operator $T_z$, which denotes multiplication by $z$, is well defined on $M$, and satisfies:

(i) There exists a $\delta > 0$ such that $\delta \|f\|_M \leq \|T_z f\|_M \leq \|f\|_M$ for all $f \in M$.

(ii) For each $n \in \mathbb{N}$, $T_z^n T_z^{n+1}(M) \subseteq T_z(M)$ (the adjoint of $T_z$ is with respect to the inner product on $M$).

Then $T_z$ acts as a weighted shift on $M$, and there exists a $b \in H^\infty$ such that

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Remark

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The subspace $bH^2$ is closed in $M \iff$ there exists a $\delta > 0$ such that

$$\delta \|f\|_M \leq \|T_z^n f\|_M \leq \|f\|_M$$

(1)

for all $f \in M$ and all $n \geq 0$. 

Example

Choose $\{\beta_n\}$ such that $c \leq \beta_n + 1 \leq \beta_n$ for some $c > 0$ and for all $n$.

Take $\beta_n = (n + 3)^{1/(n+3)}$ for $n \geq 0$.
Remark

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The subspace $bH^2$ is closed in $M \iff$ there exists a $\delta > 0$ such that
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Example

Choose $\{\beta_n\}$ such that $c \leq \beta_{n+1} \leq \beta_n$ for some $c > 0$ and for all $n$.

Take $\beta_n = (n+3)^{1/(n+3)}$ for $n \geq 0$. 
Important consequences

Corollary (Singh and Singh, 1991)

Let $M$ be a Hilbert space contained in $H^2$ as a vector subspace and such that $T_z(M) \subseteq M$ and let $T_z$ act isometrically on $M$. Then there exists a $b \in H^\infty$ such that $M = bH^2$, and $\|bf\|_M = \|f\|_2$ for all $f \in H^2$.

The above result generalizes the result of de Branges and therefore of Beurling as well. Hence our result also implies these two classical results.