A Shimorin-type analytic model for left-invertible operators

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Let $T \in \mathcal{B}(\mathcal{H})$. We say that $T$ is left-invertible if there exists $S \in \mathcal{B}(\mathcal{H})$ such that $ST = I$.

The Cauchy dual operator $T'$ of a left-invertible operator $T \in \mathcal{B}(\mathcal{H})$ is defined by

$$T' := T(T^*T)^{-1}.$$ 

An operator $T$ is left-invertible if and only if there exists a constant $c > 0$ such that $T^*T \geq cl$.

We call $T$ analytic if $\mathcal{H}_\infty := \bigcap_{i=1}^{\infty} T^i\mathcal{H} = \{0\}$. 

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The classical Wold decomposition

Theorem (H. Wold 1938)

Let \( U \) be a isometry on Hilbert space \( \mathcal{H} \). Then \( \mathcal{H} \) is the direct sum of two subspaces reducing \( U \), \( \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_p \) such that \( U|_{\mathcal{H}_u} \in \mathcal{B}(\mathcal{H}_u) \) is unitary and \( U|_{\mathcal{H}_p} \in \mathcal{B}(\mathcal{H}_p) \) is unitarily equivalent to a unilateral shift. This decomposition is unique and the canonical subspaces are defined by

\[
\mathcal{H}_u = \bigcap_{n=1}^{\infty} U^n \mathcal{H} \quad \text{and} \quad \mathcal{H}_p = \bigoplus_{n=1}^{\infty} U^n E,
\]

where \( E := \mathcal{N}(U^*) = \mathcal{H} \ominus U \mathcal{H} \).
We shall say that an operator $T \in \mathcal{B}(\mathcal{H})$ admits Wold-type decomposition, if the subspaces $\mathcal{H}_\infty := \bigcap_{n=1}^{\infty} T^n \mathcal{H}$ and $E := \mathcal{N}(T^*)$ have the following properties:

- $\mathcal{H}_\infty$ is reducing for $T$ and $T$ is unitary on $\mathcal{H}_\infty$,
- $\mathcal{H} = \mathcal{H}_\infty \oplus [E]_T$,

where $[E]_T := \bigvee \{ T^n x : x \in \mathcal{H}, n \in \mathbb{N} \}$.

**Theorem (S. Shimorin 2001)**

Assume that $T \in \mathcal{B}(\mathcal{H})$ satisfies one of the following conditions:

- $\| T^2 x \| + \| x \|^2 \leq 2 \| Tx \|^2$ for $x \in \mathcal{H}$,
- $\| T(x + y) \|^2 \leq 2(\| x \|^2 + \| Ty \|^2)$ for $x, y \in \mathcal{H}$

Then $T$ admits Wold-type decomposition.
Let \( T \in \mathcal{B}(\mathcal{H}) \) be a left-invertible analytic operator and \( E := \mathcal{N}(T^*) \).

For every \( x \in \mathcal{H} \) define a vector-valued holomorphic functions \( U_x \) as

\[
U_x(z) = \sum_{n=0}^{\infty} (P_E T'^{**n} x)z^n, \quad z \in \mathbb{D}(r(T')^{-1}),
\]

where \( T' \) is the Cauchy dual of \( T \).

The operator \( U : \mathcal{H} \ni x \rightarrow U_x \in \mathcal{H} \) is injective.

We equip the obtained space of analytic functions \( \mathcal{H} := \{ U_x : x \in \mathcal{H} \} \) with the inner product induced by \( \mathcal{H} \).

The operator \( U : \mathcal{H} \ni x \rightarrow U_x \in \mathcal{H} \) becomes a unitary isomorphism.
The space $\mathcal{H}$ is a reproducing kernel Hilbert space in the following sense: the reproducing kernel for $\mathcal{H}$ is an $\mathcal{B}(E)$-valued function of two variables $\kappa_\mathcal{H} : \Omega \times \Omega \to \mathcal{B}(E)$ such that

- for any $e \in E$ and $\lambda \in \Omega$

  $$\kappa_\mathcal{H}(\cdot, \lambda)e \in \mathcal{H},$$

- for any $e \in E$, $f \in \mathcal{H}$ and $\lambda \in \Omega$

  $$\langle f(\lambda), e \rangle_E = \langle f, \kappa_\mathcal{H}(\cdot, \lambda)e \rangle_\mathcal{H}.$$
Theorem (S. Shimorin 2001)

The space $\mathcal{H}$ is a reproducing kernel Hilbert space and the reproducing kernel $\kappa_{\mathcal{H}} : \mathbb{D}(r(T')^{-1}) \times \mathbb{D}(r(T')^{-1}) \to \mathcal{B}(E)$ is given by

$$\kappa_{\mathcal{H}}(z, \lambda) = P_E(I - \lambda T'^*)^{-1}(I - z T')^{-1}|_E.$$
Theorem (S. Shimorin 2001)

Let $T \in B(H)$ be a left-invertible analytic operator. Then the operator $T$ is unitarily equivalent to the operator $M_z$ of multiplication by $z$ on $H$ and $T^*$ is unitarily equivalent to the operator $L$ given by

$$(Lf)(z) = \frac{f(z) - f(0)}{z}, \quad f \in H.$$
Let $T \in \mathcal{B}(\mathcal{H})$ be a left-invertible operator.

Let $E$ be a subspace of $\mathcal{H}$ denote by $[E]_{T^*, T'}$ the following subspace of $\mathcal{H}$:

$$[E]_{T^*, T'} := \bigvee \left( \{ T^n x : x \in E, n \in \mathbb{N} \} \cup \{ T'^n x : x \in E, n \in \mathbb{N} \} \right),$$

We choose closed subspace $E$ such that $[E]_{T^*, T'} = \mathcal{H}$. where $T'$ is the Cauchy dual of $T$.

For each $x \in \mathcal{H}$, define a formal Laurent series $U_x$ with vector coefficients as

$$U_x(z) = \sum_{n=1}^{\infty} \left( P_E T^n x \right) \frac{1}{z^n} + \sum_{n=0}^{\infty} \left( P_E T'^n x \right) z^n,$$

where $T'$ is the Cauchy dual of $T$. 
The operator $U : \mathcal{H} \ni x \mapsto U_x \in \mathcal{H}$ is injective.

We equip the obtained space of formal Laurent series $\mathcal{H} := \{U_x : x \in \mathcal{H}\}$ with the inner product induced by $\mathcal{H}$.

The operator $U : \mathcal{H} \ni x \mapsto U_x \in \mathcal{H}$ becomes a unitary isomorphism.
Theorem (P.P. 2018)

Let \( T \in \mathcal{B}(\mathcal{H}) \) be a left-invertible operator and \( E \) be a closed subspace of \( \mathcal{H} \) such that \([E]_{T^*,T'} = \mathcal{H}\). Then the operator \( T \) is unitary equivalent to the operator \( M_z : \mathcal{H} \to \mathcal{H} \) of multiplication by \( z \) on \( \mathcal{H} \) given by

\[
(M_z f)(z) = zf(z), \quad f \in \mathcal{H},
\]

and operator \( T'^* \) is unitary equivalent to the operator \( L : \mathcal{H} \to \mathcal{H} \) given by

\[
(L f)(z) = \frac{f(z) - (P_N(M_z^*) f)(z)}{z}, \quad f \in \mathcal{H}.
\]
Theorem (P.P. 2018)

Let $T \in \mathcal{B}(\mathcal{H})$ be left-invertible and analytic, $\mathcal{H}_1$, $U_1$ be the Hilbert space and the unitary map constructed in our analytic model with $E := \mathcal{N}(T^*)$ and $\mathcal{H}_2$, $U_2$ be the Hilbert space and the unitary map obtained in Shimorin’s construction. Then $\mathcal{H}_1 = \mathcal{H}_2$ and $U_1 = U_2$. 
Theorem (P.P. 2018)

Let $T \in \mathcal{B}(\mathcal{H})$ be a left-invertible operator and $E$ be a closed subspace of $\mathcal{H}$ such that $[E]_{T^*, T'} = \mathcal{H}$. Then for every $m \in \mathbb{N}$ the following assertions hold:

(i) $T'^m E$ is a closed subspace and $[T'^m E]_{T^*, T'} = \mathcal{H}$,

(ii) the mapping $\Phi_m : \mathcal{H}_0 \rightarrow \mathcal{H}_m$ defined by

$$\Phi_m \left( \sum_{n=-\infty}^{\infty} a_n z^n \right) = \sum_{n=-\infty}^{\infty} (V_m a_{m+n}) z^n, \quad \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{H}_0$$

is a unitary isomorphism, where $\mathcal{H}_k$ for $k \in \mathbb{N}$ is the Hilbert space constructed in our analytic model with subspace $T'^k E$ and $V_k : E \rightarrow T'^k E$ for $k \in \mathbb{N}$ is defined by,

$$Ve = P_{T'^k E} T^k e, \quad e \in E.$$
Theorem (P.P. 2018)

Let $T \in \mathcal{B}(\mathcal{H})$ be a left-invertible operator and $E$ be a closed subspace of $\mathcal{H}$ such that $[E]_{T^*, T'} = \mathcal{H}$. Let

\[
    r^+ := \lim_{n \to \infty} \inf \| P_E T'^*n \|^{-\frac{1}{n}},
\]

\[
    r^- := \lim_{n \to \infty} \sup \| P_E T^n \|^\frac{1}{n}.
\]

If $r^+ > r^-$, then formal Laurent series

\[
    U_x(z) = \sum_{n=1}^{\infty} (P_E T^n x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^*n x) z^n,
\]

represent analytic functions on annulus $\mathbb{A}(r^-, r^+)$. 
Theorem (P.P. 2018)

Let $T \in \mathcal{B}(\mathcal{H})$ be a left-invertible operator and $E$ be a closed subspace of $\mathcal{H}$ such that $[E]_{T^*, T'} = \mathcal{H}$ and the series

$$U_x(z) = \sum_{n=1}^{\infty} (P_E T^n x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^* n x) z^n,$$

is convergent in $E$ on an annulus $\mathbb{A}(r^-, r^+)$ with $r^- < r^+$ and $r^-, r^+ \in [0, \infty)$ for every $x \in \mathcal{H}$. Then $\mathcal{H}$ is a reproducing kernel Hilbert space of $E$-valued holomorphic functions on $\mathbb{A}(r^-, r^+)$. The reproducing kernel $\kappa_{\mathcal{H}} : \mathbb{A}(r^-, r^+) \times \mathbb{A}(r^-, r^+) \to \mathcal{B}(E)$ associated with $\mathcal{H}$ is given by

$$\kappa_{\mathcal{H}}(z, \lambda) = \sum_{i,j \geq 1} P_E T^i T^* j |_{E} \frac{1}{z^i \bar{\lambda}^j} + \sum_{i \geq 1, j \geq 0} P_E T^i T'^j |_{E} \frac{1}{z^i \bar{\lambda}^j}$$

$$+ \sum_{i \geq 0, j \geq 1} P_E T'^i T^* j |_{EZ} \frac{1}{z^i \bar{\lambda}^j} + \sum_{i,j \geq 0} P_E T'^i T'^j |_{EZ} \bar{\lambda}^j.$$
Moreover, the following assertions hold.

(i) For any $\lambda \in A(r^-, r^+)$

$$
\sum_{n=1}^{\infty} (P_E T^n) \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} (P_E T^*n) \lambda^n \in B(\mathcal{H}, E), \tag{2}
$$

$$
\sum_{n=1}^{\infty} T^*n \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} T'n \lambda^n \in B(E, \mathcal{H}), \tag{3}
$$

(ii) The series (1), (2) and (3) converges absolutely and uniformly in operator norm on any compact set contained in $A(r^-, r^+) \times A(r^-, r^+), A(r^-, r^+) \text{ and } A(r^-, r^+)$, respectively.

(iii) The function $A(r^-, r^+) \ni \lambda \to \kappa_{\mathcal{H}}(\cdot, \bar{\lambda}) e \in \mathcal{H}, e \in E$ is holomorphic and given by

$$
\kappa_{\mathcal{H}}(\cdot, \bar{\lambda}) e = \sum_{n=1}^{\infty} U T^*n e \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} U T'n e \lambda^n, \quad \lambda \in A(r^-, r^+).
$$
Theorem (P.P. 2018)

Let $T \in \mathcal{B}(\mathcal{H})$ be a left-invertible operator and $E$ be a closed subspace of $\mathcal{H}$ such that $[E]_{T^*, T'} = \mathcal{H}$ and the series

$$U_x(z) = \sum_{n=1}^{\infty} (P_E T^n x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^* n x) z^n,$$

convergent in $E$ for every $x \in \mathcal{H}$ on open nonempty subset $\Omega \subset \mathbb{C}$. Then the following assertions hold:

(i) the point spectrum of $T$ is empty, that is $\sigma_p(T) = \emptyset$,
(ii) $\mathcal{M}_z \kappa_{\mathcal{H}}(\cdot, \mu) g = \bar{\mu} \kappa_{\mathcal{H}}(\cdot, \mu) g$, for every $\mu \in \Omega$, $g \in E$,
(iii) $\bar{\Omega} \subset \sigma_p(T^*)$,
(iv) $\bigvee \{ \mathcal{N}(T^* - \bar{\mu}) : \mu \in U \} = \mathcal{H}$, where $U \subset \Omega$ and $\text{int} \, U \neq \emptyset$. 
Composition operators in $L^2$-spaces

- $(X, \mathcal{A}, \mu)$ is a $\sigma$-finite measure space
- $\phi : X \to X$ is an $\mathcal{A}$-measurable transformation, i.e., $\phi^{-1}(\Delta) \in \mathcal{A}$ for every $\Delta \in \mathcal{A}$
- If the measure $\mu \circ \phi^{-1}$ given by $\mu \circ \phi^{-1}(\Delta) = \mu(\phi^{-1}(\Delta))$ for $\Delta \in \mathcal{A}$ is absolutely continuous with respect to $\mu$ (we say that $\mu$ is **nonsingular**), then the operator $C_\phi$ in $L^2(\mu)$ given by $D(C_\phi) = \{ f \in L^2(\mu) : f \circ \phi \in L^2(\mu) \}$, $C_\phi f = f \circ \phi, f \in D(C_\phi)$ is well-defined
- We call it a **composition** operator with symbol $\phi$
For $x \in X$ the set

$$[x]_{\phi} = \{ y \in X : \text{there exist } i, j \in \mathbb{N} \text{ such that } \phi^i(x) = \phi^j(y) \}$$

is called the orbit of $f$ containing $x$.

If $x \in X$ and $\phi^i(x) = x$ for some $i \in \mathbb{Z}_+$ then the cycle of $\phi$ containing $x$ is the set

$$C_{\phi} = \{ \phi^i(x) : i \in \mathbb{N} \}$$

We will only consider composition functions with one orbit, since an orbit induces a reducing subspace to which the restriction of the weighted composition operator is again a weighted composition operator.

Any self-map $\phi : X \to X$ induces a directed graph $(X, E_{\phi})$ given by

$$E_{\phi} = \{ (x, y) \in X \times X : x = \phi(y) \}$$
Lemma (P.P. 2018)

Let $X$ be a countable set, $w : X \to \mathbb{C}$ be a complex function on $X$ and $\varphi : X \to X$ be a transformation of $X$. Let $C_{\varphi, w}$ be a weighted composition operator in $\ell^2(X)$ and

$$E := \begin{cases} \bigoplus_{x \in \text{Gen}_{\varphi}(1,1)} \langle e_x \rangle \oplus \mathcal{N}\left( (C_{\varphi, w} |_{\ell^2(\text{Des}(x))})^* \right) & \text{if } \varphi \text{ has a cycle}, \\ \langle e_{\omega} \rangle \oplus \mathcal{N}(C_{\varphi, w}^*) & \text{otherwise}, \end{cases}$$

where $\text{Des}(x) := \bigcup_{n=0}^{\infty} \varphi(-n)(x)$ and $\omega$ is a generalized root of the tree. Then the subspace $E$ has the following properties:

(i) $[E]_{C_{\varphi, w}^*, C_{\varphi, w}} = \mathcal{H}$ and $[E]_{C_{\varphi, w}, C_{\varphi, w}^*} = \mathcal{H}$,

(ii) $E \perp C_{\varphi, w}^n E$ and $E \perp C_{\varphi, w}^n E$, $n \in \mathbb{Z}_+$. 
The non-negative number

\[
    r_{w,\varphi}^+ := \lim_{n \to \infty} \inf \left( \sum_{x \in W_n^{E,\varphi}, n \geq 0} |w'(x)w'(\varphi(x)) \cdots w'(\varphi^{(n-1)}(x))|^2 \right)^{-\frac{1}{2n}}
\]  

(5)

will be called the outer radius of convergence for \( C_{\varphi,w} \), and similarly the non-negative number

\[
    r_{w,\varphi}^- := \begin{cases} 
    \sqrt{\prod_{x \in \mathcal{C}_\varphi} |w(x)|} & \text{if } \varphi \text{ has a cycle}, \\
    \limsup_{n \to \infty} \sqrt[n]{|w(\varphi^1(\omega))w(\varphi^2(\omega)) \cdots w(\varphi^n(\omega))|} & \text{otherwise},
\end{cases}
\]

(6)

where \( \tau := \text{card } \mathcal{C}_\varphi \) will be called the inner radius of convergence for \( C_{\varphi,w} \).
Theorem (P.P. 2018)

Let $X$ be a countable set, $w : X \to \mathbb{C}$ be a complex function on $X$ and $\varphi : X \to X$ be a transformation of $X$, which has finite branching index. Let $C_{\varphi,w}$ be a left-invertible weighted composition operator in $\ell^2(X)$. If $r^{+}_{w,\varphi} > r^{-}_{w,\varphi}$, then there exist a $z$-invariant reproducing kernel Hilbert space $\mathcal{H}$ of $E$-valued holomorphic functions defined on the annulus $A(r^{-}_{w,\varphi}, r^{+}_{w,\varphi})$ and a unitary mapping $U : \ell^2(X) \to \mathcal{H}$ such that $\mathcal{M}_z U = UC_{\varphi,w}$, where $\mathcal{M}_z$ denotes the operator of multiplication by $z$ on $\mathcal{H}$, where $E$ is as in (4).
Moreover, the following assertions hold:

(i) the reproducing kernel
\[ \kappa_{\mathcal{H}} : \mathbb{A}(r_{w,\varphi}^-, r_{w,\varphi}^+) \times \mathbb{A}(r_{w,\varphi}^-, r_{w,\varphi}^+) \to \mathcal{B}(E) \]
associated with \( \mathcal{H} \) has the property that \( \kappa_{\mathcal{H}}(\cdot, w)g \in \mathcal{H} \) and
\[ \langle Uf, \kappa_{\mathcal{H}}(\cdot, w)g \rangle = \langle (Uf)(w), g \rangle \]
for \( f, g \in \ell^2(X) \).

(ii) the reproducing kernel \( \kappa_{\mathcal{H}} \) has the following form:
\[ \kappa_{\mathcal{H}}(z, \lambda) = \sum_{i,j \geq 1} A_{i,j} \frac{1}{z^i} \frac{1}{\lambda^j} + \sum_{i \geq 1, j \geq 0} B_{i,j} \frac{1}{z^i} \lambda^j \]
\[ + \sum_{i \geq 0, j \geq 1} C_{i,j} z^i \frac{1}{\lambda^j} + \sum_{i,j \geq 0} D_{i,j} z^i \lambda^j, \]
where \( A_{i,j}, B_{i,j}, C_{i,j}, D_{i,j} \in \mathcal{B}(E) \).
Theorem (P.P. 2018)

(iii) if $\varphi$ does not have a cycle, then the linear subspace generated by $E$-valued polynomials in $z$ and $\tilde{E}$-valued polynomials involving only negative powers of $z$ is dense in $\mathcal{H}$, that is

$$\bigvee \left( \{ z^n E : n \in \mathbb{N} \} \cup \left\{ \frac{1}{z^n} \tilde{E} : n \in \mathbb{Z}_+ \right\} \right) = \mathcal{H},$$

where $\tilde{E} := \bigvee \{ e_x : x \in \text{Gen}_{\varphi}(1, 1) \};$
Theorem (P.P. 2018)

if $\varphi$ has a cycle $\mathcal{C}_\varphi$ with $\tau := \text{card } \mathcal{C}_\varphi$, then there exist $\tau$ functions $f_1, \ldots, f_\tau$ on $\mathbb{A}(r^{-}_w, \varphi, r^{+}_w, \varphi)$ given by the following Laurent series

$$f_i(z) := \sum_{k=0}^{\infty} \sum_{i=1}^{\tau} \Lambda^k A_i \frac{1}{z^{k\tau+i}}, \quad i = 1, \ldots, \tau,$$

where $\Lambda := \prod_{x \in \mathcal{C}_\varphi} w(x)$ such that the linear subspace generated by $E$-valued polynomials in $z$ and the above functions is dense in $\mathcal{H}$, that is

$$\mathcal{H} = \sqrt{\{z^n E : n \in \mathbb{N}\} \cup \{f_i : i \in \{1, \ldots, \tau\}\}}.$$
Fix $m \in \mathbb{N}$ and set $X = \{0, 1, \ldots, m\}$. Let $w : X \to \mathbb{C}$ be a function and define a mapping $\varphi : X \to X$ by

$$\varphi(i) = \begin{cases} 
  i + 1 & \text{if } i < m \\
  0 & \text{if } i = m
\end{cases}$$

(see Figure 1). Set $\Lambda := w(0)w(1)\ldots w(m)$. Let $C_{\varphi,w}$ be the left-invertible composition operator in $\mathbb{C}^{m+1}$.
Example

Let $E := \text{lin} \{ e_1 \}$. It is easy to see that $[E]_{S, S^*} = \mathcal{H}$. One can verify that

$$P_E C_{\varphi, w}^{mk+r} x = \Lambda^k \left( \prod_{i=0}^{r-1} w(i) \right) x_r e_0,$$

$$P_E C_{\varphi, w}^*(mk+r) x = \frac{1}{\Lambda^k} \left( \prod_{i=m+1-r}^{m} w(i) \right)^{-1} x_{n+1-r} e_0,$$

for $r < n$, $r, k \in \mathbb{N}$. 
This shows that formal Laurent series takes the following form:

\[ U_x(z) = \sum_{k=1}^{\infty} \sum_{r=0}^{n-1} \left( \Lambda^k \left( \prod_{i=0}^{r-1} w(i) \right) x_r e_0 \right) \frac{1}{z^{nk+r}} \]

\[ + \sum_{k=0}^{\infty} \left( \sum_{r=0}^{n-1} \frac{1}{\Lambda^k} \left( \prod_{i=m+1-r}^{m} w(i) \right)^{-1} x_{n+1-r} e_0 \right) z^{nk+r}. \]

Since \( C_{\varphi,w}^* \) acts on the finite dimensional space, the spectrum of \( C_{\varphi,w}^* \) is finite. Therefore, by assertion (iii) of Theorem 8 the above series does not converge absolutely on any open subset of \( \mathbb{C} \).

Alternatively, one can prove this fact directly by calculating convergences radii.
Example (Bilateral weighted shift)

Let $S_\lambda : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be a bilateral weighted shift with weights $\{\lambda_n\}_{n \in \mathbb{Z}}$ and $\{e_n\}_{n \in \mathbb{Z}}$ be the standard orthonormal basis of $\ell^2(\mathbb{Z})$. Then

$$S_\lambda e_n = \lambda_{n+1} e_{n+1}, \quad n \in \mathbb{Z}$$

Let $E := \text{lin} \{ e_0 \}$. It is easy to see that $[E]_{S_\lambda^*, S_\lambda'} = \mathcal{H}$. It is worth noting that $\mathcal{N}(S_\lambda^*) = \{0\}$ and thus $[\mathcal{N}(S_\lambda^*)]_{S_\lambda^*, S_\lambda'} = \{0\}$. This phenomenon is quite different comparing with the case of left-invertible and analytic operators in which $[\mathcal{N}(T^*)]_{T^*, T'} = \mathcal{H}$, where $T$ is in this class.
Figure:

It is a matter of routine to verify that the Cauchy dual $S'_\lambda^*$ of $S_\lambda$ has the following form

$$S'_\lambda^* e_n = \frac{1}{\lambda_n} e_{n-1}, \quad n \in \mathbb{Z}.$$  

It is now easily seen that

$$P_E(S'_\lambda^*)^n x = \left( \prod_{i=1}^{n} \lambda_i \right)^{-1} x_n e_0, \quad n \in \mathbb{Z}_+,$$
Example

and

\[ P_E S_{\lambda}^n x = \left( \prod_{i=-n+1}^{0} \lambda_i \right) x_{-n} e_0, \quad n \in \mathbb{Z}_+. \]

Therefore, the formal Laurent series takes the form

\[ U_x(z) = \sum_{n=1}^{\infty} \left( \prod_{i=-n+1}^{0} \lambda_i \right) x_{-n} \frac{1}{z^n} + \sum_{n=0}^{\infty} \left( \prod_{i=1}^{n} \lambda_i \right)^{-1} x_n z^n. \]

One can show that

\[ r_{w, \varphi}^+ = \liminf_{n \to \infty} \sqrt[n]{\prod_{i=1}^{n} |\lambda_i|}. \]
and

\[ r_{w,\varphi} = \limsup_{n \to \infty} \sqrt[n]{\prod_{i=-n+1}^{0} |\lambda_i|}. \]

In this case, the reproducing kernel
\[ \kappa_{\mathcal{H}} : \mathbb{A}(r^-, r^+) \times \mathbb{A}(r^-, r^+) \to \mathcal{B}(E) \]

is given by

\[ \kappa_{\mathcal{H}}(z, \lambda) = \sum_{i=1}^{\infty} \prod_{i=-n+1}^{0} |\lambda_i|^2 \frac{1}{(z\overline{\lambda})^i} + \sum_{i=0}^{\infty} \left( \prod_{i=1}^{n} |\lambda_i|^2 \right)^{-1} (z\overline{\lambda})^i. \]
Example

Set $m \in \mathbb{N}$ and $X = \{0, 1, \ldots, m\} \sqcup \{(0, i) : i \in \mathbb{N}\}$. Let $w : X \to \mathbb{C}$ be a measurable function and $\varphi : X \to X$ be transformation of $X$ defined by

$$\varphi(x) = \begin{cases} 
(0, i - 1) & \text{for } x = (0, i), \ i \in \mathbb{N} \setminus \{0\}, \\
m & \text{for } x = (0, 0), \\
i - 1 & \text{for } x = i \text{ and } i \in \{1, \ldots, m\}, \\
m & \text{for } x = 0,
\end{cases}$$

(see Figure 3). Let $C_{\varphi, w} : \ell^2(X) \to \ell^2(X)$ be a left-invertible composition operator. It is easily seen that

$$C_{\varphi, w} e_x = \begin{cases} 
w((0, i + 1))e_{(0,i+1)} & \text{for } x = (0, i), \ i \in \mathbb{N} \setminus \{0\} \\
w(i + 1)e_{i+1} & \text{for } x = i \text{ and } i \in \{0, 1, \ldots, m\} \\
w(0)e_0 + w((0, 0))e_{(0,0)} & \text{for } x = m.
\end{cases}$$
It is routine to verify that $\mathcal{N}(C^*_\varphi,w) = \text{lin}\{w((0,0))e_0 - w(0)e_{(0,0)}\}$. Let $E := \text{lin}\{e_{(0,0)}\}$. One can check that this one-dimensional subspace satisfies $[E]_{T^*,T'} = \mathcal{H}$. 
Example

This implies that

\[ P_E(C^*_\varphi, w')^n x = \left( \prod_{i=1}^{n} w(0, i) \right)^{-1} x_n e_{(0,0)}, \]

\[ P_E C^{nm+r+1}_{\varphi, w} x = \Lambda^n w((0, 0)) \left( \prod_{i=0}^{r-1} w(m - i) \right) x_{m-r} e_{(0,0)}, \]

for \( r < m, r, n \in \mathbb{N} \). Hence, the Hilbert space \( \mathcal{H} \) consist of the functions of the form

\[
U_x(z) = \sum_{n=1}^{\infty} \sum_{r=0}^{k} \Lambda^k w((0, 0)) \left( \prod_{i=0}^{r-1} w(m - i) \right) x_{m-r} \frac{1}{z^{nm+r+1}} + \sum_{n=0}^{\infty} \left( \prod_{i=1}^{n} w((0, i)) \right)^{-1} x_n z^n.
\]
Example

The formulas for the inner and outer radius of convergence take the following form

\[ r_{w, \varphi}^+ = \lim \inf_{n \to \infty} \sqrt[n]{\prod_{i=1}^{n} |w((0, i))|} \]

and

\[ r_{w, \varphi}^- = m^{+1} \sqrt[m]{\prod_{i=0}^{m} |w(i)|}. \]
Example

The reproducing kernel $\kappa_H : A(r^-, r^+) \times A(r^-, r^+) \to B(E)$ takes the form

$$
\kappa_H(z, \lambda) = \sum_{i \geq 1, j \geq 1} \Lambda^i \bar{\Lambda}^j |w((1, 0))|^2 \left( \prod_{i=0}^{r-1} |w(m - i)|^2 \right) \frac{1}{z^{-im+r+1} \bar{\lambda}^{-jm+r+1}} + 
\sum_{i=0}^{\infty} \left( \prod_{i=1}^{n} |w((0, i))|^2 \right)^{-1} (z \bar{\lambda})^i.
$$
Thank you for your attention!