Spectrum of random Schrödinger operators with decaying randomness

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(Joint work with Anish mallick)
The Model

- \( \Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} \), \( d \) dimensional Laplacian,

- \( V^\omega \) is the multiplication operator on \( L^2(\mathbb{R}^d) \),

\[
(V^\omega f)(x) = V^\omega(x)f(x), \quad V^\omega(x) = Q(x) \sum_{n \in \mathbb{Z}^d} \omega_n \chi_{n+(0,1]^d}(x).
\]

\( Q(x) = O(|x|^{-\alpha}) \), \( \alpha > 0 \) for large \( x \) and \( \{\omega_n\}_n \) are iid random variables with common distribution by \( \mu \),

\[
\frac{d\mu}{dx}(x) = O(|x|^{-(1+\delta)}), \quad \delta > 0, \quad |x| \to \infty.
\]

- Consider the probability space \( (\Omega = \mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}_\Omega, \mathbb{P} = \otimes \mu) \).

Define the random operator \( H^\omega \) as

\[
H^\omega = -\Delta + V^\omega, \quad \omega = (\omega_n)_{n \in \mathbb{Z}^d} \in \Omega.
\]
The spectrum of $-\Delta$

- It is well known that $-\Delta$ is essential self-adjoint and

$$\mathcal{F}(-\Delta)\mathcal{F}^{-1} = M_{\varphi(x)}, \varphi(x) = \sum_{i=1}^{d} |x_i|^2, x \in \mathbb{R}^d.$$ 

$\mathcal{F}$ is the Fourier transform on $L^2(\mathbb{R}^d)$.

- Now we have $\sigma(-\Delta) = \sigma_{ac}(-\Delta) = [0, \infty)$.

- Let $-\Delta_L$ is the restriction of $-\Delta$ to the domain $(-L, L)^d$ with Neumann boundary condition.

$$\sigma(-\Delta_L) = \sigma_{dis}(-\Delta_L) = \left\{ \left( \frac{\pi}{2L} \right)^2 \sum_{i=1}^{d} n_i^2 : n_i \in \mathbb{N} \cup \{0\} \right\}.$$
Results (Spectrum of $H^\omega$)

- For $\alpha \delta \leq d$ we have $\sigma(H^\omega) = \sigma_{ess}(H^\omega) = \mathbb{R}$ a.e $\omega$.
- For $\alpha \delta > d$ we have $\sigma_{ess}(H^\omega) = [0, \infty)$ and $\sigma(H^\omega) \cap (-\infty, 0)$ is discrete a.e $\omega$.
  In above case 0 may be the limit point for negative eigenvalues. But for $(\alpha - 2)\delta > d$ we have $\#\{\sigma(H^\omega) \cap (-\infty, 0)\} < \infty$.
- For $\delta > 2$ and $\alpha > 1$ we have $[0, \infty) \subset \sigma_{ac}(H^\omega)$ a.e $\omega$. 
The negative spectrum of $H^\omega$ always exhibits exponential localization (Anderson Localization), independent of the choice of $\alpha$ and $\delta$.

The negative part of the spectrum always pure point i.e $(-\infty, 0) \cap \sigma(H^\omega) \subset \sigma_{pp}(H^\omega), \ a.e \ \omega$.

$H^\omega \psi_\omega = E \psi_\omega, \ \psi_\omega(x) \leq c_\omega e^{-d_\omega |x-\eta_\omega|}, \ E < 0, \ a.e \ \omega.$

$\eta_\omega$ is the localization center, $\psi_\omega$ attain its maximum at $\eta_\omega$. 
Out line of the proof

Using Weyl’s criterion together with Borel-Cantelli lemma we get (for any choice of $\alpha$ and $\delta$)

$$[0, \infty) \subset \sigma_{ess}(H^\omega), \ a.e \ \omega.$$ 

For $\alpha \delta > d$ the Dirichlet Neumann bracketing

$$\left( \bigoplus_{n \in \mathbb{Z}^d} H_{n,N}^\omega \leq H^\omega \leq \bigoplus_{n \in \mathbb{Z}^d} H_{n,D}^\omega \right)$$

will give

$$\# \{ (-\infty, -\epsilon) \cap \sigma(H^\omega) \} < \infty, \ a.e \ \omega, \ \forall \ \epsilon > 0.$$ 

For $\alpha \delta > d$ we have $[0, \infty)$ is the essential spectrum and below zero there is the discrete spectrum a.e $\omega$. 

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For $\alpha \delta > d$ still 0 may be the limit point for the negatives eigenvalues.

But for $(\alpha - 2)\delta > d$ we can show

$$H^\omega \geq -\Delta - \frac{M^\omega}{1 + |x|^{\epsilon}}, \; \epsilon > 2, \; a.e \; \omega.$$ 

Let $H = -\Delta - V$ with $V(x) = O(|x|^{\epsilon}), \; \epsilon > 2$. The number of negative eigenvalues of $H$ is finite.

Now we get

$$\# \{(\neg \infty, 0) \cap \sigma(H^\omega)\} < \infty, \; a.e \; \omega.$$
Using min-max principle we can show that

\[ \bigcup_{\lambda \in \mathbb{R}} \sigma(-\Delta + \lambda \chi_{(0,1]^d}) = \mathbb{R}. \]

For \( \alpha \delta \leq d \)

\[ \bigcup_{\lambda \in \mathbb{R}} \sigma(-\Delta + \lambda \chi_{(0,1]^d}) \subseteq \sigma_{\text{ess}}(H^\omega), \ a.e \ \omega. \]

The above two will imply

\[ \sigma(H^\omega) = \sigma_{\text{ess}}(H^\omega) = \mathbb{R}, \ a.e \ \omega, \ for \ \alpha \delta \leq d. \]
If the potential decay fast enough, $\delta > 2$ and $\alpha > 1$ then we verified the following:

$$\int_{\mathbb{R}^d} (1 + |x|)^{-2m} (V^\omega(x))^2 \, dx < \infty, \text{ a.e } \omega, \text{ for some } m > 0,$$

$$\int_1^\infty \left( \int_{a < |x| < b} (V^\omega(xt))^2 \, dx \right) dt < \infty, \text{ a.e } \omega, \text{ } 0 < a < b < \infty.$$ 

With above two estimation (Cook’s Method, scattering theory, existence of wave operators) will ensure that $[0, \infty) \subset \sigma_{ac}(H^\omega)$, a.e $\omega$. 

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Negative part of the spectrum (Wegner estimate)

- Let $H^\omega_{\Lambda_L(x)}$ be the restriction of $H^\omega$ to the cube $\Lambda_L(x)$ with center at $x$ and side length $L$. Set

$$\Omega_L = \{\omega : |V^\omega(n)| < L^a, \ n \in \Lambda_L(0)\}, \ a > 0, \ |V^\omega(n)| \approx \frac{\omega n}{|n|^\alpha}.$$

- Wegner estimate for $E < 0$

$$\sup_{n \in \mathbb{Z}^d} \mathbb{P}\left(\text{dist}(\sigma(H^\omega_{\Lambda_L(n)}), E) < \eta \left| \Omega_L \right) \leq C \eta^s L^{d+\gamma a}.\right.$$

- The above estimate follows from

$$\mathbb{E}\left(\text{Tr}(E^\omega_{H^\omega_{\Lambda_L(n)}}(I))\right) \leq C |I|^s L^{d+\gamma a}, \ C, \gamma, a > 0, \ s \in (0, 1].$$
The Initial scale estimate for $E < 0$ is given by $(c, m, b > 0)$

$$
\mathbb{P} \left( \left\| \chi_{\partial \Lambda_L} \left( H_{\Lambda_L}^\omega(n) - E \right)^{-1} \chi_{\Lambda_L^{b/3}(0)} \right\| \leq ce^{-mL} \right) \geq 1 - \frac{1}{L^b}.
$$

Once we have Wegner estimate and Initial scale estimate we can use Bootstrap Multiscale analysis (Germinet-Klein) and show that $(-\infty, 0)$ exhibits exponential localization.

This Bootstrap Multiscale analysis is an induction method and to start the induction all we need the Wegner estimate and the Initial scale estimate.
We can see that $H^\omega$ is densely defined with domain $C_\infty^\omega(\mathbb{R}^d)$ a.e $\omega$.

For $(2 + \alpha)\delta > d$ we have essential self-adjointness of $H^\omega$.

The above choice of $\alpha$ and $\delta$ we can show that

$$V^\omega(x) \leq M^\omega(1 + |x|)^{2-\epsilon}, \epsilon > 0, \text{ a.e } \omega.$$ 

It is known that if $V^-(x) = o(|x|^{2-\epsilon})$ then $-\Delta + V$ is essential self-adjoint on $L^2(\mathbb{R}^d)$. Here $V^-(x) = \min\{0, V(x)\}$. 

Spectrum of random Schrödinger operators with decaying potentials
In Mathematical Physics there is a phenomenon called existence of extended states in low disorder.

Define $H_{\lambda}^\omega$ on $L^2(\mathbb{R}^d)$ by

$$H_{\lambda}^\omega = -\Delta + \lambda \sum_{n \in \mathbb{Z}^d} \omega_n u(x - n),$$

$u$ is compactly supported and $u \in L^\infty(\mathbb{R}^d)$, $\{\omega_n\}$ are iid random variables and $\lambda > 0$.

It is expected that for small enough $\lambda$

$$\emptyset \neq \sigma_{ac}(H_{\lambda}^\omega) \subset [0, \infty), \text{ a.e. } \omega.$$


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Thank You