On some extension of pairs of commuting isometries.

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notation

- $\mathcal{B}(H)$ - the algebra of bounded linear operators on a separable, complex Hilbert space $H$,
- $\text{Lat}(S)$ - the lattice of $S$ - invariant subspaces, $S \in \mathcal{B}(H)$,
- $L^2_{\mathcal{H}}(\mathbb{T})$ - the space of square integrable, $\mathcal{H}$ valued functions, where $\mathcal{H}$ is a complex Hilbert space,
- $H^2_{\mathcal{H}}(\mathbb{T})$ - Hardy space of $\mathcal{H}$ valued functions,
- $M_z \in \mathcal{B}(L^2_{\mathcal{H}}(\mathbb{T}))$, $T_z \in \mathcal{B}(H^2_{\mathcal{H}}(\mathbb{T}))$ operators of multiplication by the independent variable "$z$",
$T_z \in B(H^2_{\mathcal{H}}(\mathbb{T}))$ is a model of a unilateral shift of multiplicity \(\dim \mathcal{H}\).

\(\phi : \mathbb{T} \mapsto B(\mathcal{H})\) is an inner function iff \(\phi(z)\) are partial isometries with the same initial space for almost every \(z\).

\(M_\phi \in B(H^2_{\mathcal{H}}(\mathbb{T}))\) where \(M_\phi f : z \mapsto \phi(z)f(z)\).

**Theorem (Beurling-Lax-Halmos, 1961)**

Invariant subspaces of \(T_z \in H^2_{\mathcal{H}}(\mathbb{T})\) are precisely subspaces of the form

\[M_\phi H^2_{\mathcal{H}}(\mathbb{T})\]

where \(\phi\) is an inner function.
Beurling-Lax-Halmos theorem

\( T_z \in B(H^2_\mathcal{H}(\mathbb{T})) \) is a model of a unilateral shift of multiplicity \( \dim \mathcal{H} \).

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Let \((S_1, S_2) \in \mathcal{B}(H)\) be a pair of commuting unilateral shifts.

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Lat(S_1, S_2) - \text{the lattice of joint invariant subspaces.}
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Lat(S_1, S_2) = Lat(S_1) \cap Lat(S_2)
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S_i \cong T_z \in \mathcal{B}(H^2_{\mathcal{H}_i}(\mathbb{T})) \text{ where } \mathcal{H}_i \cong \ker S_i^* \text{ for } i = 1, 2.
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(\(V_1, V_2\)) \(\in \mathcal{B}(H)\) - a pair of commuting isometries,

(\(\tilde{V}_1, \tilde{V}_2\)) \(\in \mathcal{B}(\tilde{H})\) - an isometric extension of \((V_1, V_2)\).

Then:

\[ H \in \text{Lat}(\tilde{V}_1, \tilde{V}_2), \]

\[ \text{Lat}(V_1, V_2) = \{M \cap H : M \in \text{Lat}(\tilde{V}_1, \tilde{V}_2)\} \]
description via extension

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Aim:

1. for a given relatively prime, positive integers $m, n$ extend an arbitrary pair of isometries to a pair

   $$ (U^k V^m, U^l V^n), $$

   where:
   - $U$ is a unitary operator commuting with an isometry $V$, and
   - $km - ln = 1$,

2. describe a model of the pair

   $$ (U^k V^m, U^l V^n), $$

3. describe

   $$ \text{Lat}(U^k V^m, U^l V^n). $$

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     \[(U^k V^m, U^l V^n),\]

2. describe \(\text{Lat}(U^k V^m, U^l V^n)\).

   (partial results)
Proposition

For any pair of commuting isometries \((V_1, V_2) \in \mathcal{B}(H)\) and positive integers \(m, n\), there is an extension to a commuting pair of isometries \((\hat{V}_1, \hat{V}_2)\) on a Hilbert space \(\hat{H}\) where

\[\hat{V}_2^* n \hat{V}_1^m\]

is a unitary operator commuting with \(\hat{V}_1, \hat{V}_2\). Moreover, the extension may be chosen to be minimal.
\[ \mathcal{M} = \{ f \in L^2(\mathbb{T}^2) : \hat{f}_{i,j} = 0 \text{ for } (i, j) \in \mathbb{Z}^2 \setminus Z \} \subset L^2(\mathbb{T}^2) \] where \( Z \) is as in the picture.

For \( V_1 = M_{z_1}|_\mathcal{M}, V_2 = M_{z_2}|_\mathcal{M} \) a minimal extension \((\hat{V}_1, \hat{V}_2)\) such that \( \hat{V}_2^n \hat{V}_1^m \) is unitary is \( M_{z_1}, M_{z_2} \) for any \( m, n \).
\[ \mathcal{M} = \{ f \in L^2(\mathbb{T}^2) : \hat{f}_{i,j} = 0 \text{ for } (i, j) \in \mathbb{Z}^2 \setminus \mathbb{Z} \} \subset L^2(\mathbb{T}^2) \text{ where } \mathbb{Z} \text{ is as in the picture} \]

For \( V_1 = M_{z_1} |_{\mathcal{M}}, \ V_2 = M_{z_2} |_{\mathcal{M}} \) a minimal extension \( (\hat{V}_1, \hat{V}_2) \) such that \( \hat{V}_2^n \hat{V}_1^m \) is unitary is \( M_{z_1}, M_{z_2} \) for any \( m, n \).
A pair of commuting isometries \((V_1, V_2)\) on a Hilbert space \(H\) such that for some relatively prime, positive integers \(m, n\) the operator 

\[
V_2^{*n} V_1^m
\]

is unitary may be extended to a pair 

\[
(\tilde{U}^k \tilde{V}^n, \tilde{U}^l \tilde{V}^m)
\]

where:
- \(\tilde{U}\) is a unitary operator commuting with an isometry \(\tilde{V}\),
- \(H \in \text{Lat}(\tilde{V}^m, \tilde{V}^n)\) and
- \((k, l)\) are unique integers such that \(0 < k < n, 0 \leq l < m\) and \(km - ln = 1\).

Moreover, the extension may be chosen to be minimal, and for a minimal extension if \(V_1, V_2\) are unilateral shifts, then \(\tilde{V}\) is a unilateral shift.
Any pair of commuting isometries \((V_1, V_2)\), for any relatively prime, positive integers \(m, n\) may be extended to a pair
\[
(\hat{U}^k \hat{V}^n, \hat{U}^l \hat{V}^m)
\]
where \(\hat{U}\) is a unitary operator commuting with an isometry \(\hat{V}\) and \((k, l)\) are unique integers such that \(0 < k < n, 0 \leq l < m\) and \(km - ln = 1\). Moreover, the extension may be chosen minimal.
Let \( m, n \) be relatively prime, positive integers and \( km - ln = 1 \). Any pair of the form
\[
(U^k V^n, U^l V^m)
\]
where \( U \) is a unitary operator commuting with an isometry \( V \) is unitarily equivalent to:
\[
(U_1 \oplus (T^n_Z \otimes U^k), U_2 \oplus (T^m_Z \otimes U^l))
\]
on the Hilbert space \( H_u \oplus (H^2(T) \otimes \mathcal{H}) \) for the respective unitary operators \( U_1, U_2 \in \mathcal{B}(H_u), U \in \mathcal{B}(\mathcal{H}) \).
Let \( T_z \in \mathcal{B}(H^2_{\mathcal{H}}(\mathbb{T})) \) and \( m, n \) be relatively prime, positive integers. The subspaces jointly invariant under \( (T^m_z, T^n_z) \) are precisely those of the form

\[
M_\phi \left( H_0 \oplus (I - P)H^2_{\mathcal{H}0}(\mathbb{T}) \right)
\]

where

- \( P \in \mathcal{B}(H^2_{\mathcal{H}}(\mathbb{T})) \) is an orthogonal projection on the space of polynomials of degree at most \( mn - m - n \),
- \( \phi \) is an inner function with initial space \( \mathcal{H}_0 \) and \( H_0 \subset PH^2_{\mathcal{H}0}(\mathbb{T}) \) invariant under \( PT^m_z, PT^n_z \).

- \( PT^3_z = PT^2_z = 0 \) (the case \( m = 3, n = 2 \)),
- if \( \dim \mathcal{H}_0 < \infty \) then \( \dim PH^2_{\mathcal{H}0}(\mathbb{T}) < \infty \)
\[ \text{Lat}(UV, V) \ (m = n = 1) \]

\[ H^2_{\mathcal{H}}(\mathbb{T}) \cong H^2(\mathbb{T}) \otimes \mathcal{H} \]
\[ V \cong T_z \otimes I, \]
\[ U \cong I \otimes \mathcal{U}. \]

**Theorem**

The subspaces jointly invariant under \((T_z \otimes I, T_z \otimes \mathcal{U})\) are precisely those of the form \(M_\phi(H^2(\mathbb{T}) \otimes \mathcal{H})\) where \(\phi\) is an inner function satisfying

\[ M_\phi(H^2(\mathbb{T}) \otimes \mathcal{H}) = WM_\psi(H^2(\mathbb{T}) \otimes \mathcal{H}) \]

with some other inner function \(\psi\) and \(W = \sum_{i \geq 0} P_{cz^i} \otimes \mathcal{U}^i\).
$T_{z_1}, T_{z_2} \in \mathcal{B}(H^2(\mathbb{T}^2))$ and $M_{z_1}, M_{z_2} \in \mathcal{B}(L^2(\mathbb{T}^2))$

$(T_{z_1}, T_{z_2})$ extends to a pair of unilateral shifts $(\tilde{T}_{z_1}, \tilde{T}_{z_2})$ such that

$$\tilde{T}_{z_1}^* \tilde{T}_{z_2} \text{ is unitary,}$$

where

$$\tilde{T}_{z_\alpha} = M_{z_\alpha} | \mathcal{M} \text{ for } \alpha = 1, 2$$

and $\mathcal{M} := \{f \in L^2(\mathbb{T}^2) : \hat{f}_{i,j} = 0 \text{ for } j < -i\}$.

$$\operatorname{Lat}(\tilde{T}_{z_1}, \tilde{T}_{z_2}) = \operatorname{Lat}(\tilde{T}_{z_1}) \cap W\operatorname{Lat}(\tilde{T}_{z_1})$$

where $W \in \mathcal{B}(\mathcal{M})$ is defined by

$$Wz_1^i z_2^j = z_1^{2i+j} z_2^{-i}.$$
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finite multiplicity

Consider \((V_1, V_2)\) a pair of unilateral shifts such that \(U := V_2^* V_1^n\) is unitary. Then

\[ V_1^m \simeq T_z \otimes I, \quad V_2^n \simeq T_z \otimes U, \quad U = I \otimes U. \]

If \(V_1, V_2\) are of finite multiplicity then \(U\) is a unitary operator on a finite dimensional space. Eigenvalues/eigenspaces of \(U\) corresponds to those of \(U := I \otimes U\) which commutes with \(V_1, V_2\).
Remark

Let a pair of commuting unilateral shifts \((V_1, V_2)\) on \(H\) satisfy

\[ V_2^{*n} V_1^m = \lambda I \]

for relatively prime, positive integers \(m, n\) and a complex number \(\lambda\). Then there is a unilateral shift \(\tilde{V} \in \mathcal{B}(\tilde{H})\) such that \(H \subset \tilde{H}\) and

\[ \text{Lat}(V_1, V_2) = \{ H \cap N : N \in \text{Lat}(\tilde{V}^m, \tilde{V}^m) \} \]

where \(\text{Lat}(\tilde{V}^n, \tilde{V}^m)\) is described.
Thank You!