Characterization of invariant subspaces in the polydisc

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(Joint work with Aneesh M., Jaydeb Sarkar & Sankar T. R.)

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Characterization of invariant subspaces . . .

Aim

- To give a complete characterization of (joint) invariant subspaces for \((M_{z_1}, \ldots, M_{z_n})\) on the Hardy space \(H^2(\mathbb{D}^n)\) over the unit polydisc \(\mathbb{D}^n\) in \(\mathbb{C}^n\), \(n > 1\).
- To discuss about a complete set of unitary invariants for invariant subspaces as well as unitarily equivalent invariant subspaces.
- To classify a large class of \(n\)-tuples of commuting isometries.
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- To discuss about a complete set of unitary invariants for invariant subspaces as well as unitarily equivalent invariant subspaces.
- To classify a large class of $n$-tuples of commuting isometries.
Unit polydisc $\mathbb{D}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1, \ldots, n\}$.

Hardy space $H^2(\mathbb{D}) = \{f = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$.

Vector-valued Hardy space

$$H^2_{\mathcal{E}}(\mathbb{D}) = \{f = \sum_{n=0}^{\infty} a_n z^n : a_n \in \mathcal{E} \text{ and } \sum_{n=0}^{\infty} \|a_n\|_{\mathcal{E}}^2 < \infty\},$$

where $\mathcal{E}$ is some Hilbert space.

$M_z$ denote the multiplication operator on $H^2_{\mathcal{E}}(\mathbb{D})$ defined by

$$(M_z f)(w) = w f(w) \quad (f \in H^2_{\mathcal{E}}(\mathbb{D}), w \in \mathbb{D}).$$

$H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ denote the space of bounded $\mathcal{B}(\mathcal{E})$-valued holomorphic functions on $\mathbb{D}$. 
Invariant subspaces: Motivation

A closed subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$ is said to be invariant subspace under $T \in B(\mathcal{H})$ if $T(\mathcal{M}) \subseteq \mathcal{M}$. 

One of the most famous open problems in operator theory and function theory is the so-called invariant subspace problem: Does every bounded linear operator have a non-trivial closed invariant subspace?
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- Given an invariant subspace $\mathcal{M}$ of $T \in B(\mathcal{H})$, one can view $T$ as an operator matrix with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$

$$
\begin{bmatrix}
T|_\mathcal{M} & * \\
0 & *
\end{bmatrix}.
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One of the most famous open problems in operator theory and function theory is the so-called invariant subspace problem: Does every bounded linear operator have a non-trivial closed invariant subspace?
The celebrated Beurling theorem (1949) says that a non-zero closed subspace \( S \) of \( H^2(\mathbb{D}) \) is invariant for \( M_z \) if and only if there exists an inner function \( \theta \in H^\infty(\mathbb{D}) \) such that
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S = \theta H^2(\mathbb{D}).
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The celebrated Beurling theorem (1949) says that a non-zero closed subspace $S$ of $H^2(\mathbb{D})$ is invariant for $M_z$ if and only if there exists an inner function $\theta \in H^\infty(\mathbb{D})$ such that
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One may now ask whether an analogous characterization holds for (joint) invariant subspaces for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2(\mathbb{D}^n)$, $n > 1$, i.e.,
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  \[ S = \psi H^2(\mathbb{D}^n), \]

where $\psi \in H^\infty(\mathbb{D}^n)$ is an inner function.
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- Rudin’s pathological examples (Rudin (1969)): There exist invariant subspaces $S_1$ and $S_2$ for $(M_{z_1}, M_{z_2})$ on $H^2(\mathbb{D}^2)$ such that
  (1) $S_1$ is not finitely generated, and
  (2) $S_2 \cap H^\infty(\mathbb{D}^2) = \{0\}$. 
Invariant subspaces: Motivation

To understand the structure of a large class of operators we need to understand the structure of shift invariant subspaces for the vector-valued Hardy space over the unit disc.

Theorem (Beurling-Lax-Halmos)

A non-zero closed subspace \( S \) of \( H^2_\mathcal{E}(\mathbb{D}) \) is invariant for \( M_z \) if and only if there exists a closed subspace \( \mathcal{F} \subseteq \mathcal{E} \) and an inner function \( \Theta \in H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D}) \) such that

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S = \Theta H^2_\mathcal{F}(\mathbb{D}).
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Invariant subspaces: Motivation

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Theorem (Beurling-Lax-Halmos)

A non-zero closed subspace $S$ of $H^2(E(D))$ is invariant for $M_z$ if and only if there exists a closed subspace $F \subseteq E$ and an inner function $\Theta \in H^\infty(\mathcal{B}(F,E))$ such that

$$S = \Theta H^2_F(D).$$

Idea

- Identify Hardy space over polydisc $H^2(D^{n+1})$ to the $H^2(D^n)$-valued Hardy space over disc $H^2_{H^2(D^n)}(D)$. 
Invariant subspaces: Motivation

- To understand the structure of a large class of operators we need to understand the structure of shift invariant subspaces for the vector-valued Hardy space over the unit disc.

**Theorem (Beurling-Lax-Halmos)**

A non-zero closed subspace $S$ of $H^2_E(\mathbb{D})$ is invariant for $M_z$ if and only if there exists a closed subspace $\mathcal{F} \subseteq \mathcal{E}$ and an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{F}, \mathcal{E})}(\mathbb{D})$ such that

$$S = \Theta H^2_{\mathcal{F}}(\mathbb{D}).$$

**Idea**

- Identify Hardy space over polydisc $H^2(\mathbb{D}^{n+1})$ to the $H^2(\mathbb{D}^n)$-valued Hardy space over disc $H^2_{H^2(\mathbb{D}^n)}(\mathbb{D})$.

- Represent $(M_{z_1}, M_{z_2}, \ldots, M_{z_{n+1}})$ on $H^2(\mathbb{D}^{n+1})$ to $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{H^2(\mathbb{D}^n)}(\mathbb{D})$, where $\kappa_i \in H^\infty_{\mathcal{B}(H^2(\mathbb{D}^n))}(\mathbb{D})$, $i = 1, \ldots, n$, is a constant as well as simple and explicit $\mathcal{B}(H^2(\mathbb{D}^n))$-valued analytic function.
A closed subspace $S \subseteq H^2_{\mathcal{E}}(\mathbb{D}^n)$ is called a (joint) invariant subspace for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ if

$$z_i S \subseteq S,$$

for all $i = 1, \ldots, n.$
A closed subspace $S \subseteq H^2_\mathbb{C}(\mathbb{D}^n)$ is called a \textit{(joint) invariant subspace} for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2_\mathbb{C}(\mathbb{D}^n)$ if

$$z_i S \subseteq S,$$

for all $i = 1, \ldots, n$.

An isometry $V$ on $\mathcal{H}$ is called a pure isometry (or shift) if $V^* m \to 0$ in SOT.
Basic definitions

- A closed subspace \( S \subseteq H^2_\mathbb{D}(\mathbb{D}^n) \) is called a (joint) invariant subspace for 
  \((M_{z_1}, \ldots, M_{z_n})\) on \( H^2_\mathbb{D}(\mathbb{D}^n) \) if 
  \[ z_i S \subseteq S, \]
  for all \( i = 1, \ldots, n. \)
- An isometry \( V \) on \( \mathcal{H} \) is called a pure isometry (or shift) if \( V^{*m} \to 0 \) in SOT.
- Let \( V \) be a pure isometry on \( \mathcal{H} \). Then 
  \[ \mathcal{H} = \bigoplus_{m=0}^{\infty} V^m \mathcal{W}, \]
  where \( \mathcal{W} = \ker V^* = \mathcal{H} \ominus V\mathcal{H}. \)
Basic definitions

- A closed subspace $S \subseteq H^2_\mathcal{E}(\mathbb{D}^n)$ is called a (joint) invariant subspace for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2_\mathcal{E}(\mathbb{D}^n)$ if $z_i S \subseteq S$, for all $i = 1, \ldots, n$.
- An isometry $V$ on $\mathcal{H}$ is called a pure isometry (or shift) if $V^* m \to 0$ in SOT.
- Let $V$ be a pure isometry on $\mathcal{H}$. Then
  $$\mathcal{H} = \bigoplus_{m=0}^{\infty} V^m \mathcal{W},$$
  where $\mathcal{W} = \ker V^* = \mathcal{H} \ominus V \mathcal{H}$.
- The natural map $\Pi_V : \mathcal{H} \to H^2_{\mathcal{W}}(\mathbb{D})$ defined by
  $$\Pi_V(V^m \eta) = z^m \eta,$$
  for all $m \geq 0$ and $\eta \in \mathcal{W}$, is a unitary operator and
  $$\Pi_V V = M_z \Pi_V.$$

We call $\Pi_V$ the Wold-von Neumann decomposition of the shift $V$. 

Commutator of shift

Theorem 1 (Maji, Sarkar, & Sankar’ 18)

Let $V$ be a pure isometry on $H$, and let $C$ be a bounded operator on $H$. Let $P_iV$ be the Wold-von Neumann decomposition of $V$. Let $\mathcal{W} = \text{Ker}(V^*)$ and assume that $M = \Pi_V C \Pi_V^*$. Then

$$CV = VC,$$

if and only if

$$M = M_\Theta,$$

where

$$\Theta(z) = P_\mathcal{W}(I_H - zV^*)^{-1} C |_{\mathcal{W}} \quad (z \in \mathbb{D}).$$

Remark: In addition if $CV^* = V^*C$, then $\Theta(z) = C|_W = \Theta(0) \quad (z \in \mathbb{D})$, as $C(I - VV^*) = (I - VV^*)C$ and $V^*m|_W = 0$ for all $m \geq 1$. 

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Theorem 1 (Maji, Sarkar, & Sankar’ 18)

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Remark

In addition if $CV^* = V^* C$, then

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as $C(I - VV^*) = (I - VV^*)C$ and $V^m \big|_{\mathcal{W}} = 0$ for all $m \geq 1$. 
Preparation for main result

For the sake of simplicity we discuss firstly for \( n = 2 \), i.e., vector-valued Hardy space over the bidisc \( H^2_\mathcal{E}(\mathbb{D}^2) \).
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- We now identify \( H^2_\mathcal{E}(\mathbb{D}^2) \) with \( H^2_{H^2_\mathcal{E}(\mathbb{D})}(\mathbb{D}) \) by the canonical unitaries

\[
H^2_\mathcal{E}(\mathbb{D}^2) \xrightarrow{\hat{U}} H^2(\mathbb{D}) \otimes H^2_\mathcal{E}(\mathbb{D}) \xrightarrow{\tilde{U}} H^2_{H^2_\mathcal{E}(\mathbb{D})}(\mathbb{D})
\]

where

\[
\hat{U}(z_1^{k_1}z_2^{k_2}\eta) = z_1^{k_1} \otimes (z_2^{k_2}\eta), \quad (k_1, k_2 \geq 0, \eta \in \mathcal{E})
\]

and

\[
\tilde{U}(z^k \otimes \zeta) = z^k \zeta, \quad (k \geq 0, \zeta \in H^2_\mathcal{E}(\mathbb{D}))
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and

$$\tilde{U}(z^k \otimes \zeta) = z^k \zeta, \quad (k \geq 0, \zeta \in H^2_E(\mathbb{D})).$$

- Set $U = \tilde{U} \hat{U}$. Then it follows that $U : H^2_E(\mathbb{D}^2) \to H^2_{H^2_E(\mathbb{D})}(\mathbb{D})$ is a unitary operator. Since

$$\hat{U} M_{z_1} = (M_z \otimes I_{H^2_E(\mathbb{D})}) \hat{U} \quad \text{and} \quad \hat{U} M_{z_2} = (I_{H^2(\mathbb{D})} \otimes M_{z_1}) \hat{U},$$

we have $UM_{z_1} = M_z U$, and $UM_{z_2} = M_{\kappa_1} U$, where $\kappa_1(w) = M_{z_1}$ for $w \in \mathbb{D}$. 
Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{E}_n = H^2(\mathbb{D}^n) \otimes \mathcal{E}$ for $n \geq 1$. Let $\kappa_i \in H^\infty_{\mathcal{B}(\mathcal{E}_n)}(\mathbb{D})$ denote the $\mathcal{B}(\mathcal{E}_n)$-valued constant function on $\mathbb{D}$ defined by

$$\kappa_i(w) = M_{z_i} \in \mathcal{B}(\mathcal{E}_n),$$

for all $w \in \mathbb{D}$, and let $M_{\kappa_i}$ denote the multiplication operator on $H^2_{\mathcal{E}_n}(\mathbb{D})$ defined by

$$M_{\kappa_i} f = \kappa_i f,$$

for all $f \in H^2_{\mathcal{E}_n}(\mathbb{D})$ and $i = 1, \ldots, n$. Then

Theorem 2 (Maji, Aneesh, Sarkar, & Sankar’ 18)

(i) $(M_{z_1}, M_{z_2}, \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent.
Let $S \subseteq H^2_{H^2_{\mathcal{E}}(\mathbb{D})} \mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1})$ on $H^2_{H^2_{\mathcal{E}}(\mathbb{D})} \mathbb{D})$. Set

$$V = M_z|_S \text{ and } V_1 = M_{\kappa_1}|_S.$$
Preparation for main result

Let $S \subseteq H^2_{H^2(D)}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1})$ on $H^2_{H^2(D)}(\mathbb{D})$. Set

$$V = M_z|_S \quad \text{and} \quad V_1 = M_{\kappa_1}|_S.$$ 

Let $\Pi_V : S \to H^2_{\mathcal{W}}(\mathbb{D})$ be the Wold-von Neumann decomposition of $V$ on $S$. Then

$$\Pi_V V \Pi_V^* = M_z \quad \text{and} \quad \Pi_V V_1 \Pi_V^* = M_{\Phi_1},$$

where

$$\Phi_1(w) = P_{\mathcal{W}}(I_S - wV^*)^{-1}V_1|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$, $\Phi_1 \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$. 

Let $i_S$ denote the inclusion map $i_S : S \hookrightarrow H^2_{H^2(D)}(\mathbb{D})$. Then

$$H^2_{\mathcal{W}}(\mathbb{D}) \xrightarrow{\Pi_V} S \xleftarrow{i_S} H^2_{H^2(D)}(\mathbb{D}) \text{ i.e., } \Pi_S = i_S \circ \Pi_V : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{H^2(D)}(\mathbb{D}) \text{ is an isometry and } \text{ran } \Pi_S = S.$$
Preparation for main result

Let $S \subseteq H^2_{H^2_\mathbb{D}}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1})$ on $H^2_{H^2_\mathbb{D}}(\mathbb{D})$. Set

$$V = M_z|_S \quad \text{and} \quad V_1 = M_{\kappa_1}|_S.$$

Let $\Pi_V : S \rightarrow H^2_{\mathcal{W}}(\mathbb{D})$ be the Wold-von Neumann decomposition of $V$ on $S$. Then

$$\Pi_V V \Pi^*_V = M_z \quad \text{and} \quad \Pi_V V_1 \Pi^*_V = M_{\Phi_1},$$

where

$$\Phi_1(w) = P_{\mathcal{W}}(I_S - wV^*)^{-1}V_1|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$, $\Phi_1 \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$.

Let $i_S$ denote the inclusion map $i_S : S \hookrightarrow H^2_{H^2_\mathbb{D}}(\mathbb{D})$. Then

$$H^2_{\mathcal{W}}(\mathbb{D}) \xrightarrow{\Pi^*_V} S \xrightarrow{i_S} H^2_{H^2_\mathbb{D}}(\mathbb{D})$$

i.e., $\Pi_S = i_S \circ \Pi^*_V : H^2_{\mathcal{W}}(\mathbb{D}) \rightarrow H^2_{H^2_\mathbb{D}}(\mathbb{D})$ is an isometry and

$$\text{ran } \Pi_S = S.$$
Main result

We have invariant subspace result on vector-valued Hardy space over polydisc setting:

**Theorem 2 (Maji, Aneesh, Sarkar, & Sankar’ 18)**

(ii) Let $E$ be a Hilbert space, $S \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be a closed subspace, and let $W = S \ominus zS$. Then $S$ is invariant for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ if and only if $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is an *n-tuple of commuting shifts* on $H^2_W(\mathbb{D})$ and there exists an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W},\mathcal{E}_n)}(\mathbb{D})$ such that

$$S = \Theta H^2_W(\mathbb{D}),$$

and

$$\kappa_i \Theta = \Theta \Phi_i,$$

where

$$\Phi_i(w) = P_W(l_S - wP_SM_z^*)^{-1}M_{\kappa_i}|_W,$$

for all $w \in \mathbb{D}$ and $i = 1, \ldots, n$. 

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Remarks

One obvious necessary condition for a closed subspace \( S \) of \( H^2_{\mathbb{C}^n}(\mathbb{D}) \) to be (joint) invariant for \( (M_z, M_{\kappa_1}, \ldots, M_{\kappa_n}) \) is that \( S \) is invariant for \( M_z \), and, consequently

\[
S = \Theta H^2_{\mathcal{W}}(\mathbb{D}),
\]

where \( \mathcal{W} = S \ominus zS \) and \( \Theta \in H^\infty_{B(\mathcal{W}, \mathbb{C})}(\mathbb{D}) \) is the Beurling, Lax and Halmos inner function.
Remarks

- One obvious necessary condition for a closed subspace $S$ of $H^2_{\mathbb{C}^n}(\mathbb{D})$ to be (joint) invariant for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ is that $S$ is invariant for $M_z$, and, consequently

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- Again $\kappa_i S \subseteq S$, $\kappa_i \Theta = \Theta \Gamma_i,$

  for some $\Gamma_i \in \mathcal{B}(H^2_{\mathcal{W}}(\mathbb{D})), i = 1, \ldots, n$ (by Douglas’s range inclusion theorem).
Remarks

- One obvious necessary condition for a closed subspace $S$ of $H^2_{\mathcal{E}_n}(\mathbb{D})$ to be (joint) invariant for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ is that $S$ is invariant for $M_z$, and, consequently

$$S = \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$

where $\mathcal{W} = S \ominus zS$ and $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}, \mathcal{E}_n)}(\mathbb{D})$ is the Beurling, Lax and Halmos inner function.

- Again $\kappa_i S \subseteq S$, $\implies$ 

$$\kappa_i \Theta = \Theta \Gamma_i,$$

for some $\Gamma_i \in \mathcal{B}(H^2_{\mathcal{W}}(\mathbb{D})), i = 1, \ldots, n$ (by Douglas’s range inclusion theorem).

- In the above theorem, we prove that $\Gamma_i$ is explicit, that is

$$\Gamma_i = \Phi_i \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D}),$$

for all $i = 1, \ldots, n$, and $(\Gamma_1, \ldots, \Gamma_n)$ is an $n$-tuple of commuting shifts on $H^2_{\mathcal{W}}(\mathbb{D})$. 
Uniqueness

Let $S$ be an invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{E_n}(\mathbb{D})$. Then $S = \Theta H^2_{W}(\mathbb{D})$ and

$$\kappa_i \Theta = \Theta \Phi_i \quad (i = 1, \ldots, n),$$

from the above Theorem.
Let $S$ be an invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Then $S = \Theta H^2_{\mathcal{W}}(\mathbb{D})$ and

$$\kappa_i \Theta = \Theta \Phi_i \quad (i = 1, \ldots, n),$$

from the above Theorem.

Now suppose $S = \tilde{\Theta} H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ and $\kappa_i \tilde{\Theta} = \tilde{\Theta} \tilde{\Phi}_i$ for some Hilbert space $\tilde{\mathcal{W}}$, inner function $\tilde{\Theta} \in H^\infty_{B(\tilde{\mathcal{W}})}(\mathbb{D})$ and shift $M_{\tilde{\Phi}_i}$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$, $i = 1, \ldots, n$. 
Uniqueness

Let $S$ be an invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Then $S = \Theta H^2_{\mathcal{W}}(\mathbb{D})$ and

$$\kappa_i \Theta = \Theta \Phi_i \quad (i = 1, \ldots, n),$$

from the above Theorem.

Now suppose $S = \tilde{\Theta} H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ and $\kappa_i \tilde{\Theta} = \tilde{\Theta} \tilde{\Phi}_i$ for some Hilbert space $\tilde{\mathcal{W}}$, inner function $\tilde{\Theta} \in H^\infty_{B(\tilde{\mathcal{W}})}(\mathbb{D})$ and shift $M_{\tilde{\Phi}_i}$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$, $i = 1, \ldots, n$.

Then there exists a unitary operator $\tau : \mathcal{W} \rightarrow \tilde{\mathcal{W}}$ such that

$$\Theta = \tilde{\Theta} \tau,$$

and

$$\tau \Phi_i = \tilde{\Phi}_i \tau,$$

for all $i = 1, \ldots, n$. 
Theorem (Maji, Aneesh, Sarkar, & Sankar’ 18)

Let $\mathcal{E}$ be a Hilbert space, and let $S_1 = \Theta_1 H^2_{W_1}(\mathbb{D})$ and $S_2 = \Theta_2 H^2_{W_2}(\mathbb{D})$ be two invariant subspaces for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $W_j = S_j \ominus zS_j$ for $j = 1, 2$. Let

$$\Phi_{j,i}(w) = P_{W_j}(I_{S_j} - wP_{S_j}M_z^*)^{-1}M_{\kappa_i}|_{W_j},$$

for all $w \in \mathbb{D}$, $j = 1, 2$, and $i = 1, \ldots, n$. Then $S_1 \subseteq S_2$ if and only if there exists an inner multiplier $\Psi \in H^\infty_{\mathbb{B}(W_1, W_2)}(\mathbb{D})$ such that $\Theta_1 = \Theta_2 \Psi$ and $\Psi \Phi_{1,i} = \Phi_{2,i} \Psi$ for all $i = 1, \ldots, n$. 
Theorem (Maji, Aneesh, Sarkar, & Sankar’ 18)

Let $\mathcal{E}$ be a Hilbert space, and let $S_1 = \Theta_1 H^2_{\mathcal{W}_1}(\mathbb{D})$ and $S_2 = \Theta_2 H^2_{\mathcal{W}_2}(\mathbb{D})$ be two invariant subspaces for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $\mathcal{W}_j = S_j \ominus zS_j$ for $j = 1, 2$. Let

$$\Phi_{j,i}(w) = P_{\mathcal{W}_j}(I_{S_j} - wP_{S_j}M_z^*)^{-1}M_{\kappa_i}|_{\mathcal{W}_j},$$

for all $w \in \mathbb{D}$, $j = 1, 2$, and $i = 1, \ldots, n$.

Then $S_1 \subseteq S_2$ if and only if there exists an inner multiplier $\Psi \in H^\infty_{\mathcal{B}(\mathcal{W}_1, \mathcal{W}_2)}(\mathbb{D})$ such that $\Theta_1 = \Theta_2 \Psi$ and $\Psi \Phi_{1,i} = \Phi_{2,i} \Psi$ for all $i = 1, \ldots, n$. 
Unitarily equivalent invariant subspaces

**Definition**

Let $S$ and $	ilde{S}$ be invariant subspaces for the $(n + 1)$-tuples of multiplication operators $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{E_n}(\mathbb{D})$ and $H^2_{\tilde{E}_n}(\mathbb{D})$, respectively. We say that $S$ and $	ilde{S}$ are unitarily equivalent, and write $S \sim \tilde{S}$, if there is a unitary map $U : S \to \tilde{S}$ such that

$$UM_z|_S = M_z|_{\tilde{S}}U \quad \text{and} \quad UM_{\kappa_i}|_S = M_{\kappa_i}|_{\tilde{S}}U,$$

for all $i = 1, \ldots, n$. 

**Identification**

There exists a unitary operator $U_{E_n} : H^2_{E_n}(\mathbb{D}) \to H^2_{E_n}(\mathbb{D})$ such that

$$U_{E_n} M_z = M_z U_{E_n},$$

and

$$U_{E_n} M_{\kappa_i} = M_{\kappa_i} U_{E_n},$$

for all $i = 1, \ldots, n$. 

Amit Maji (IIIT G)
Unitarily equivalent invariant subspaces

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Let $S$ and $\tilde{S}$ be invariant subspaces for the $(n+1)$-tuples of multiplication operators $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $H^2_{\mathcal{E}_n}(\mathbb{D})$, respectively. We say that $S$ and $\tilde{S}$ are unitarily equivalent, and write $S \cong \tilde{S}$, if there is a unitary map $U : S \to \tilde{S}$ such that

$$UM_z|_S = M_z|_{\tilde{S}} U \quad \text{and} \quad UM_{\kappa_i}|_S = M_{\kappa_i}|_{\tilde{S}} U,$$

for all $i = 1, \ldots, n$.

**Identification**

There exists a unitary operator $U_\mathcal{E} : H^2_{\mathcal{E}_n}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{E}_n}(\mathbb{D})$ such that

$$U_\mathcal{E} M_{z_1} = M_z U_\mathcal{E},$$

and

$$U_\mathcal{E} M_{z_{i+1}} = M_{\kappa_i} U_\mathcal{E},$$

for all $i = 1, \ldots, n$. 
Intertwining maps

Let $\mathcal{F}$ be another Hilbert space, and let $X : H^2_\mathcal{E}(\mathbb{D}^{n+1}) \to H^2_\mathcal{F}(\mathbb{D}^{n+1})$ be a bounded linear operator such that

$$XM_{zi} = M_{zi}X,$$

for all $i = 1, \ldots, n + 1$. Set

$$X_n = U_{\mathcal{F}}XU_{\mathcal{E}}^*.$$

Then $X_n : H^2_\mathcal{E}(\mathbb{D}) \to H^2_\mathcal{F}(\mathbb{D})$ is bounded and

$$X_nM_z = M_zX_n \quad \text{and} \quad X_nM_{\kappa_i} = M_{\kappa_i}X_n,$$

for all $i = 1, \ldots, n$.

**Definition**

Any map satisfying (0.2) will be referred to *module maps*. 
Unitarily equivalent invariant subspaces

Theorem (Maji, Aneesh, Sarkar, & Sankar’ 18)

Let $S \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$. Then $S \cong H^2_{\mathcal{E}_n}(\mathbb{D})$ if and only if there exists an isometric module map $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ such that

$$S = X_n H^2_{\mathcal{E}_n}(\mathbb{D}).$$

Moreover, in this case

$$\dim \mathcal{E} \leq \dim \mathcal{F}.$$
Unitarily equivalent invariant subspaces

**Theorem (Maji, Aneesh, Sarkar, & Sankar’ 18)**

Let $S \subseteq H^2_{\mathbb{F}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathbb{F}_n}(\mathbb{D})$. Then $S \cong H^2_{\mathcal{E}_n}(\mathbb{D})$ if and only if there exists an isometric module map $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathbb{F}_n}(\mathbb{D})$ such that

$$S = X_n H^2_{\mathcal{E}_n}(\mathbb{D}).$$

Moreover, in this case

$$\dim \mathcal{E} \leq \dim \mathcal{F}.$$  

**Corollary**

Let $S \subseteq H^2_{\mathcal{H}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{H}_n}(\mathbb{D})$. Then $S \cong H^2_{\mathcal{H}_n}(\mathbb{D})$ if and only if there exists an isometric module map $X_n : H^2_{\mathcal{H}_n}(\mathbb{D}) \to H^2_{\mathcal{H}_n}(\mathbb{D})$ such that

$$S = X_n(H^2_{\mathcal{H}_n}(\mathbb{D})).$$

The above corollary was first observed by Agrawal, Clark and Douglas (1986).
A complete set of unitary invariants

Definition

Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be Hilbert spaces, and let $\{\psi_1, \ldots, \psi_n\} \subseteq H^\infty_{B(\mathcal{E})}(\mathbb{D})$ and $\{\tilde{\psi}_1, \ldots, \tilde{\psi}_n\} \subseteq H^\infty_{\tilde{B}(\tilde{\mathcal{E}})}(\mathbb{D})$. We say that $\{\psi_1, \ldots, \psi_n\}$ and $\{\tilde{\psi}_1, \ldots, \tilde{\psi}_n\}$ coincide if there exists a unitary operator $\tau : \mathcal{E} \to \tilde{\mathcal{E}}$ such that

$$\tau \psi_i(z) = \tilde{\psi}_i(z) \tau,$$

for all $z \in \mathbb{D}$ and $i = 1, \ldots, n$. 
A complete set of unitary invariants

Definition

Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be Hilbert spaces, and let $\{\psi_1, \ldots, \psi_n\} \subseteq H^\infty_B(\mathcal{E})$ and $\{\tilde{\psi}_1, \ldots, \tilde{\psi}_n\} \subseteq H^\infty_B(\tilde{\mathcal{E}})$. We say that $\{\psi_1, \ldots, \psi_n\}$ and $\{\tilde{\psi}_1, \ldots, \tilde{\psi}_n\}$ coincide if there exists a unitary operator $\tau : \mathcal{E} \to \tilde{\mathcal{E}}$ such that

$$\tau \psi_i(z) = \tilde{\psi}_i(z) \tau,$$

for all $z \in \mathbb{D}$ and $i = 1, \ldots, n$.

Theorem (Maji, Aneesh, Sarkar, & Sankar’ 18)

Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be Hilbert spaces. Let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ and $\tilde{\mathcal{S}} \subseteq H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$ be invariant subspaces for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$, respectively. Then $\mathcal{S} \cong \tilde{\mathcal{S}}$ if and only if $\{\phi_1, \ldots, \phi_n\}$ and $\{\tilde{\phi}_1, \ldots, \tilde{\phi}_n\}$ coincide.
Representations of model isometries

Question

Given a Hilbert space $\mathcal{E}$, characterize $(n+1)$-tuples of commuting shifts on Hilbert spaces that are unitarily equivalent to $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n} (\mathbb{D})$. 

Answer

Answer to this question is related to (numerical invariant) the rank of an operator associated with the Szegő kernel on $\mathbb{D}^n+1$. 

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Characterization of invariant subspaces in the polydisc
Question

Given a Hilbert space $\mathcal{E}$, characterize $(n + 1)$-tuples of commuting shifts on Hilbert spaces that are unitarily equivalent to $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$.

Answer

Answer to this question is related to (numerical invariant) the rank of an operator associated with the Szegö kernel on $\mathbb{D}^{n+1}$.
Representations of model isometries

**Definition**

The *defect operator* corresponding to an $m$-tuple of commuting contractions $(T_1, \ldots, T_m)$ on a Hilbert space $\mathcal{H}$ is defined (see, Guo & Yang (2004)) as

$$S_m^{-1}(T_1, \ldots, T_m) = \sum_{0 \leq |k| \leq m} (-1)^{|k|} T_1^{k_1} \ldots T_m^{k_m} T_{1}^{*k_1} \ldots T_{m}^{*k_m},$$

where $|k| = k_1 + k_2 + \ldots + k_m$, $0 \leq k_i \leq 1$, $i = 1, \ldots, m$. 

We say that $(T_1, \ldots, T_m)$ is of rank $p$ ($p \in \mathbb{N} \cup \{\infty\}$) if $\text{rank } [S_m^{-1}(T_1, \ldots, T_m)] = p$, and we write $\text{rank } (T_1, \ldots, T_m) = p$. 

Amit Maji (IIIT G)  
Characterization of invariant subspaces in the polydisc
Representations of model isometries

Definition

The **defect operator** corresponding to an \( m \)-tuple of commuting contractions \((T_1, \ldots, T_m)\) on a Hilbert space \( \mathcal{H} \) is defined (see, Guo & Yang (2004)) as

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\]

where \( |k| = k_1 + k_2 + \ldots + k_m, \ 0 \leq k_i \leq 1, \ i = 1, \ldots, m. \)

Definition

We say that \((T_1, \ldots, T_m)\) is of rank \( p \) \((p \in \mathbb{N} \cup \{\infty\})\) if

\[
\text{rank} [\mathbb{S}_m^{-1}(T_1, \ldots, T_m)] = p,
\]

and we write

\[
\text{rank} (T_1, \ldots, T_m) = p.
\]
Let \((V, V_1, \ldots, V_n)\) be an \((n+1)\)-tuple of doubly commuting shifts on \(\mathcal{H}\). Then Sarkar (2014) proved that \((V, V_1, \ldots, V_n)\) on \(\mathcal{H}\) and \((M_{z_1}, \ldots, M_{z_{n+1}})\) on \(H^2_D(D^{n+1})\) are unitarily equivalent, where

\[
D = \text{ran } S_{n+1}^{-1}(V, V_1, \ldots, V_n) = \left( \bigcap_{i=1}^{n} \ker V_i^* \right) \cap \ker V^*.
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Representations of model isometries

Let \((V, V_1, \ldots, V_n)\) be an \((n + 1)\)-tuple of doubly commuting shifts on \(\mathcal{H}\). Then Sarkar (2014) proved that \((V, V_1, \ldots, V_n)\) on \(\mathcal{H}\) and \((M_{z_1}, \ldots, M_{z_{n+1}})\) on \(H^2_D(\mathbb{D}^{n+1})\) are unitarily equivalent, where

\[
\mathcal{D} = \text{ran } S^{-1}_{n+1}(V, V_1, \ldots, V_n) = \left( \bigcap_{i=1}^{n} \ker V_i^* \right) \cap \ker V^*.
\]

Theorem (Maji, Aneesh, Sarkar, & Sankar’ 18)

Let \((V, V_1, \ldots, V_n)\) be an \((n + 1)\)-tuple of doubly commuting shifts on some Hilbert space \(\mathcal{H}\). Let \(\mathcal{W} = \mathcal{H} \ominus V\mathcal{H}\), and let

\[
\Psi_i(z) = V_i|_{\mathcal{W}} \quad (i = 1, \ldots, n),
\]

for all \(z \in \mathbb{D}\). Then \((V, V_1, \ldots, V_n)\) on \(\mathcal{H}\), \((M_z, M_{\Psi_1}, \ldots, M_{\Psi_n})\) on \(H^2_{\mathcal{W}}(\mathbb{D})\), and \((M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})\) on \(H^2_{\mathcal{E}_n}(\mathbb{D})\) are unitarily equivalent, where \(\mathcal{E}\) is a Hilbert space and \(\text{dim } \mathcal{E} = \text{rank } (V, V_1, \ldots, V_n)\).


Thank You!