Composition operators which are similar to an isometry on various Banach spaces $X \hookrightarrow \text{Hol}(\mathbb{D})$

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(Joint work with W. Arendt, I. Chalendar, S. Srivastava)
Let $\operatorname{Hol}(\mathbb{D})$ denote the space of all holomorphic functions on $\mathbb{D}$, where $\mathbb{D}$ is the open unit disc of $\mathbb{C}$. Then
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1. $(X, \| \cdot \|_X)$ is a Banach space,
2. $X \subset \text{Hol}(\mathbb{D})$. 

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Composition operators similar to an isometry
Let $\operatorname{Hol}(\mathbb{D})$ denote the space of all holomorphic functions on $\mathbb{D}$, where $\mathbb{D}$ is the open unit disc of $\mathbb{C}$. Then $(X, \| \cdot \|_X)$ is called a \textit{Banach space of holomorphic functions} if

- $(X, \| \cdot \|_X)$ is a Banach space,
- $X \subset \operatorname{Hol}(\mathbb{D})$,
- the point evaluations $\delta_z \in X'$, $z \in \mathbb{D}$. 
Let \( \text{Hol}(\mathbb{D}) \) denote the space of all holomorphic functions on \( \mathbb{D} \), where \( \mathbb{D} \) is the open unit disc of \( \mathbb{C} \). Then \((X, \|\cdot\|_X)\) is called a *Banach space of holomorphic functions* if

- \((X, \|\cdot\|_X)\) is a Banach space,
- \(X \subset \text{Hol}(\mathbb{D})\),
- the point evaluations \( \delta_z \in X', \ z \in \mathbb{D} \).

We denote this by

\[ X \hookrightarrow \text{Hol}(\mathbb{D}). \]
Examples of $X \hookrightarrow \text{Hol}(\mathbb{D})$

- $H^p(\mathbb{D})$, $1 \leq p \leq \infty$ (Hardy spaces)
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- $A^p_\alpha(\mathbb{D}), \alpha > -1, 1 \leq p < \infty$ (*Standard weighted Bergman spaces*)

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- $\mathcal{D}$ (*Dirichlet space*)
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- $W(\mathbb{D})$ (*Wiener algebra*)
- $A(\mathbb{D})$ (*Disc algebra*)
- $\mathcal{B}$ (*Bloch space*)
- $\mathcal{B}_\alpha$, $0 < \alpha < \infty$ (*Bloch type spaces*)
Let \( \varphi : \mathbb{D} \rightarrow \mathbb{D} \) be holomorphic. Then the \textit{composition operator} \( C_\varphi : \text{Hol}(\mathbb{D}) \rightarrow \text{Hol}(\mathbb{D}) \) is defined by

\[
C_\varphi f = f \circ \varphi \quad \text{for all } f \in \text{Hol}(\mathbb{D}).
\]
Composition operators on $X \hookrightarrow \text{Hol}(\mathbb{D})$

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- Let $X \hookrightarrow \text{Hol}(\mathbb{D})$. Then by the Closed Graph Theorem, $C_\varphi X \subset X$, if and only if, $C_\varphi \in \mathcal{L}(X)$. 

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Composition operators similar to an isometry
Composition operators on \( X \hookrightarrow \text{Hol}(\mathbb{D}) \)

- Let \( \varphi : \mathbb{D} \rightarrow \mathbb{D} \) be holomorphic. Then the *composition operator* \( C_\varphi : \text{Hol}(\mathbb{D}) \rightarrow \text{Hol}(\mathbb{D}) \) is defined by

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- If \( \|C_\varphi f\|_X = \|f\|_X \) for all \( f \in X \), then \( C_\varphi \) is called an isometry of \( X \).
Composition operators on $X \hookrightarrow \text{Hol}(\mathbb{D})$

- Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then the composition operator $C_\varphi : \text{Hol}(\mathbb{D}) \to \text{Hol}(\mathbb{D})$ is defined by

$$C_\varphi f = f \circ \varphi \quad \text{for all } f \in \text{Hol}(\mathbb{D}).$$

- Let $X \hookrightarrow \text{Hol}(\mathbb{D})$. Then by the Closed Graph Theorem, $C_\varphi X \subset X$, if and only if, $C_\varphi \in \mathcal{L}(X)$.

- If $\|C_\varphi f\|_X = \|f\|_X$ for all $f \in X$, then $C_\varphi$ is called an isometry of $X$.

- Moreover if there exists an invertible $S \in \mathcal{L}(X)$ such that $C_\varphi = S^{-1}VS$, where $V$ is an isometry of $X$, then $C_\varphi$ is said to be similar to an isometry of $X$. 

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Composition operators similar to an isometry
Composition operators similar to an isometry of $H^p$

Let $\varphi$ be a holomorphic self map of $D$. The following assertions are equivalent on $H^p$, $1 \leq p < \infty$:

(i) $C_\varphi$ is similar to an isometry of $H^p$;

(ii) $\varphi$ is inner and has a fixed point in $D$. 

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Composition operators similar to an isometry
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Theorem (Bayart, 2002)

Let $\varphi$ be a holomorphic self map of $\mathbb{D}$. The following assertions are equivalent on $H^p$, $1 \leq p < \infty$:

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Composition operators similar to an isometry
Theorem (ACKS-2018)

The following assertions are equivalent on $X$.

(i) $C^n \phi$ converges strongly;

(ii) $\phi$ is not inner and there is $b \in D$ s.t. $\phi(b) = b$;

(iii) $C^n \phi$ converges uniformly.

In that case, $C^n \phi$ converges to $Pf = f(b) \mathbf{1}_D$ for all $f \in X$.

Theorem (Cowen, MacCluer, 95)

Let $\phi$ be a holomorphic self map of $D$. Then a composition operator $C_{\phi}$ is an isometry of $H^p$, $1 \leq p < \infty$ if and only if $\phi$ is an inner function and $\phi(0) = 0$. 

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In that case, $C^n_\varphi$ converges to $P$, where $Pf = f(b)1_{\mathbb{D}}$ for all $f \in X$. 
### Theorem (ACKS-2018)

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### Theorem (Cowen, MacCluer, 95)

Let $\varphi$ be a holomorphic self map of $\mathbb{D}$. Then a composition operator $C_\varphi$ is an isometry of $H^p$, $1 \leq p < \infty$ if and only if $\varphi$ is an inner function and $\varphi(0) = 0$. 
Let $X \in \{ A^p_\beta \ (1 \leq p < \infty, \ \beta > -1), \ H^\infty_{\nu_q} \ (q > 0), \ B_0, \ B^\alpha \ (\alpha > 0, \ \alpha \neq 1) \ \}$. 
Composition operators similar to an isometry

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Theorem (ACKS-2018)
Let \( \phi \) be a holomorphic self map of \( D \). Consider the composition operator \( C_\phi \) on \( X \). The following assertions are equivalent:

(i) \( C_\phi \) is similar to an isometry of \( X 
(ii) \phi \) is an elliptic automorphism.
Composition operators similar to an isometry

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**Theorem (ACKS-2018)**

Let \( \varphi \) be a holomorphic self map of \( \mathbb{D} \). Consider the composition operator \( C_\varphi \) on \( X \). The following assertions are equivalent:

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(ii) \( \varphi \) is an elliptic automorphism.
Sketch of the proof \( (X = A^p_\beta \text{ or } H^{\infty}_{\nu q} \text{ or } \mathcal{B}^\alpha, \alpha > 1) \)

Let \( X \) be \( A^p_\beta \text{ or } H^{\infty}_{\nu q} \text{ or } \mathcal{B}^\alpha \) and \( \phi : D \to D \) be holomorphic.

Theorem (ACKS-2018)

The following assertions are equivalent on \( X \).

(i) \( C^n \phi \) converges strongly;

(ii) \( \phi \) is not an automorphism and there is \( b \in D \) s.t. \( \phi(b) = b \);

(iii) \( C^n \phi \) converges uniformly.

In that case, \( C^n \phi \) converges to \( P \), where \( Pf = f(b)^{-1} \) for all \( f \in X \).


\( C \phi \) is an isometry of \( X \) if and only if \( \phi \) is a rotation.
Let $X$ be $A^p_\beta$ or $H^\infty_{\nu_q}$ or $\mathcal{B}^\alpha$ ($\alpha > 1$) and $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic.
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(i) $C^n_\varphi$ converges strongly;

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In that case, $C^n_\varphi$ converges to $P$, where $Pf = f(b)1_{\mathbb{D}}$ for all $f \in X$.


$C_\varphi$ is an isometry of $X$ if and only if $\varphi$ is a rotation.
Let $X$ be $A^p_\beta$ or $H^\infty_{\nu_q}$ or $B^\alpha$ ($\alpha > 1$) and $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic.

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$C_\varphi$ is an isometry of $X$ if and only if $\varphi$ is a rotation.
Sketch of the proof \((X = \mathcal{B}_0)\)

Let \(\phi\) be a holomorphic self map of \(D\) such that \(\phi \in \mathcal{B}_0\).

**Theorem (ACKS-2018)**

The following assertions are equivalent on \(\mathcal{B}_0\):

1. \(C_{\phi}\) is an isometry of \(\mathcal{B}_0\);
2. \(\phi(0) = 0\) and \(\tau_\infty \phi = 1\);
3. \(\phi\) is a rotation, where \(\tau_\infty \phi := \sup_{z \in D} \frac{|z|}{|\phi(z)|} - \frac{|z|}{|\phi'(z)|}\).

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Composition operators similar to an isometry
Let $\varphi$ be a holomorphic self map of $\mathbb{D}$ such that $\varphi \in \mathcal{B}_0$. 
Let $\varphi$ be a holomorphic self map of $\mathbb{D}$ such that $\varphi \in B_0$.

**Theorem (ACKS-2018)**

The following assertions are equivalent on $B_0$:

(i) $C_\varphi$ is an isometry of $B_0$;
(ii) $\varphi(0) = 0$ and $\tau_\varphi^\infty = 1$;
(iii) $\varphi$ is a rotation, where

$$
\tau_\varphi^\infty := \sup_{z \in \mathbb{D}} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} |\varphi'(z)|
$$
Sketch of the proof \((X = B_0)\)

Theorem (Rodríguez, 99)

Suppose that \(\phi\) is a holomorphic self map of \(D\). Then
\[
\|C\phi\|_{e, B_0} \leq \tau_\infty \phi \leq 1.
\]
Moreover, if \(\phi \in B_0\), then
\[
\|C\phi\|_{e, B_0} \leq \tau_\infty \phi \leq 1.
\]

Theorem (ACKS-2017)

Let \(X \hookrightarrow \text{Hol}(D)\) and \(\phi : D \to D\) be holomorphic s.t. \(C\phi(X) \subset X\) and that there exists \(b \in D\) s.t.
\[
\lim_{n \to \infty} \phi^n(z) = b \quad \text{for all } z \in D.
\]
Then the following assertions are equivalent:

(i) \(C_n\phi\) converges in \(L(X)\) as \(n \to \infty\);

(ii) \(\rho(C\phi) < 1\).

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Composition operators similar to an isometry
Theorem (Rodríguez, 99)

Suppose that $\varphi$ is a holomorphic self map of $\mathbb{D}$. Then

$$\| C_\varphi \|_{e,B} \leq \tau_\varphi^\infty \leq 1.$$  

Moreover, if $\varphi \in B_0$, then

$$\| C_\varphi \|_{e,B_0} \leq \tau_\varphi^\infty \leq 1.$$
Sketch of the proof ($X = \mathcal{B}_0$)

**Theorem (Rodríguez, 99)**

Suppose that $\varphi$ is a holomorphic self map of $\mathbb{D}$. Then

$$\|C_\varphi\|_{e, \mathcal{B}} \leq \tau_\varphi \leq 1.$$  

Moreover, if $\varphi \in \mathcal{B}_0$, then

$$\|C_\varphi\|_{e, \mathcal{B}_0} \leq \tau_\varphi \leq 1.$$  

**Theorem (ACKS-2017)**

Let $X \hookrightarrow \text{Hol}(\mathbb{D})$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic s.t. $C_\varphi(X) \subset X$ and that there exists $b \in \mathbb{D}$ s.t. $\lim_{n \to \infty} \varphi_n(z) = b$ for all $z \in \mathbb{D}$. Then the following assertions are equivalent:

(i) $C^n_\varphi$ converges in $\mathcal{L}(X)$ as $n \to \infty$;

(ii) $r_e(C_\varphi) < 1.$
Sketch of the proof ($X = B_0$)

Theorem (Allen, Collona, 2009)

Suppose that $\phi$ is a holomorphic self map of $D$. Then the operator $C_\phi$ on $B$ is isometric if and only if $\phi(0) = 0$ and one of the following equivalent conditions holds:

(i) $\tau_\infty \phi = 1$;

(ii) $\phi$ either is a rotation or for every $w \in D$, there exists $(a_n) \subset D$ such that $|a_n| \to 1$, $\phi(a_n) \to w$, and $\tau_\phi(a_n) \to 1$ as $n \to \infty$.

(iii) $\phi$ either is a rotation or the zeros of $\phi$ form an infinite sequence $(z_k)$ in $D$ s.t. $\limsup_{k \to \infty} (1 - |z_k|^2) |\phi'(z_k)| = 1$.

(iv) $\phi$ either is a rotation or $\phi = gB$, where $g$ is a non-vanishing analytic function mapping $D$ into itself and $B$ is an infinite Blaschke product whose zero set $Z$ contains a sequence $(z_k)$ such that $|g(z_k)| \to 1$ when $k \to \infty$ and $\lim_{k \to \infty} \prod_{\xi \in Z, \xi \neq z_k} \frac{|z_k - \xi|}{1 - \xi z_k} = 1$.
Sketch of the proof \((X = \mathcal{B}_0)\)

Theorem (Allen, Collona, 2009)

Suppose that \(\varphi\) is a holomorphic self map of \(\mathbb{D}\). Then the operator \(C_\varphi\) on \(\mathcal{B}\) is isometric if and only if \(\varphi(0) = 0\) and one of the following equivalent conditions holds:

(i) \(\tau_\varphi^\infty = 1\);

(ii) \(\varphi\) either is a rotation or for every \(w \in \mathbb{D}\), there exists \((a_n) \subset \mathbb{D}\) such that \(|a_n| \to 1\), \(\varphi(a_n) \to w\), and \(\tau_\varphi(a_n) \to 1\) as \(n \to \infty\).

(iii) \(\varphi\) either is a rotation or the zeros of \(\varphi\) form an infinite sequence \((z_k)\) in \(\mathbb{D}\) s.t. \(\lim\sup_{k \to \infty} (1 - |z_k|^2)|\varphi'(z_k)| = 1\).

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\[
\lim_{k \to \infty} \prod_{\xi \in Z, \xi \neq z_k} \left| \frac{z_k - \xi}{1 - \xi z_k} \right| = 1.
\]
Sketch of the proof \((X = B_0)\)

Theorem (ACKS-2018)

The following assertions are equivalent on \(B_0\).

(i) \(C_n\phi\) converges weakly;
(ii) \(\phi\) is not an automorphism and there is \(b \in D\) s.t. \(\phi(b) = b\);
(iii) \(C_n\phi\) converges uniformly.

In that case, \(C_n\phi\) converges to \(P\) as \(n \to \infty\), where \(P = f(b)\).
The following assertions are equivalent on $\mathcal{B}_0$.

(i) $C^n \varphi$ converges weakly;

(ii) $\varphi$ is not an automorphism and there is $b \in \mathbb{D}$ s.t. $\varphi(b) = b$;

(iii) $C^n \varphi$ converges uniformly.

In that case, $C^n \varphi$ converges to $P$ as $n \to \infty$, where $Pf = f(b)1_{\mathbb{D}}$. 
Sketch of the proof ($X = B^\alpha$, $0 < \alpha < 1$)

For $\alpha > 0$ and $\phi$ a holomorphic self map of $D$, let $\tau_{\infty} \phi, \alpha < \infty$, where $\tau_{\infty} \phi, \alpha := \sup_{z \in D} (1 - |z|^2)^{\alpha} |\phi'(z)| (1 - |\phi(z)|^2)^{\alpha}$.

Theorem (ACKS-2018)

Suppose there is $b \in D$ s.t. $\phi(b) = b$. The following assertions are equivalent on $B^\alpha, 0 < \alpha < 1$.

(i) $C_n \phi$ converges strongly;

(ii) there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(D) \subset D$;

(iii) $C_n \phi$ converges uniformly;

(iv) $C \phi$ is mean ergodic.

In that case, $C_n \phi$ converges to $P_{\mathcal{F}} f = f(b)$.
Sketch of the proof ($X = B^\alpha$, $0 < \alpha < 1$)

For $\alpha > 0$ and $\varphi$ a holomorphic self map of $\mathbb{D}$, let $\tau_{\varphi,\alpha} < \infty$, where $\tau_{\varphi,\alpha} := \sup_{z \in \mathbb{D}} \frac{(1-|z|^2)^\alpha |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}$.
Sketch of the proof \((X = \mathcal{B}^\alpha, 0 < \alpha < 1)\)

For \(\alpha > 0\) and \(\varphi\) a holomorphic self map of \(\mathbb{D}\), let \(\tau_{\varphi,\alpha}^\infty < \infty\), where

\[
\tau_{\varphi,\alpha}^\infty := \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\alpha |\varphi'(z)|}{(1 - |\varphi(z)|^2)^\alpha}.
\]

**Theorem (ACKS-2018)**

*Suppose there is \(b \in \mathbb{D}\) s.t. \(\varphi(b) = b\). The following assertions are equivalent on \(\mathcal{B}^\alpha, 0 < \alpha < 1\).*

(i) \(C^n_\varphi\) converges strongly;

(ii) there exists \(n_0 \in \mathbb{N}\) such that \(\varphi_{n_0}(\overline{\mathbb{D}}) \subset \mathbb{D}\);

(iii) \(C^n_\varphi\) converges uniformly;

(iv) \(C_\varphi\) is mean ergodic.

*In that case, \(C^n_\varphi\) converges to \(P\), where \(Pf = f(b)1_\mathbb{D}\).*
Sketch of the proof $(X = \mathcal{B}^\alpha, 0 < \alpha < 1)$

Theorem (ACKS-2018)

There exist positive constants $k_\alpha$ and $K_\alpha$ depending only on $\alpha$ such that

$$k_\alpha \tau_{\infty} \phi,\alpha \leq \|C_\phi\|_{L(B^{\alpha})} \leq K_\alpha \tau_{\infty} \phi,\alpha.$$ 

Theorem (Zorboska, 2007)

Let $0 < \alpha < 1$ and let $\phi$ be a holomorphic self map of $D$. Then $C_\phi$ is an isometry of $B^\alpha$ if and only if $\phi$ is a rotation.

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Composition operators similar to an isometry
Sketch of the proof ($X = \mathcal{B}^\alpha, 0 < \alpha < 1$)

Theorem (ACKS-2018)

There exist positive constants $k_\alpha$ and $K_\alpha$ depending only on $\alpha$ such that

$$k_\alpha \tau_{\varphi,\alpha}^\infty \leq \| C_\varphi \|_{\mathcal{L}(\mathcal{B}^\alpha)} \leq K_\alpha \tau_{\varphi,\alpha}^\infty.$$
Sketch of the proof ($X = B^\alpha$, $0 < \alpha < 1$)

**Theorem (ACKS-2018)**

There exist positive constants $k_\alpha$ and $K_\alpha$ depending only on $\alpha$ such that

$$k_\alpha \tau^\infty_{\varphi,\alpha} \leq \|C_\varphi\|_{L(B^\alpha)} \leq K_\alpha \tau^\infty_{\varphi,\alpha}.$$ 

**Theorem (Zorboska, 2007)**

Let $0 < \alpha < 1$ and let $\varphi$ be a holomorphic self map of $\mathbb{D}$. Then $C_\varphi$ is an isometry of $B^\alpha$ if and only if $\varphi$ is a rotation.
Composition operators similar to an isometry of $B$.

Theorem (ACKS-2018)

Let $\phi$ be a holomorphic self map of $D$. Consider the composition operator $C_\phi$ on $B$. The following assertions are equivalent:

(i) $C_\phi$ is similar to an isometry;

(ii) $\phi$ has a fixed point $b \in D$ and $\tau_\phi \infty = 1$.
Those composition operators which are similar to an isometry of $\mathcal{B}$ is characterized as follows:

Theorem (ACKS-2018)

Let $\phi$ be a holomorphic self map of $D$. Consider the composition operator $C_\phi$ on $\mathcal{B}$. The following assertions are equivalent:

(i) $C_\phi$ is similar to an isometry;

(ii) $\phi$ has a fixed point $b \in D$ and $\tau_\infty \phi = 1$.
Those composition operators which are similar to an isometry of $\mathcal{B}$ is characterized as follows:

**Theorem (ACKS-2018)**

Let $\varphi$ be a holomorphic self map of $\mathbb{D}$. Consider the composition operator $C_\varphi$ on $\mathcal{B}$. The following assertions are equivalent:

(i) $C_\varphi$ is similar to an isometry;

(ii) $\varphi$ has a fixed point $b \in \mathbb{D}$ and $\tau_\varphi^\infty = 1$. 

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Composition operators similar to an isometry
Sketch of the proof

Theorem (ACKS-2018)

Let \( \phi : D \to D \) be holomorphic. The following assertions are equivalent on \( B \).

(i) \( C_n \phi \) converges strongly;

(ii) \( \tau_\infty \phi < 1 \) and there is \( b \in D \) such that \( \phi(b) = b \);

(iii) \( C_n \phi \) converges uniformly.

In that case, \( C_n \phi \) converges to \( P_f = f(b) \) for all \( f \in B \).

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Composition operators similar to an isometry
Theorem (ACKS-2018)

Let \( \varphi : \mathbb{D} \rightarrow \mathbb{D} \) be holomorphic. The following assertions are equivalent on \( \mathcal{B} \).

(i) \( C^n_\varphi \) converges strongly;

(ii) \( \tau^\infty_\varphi < 1 \) and there is \( b \in \mathbb{D} \) such that \( \varphi(b) = b \);

(iii) \( C^n_\varphi \) converges uniformly.

In that case, \( C^n_\varphi \) converges to \( P \), where \( Pf = f(b)1_\mathbb{D} \) for all \( f \in \mathcal{B} \).
Composition operators similar to an isometry of $D$

Theorem (ACKS-2018)

Let $\phi$ be a univalent and holomorphic self map of $D$ such that $n\phi$ is essentially radial. The following assertions are equivalent on $D$.

(i) $C\phi$ is similar to an isometry of $D$;

(ii) $\phi$ is a full map with a fixed point $b \in D$. 

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Composition operators similar to an isometry of $D$
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**Theorem (ACKS-2018)**

Let $\varphi$ be a univalent and holomorphic self map of $\mathbb{D}$ such that $n_\varphi$ is essentially radial. The following assertions are equivalent on $\mathcal{D}$.

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Composition operators similar to an isometry
Sketch of the proof

Theorem (Martín, Vukotić, 2006)
A composition operator \( C_\phi \) is an isometry of \( D \) if and only if \( \phi \) is a univalent full map of \( D \) that fixes the origin.

Theorem (ACKS-2018)
Let \( \phi \) be a univalent and holomorphic self map of \( D \) such that the counting function \( n_\phi \) is essentially radial. The following assertions are equivalent on \( D \).

(i) \( C_n \phi \) converges strongly;

(ii) \( \phi \) is not a full map of \( D \) and there is \( b \in D \) with \( \phi(b) = b \);

(iii) \( C_n \phi \) converges uniformly.

In that case, \( C_n \phi \) converges to \( P_f = f(b) \) for all \( f \in D \).

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Composition operators similar to an isometry
Theorem (Martín, Vukotić, 2006)

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**Theorem (Martín, Vukotić, 2006)**

A composition operator $C_\varphi$ is an isometry of $\mathbb{D}$ if and only if $\varphi$ is a univalent full map of $\mathbb{D}$ that fixes the origin.

**Theorem (ACKS-2018)**

Let $\varphi$ be a univalent and holomorphic self map of $\mathbb{D}$ such that the counting function $n_\varphi$ is essentially radial. The following assertions are equivalent on $\mathbb{D}$.

(i) $C_\varphi^n$ converges strongly;

(ii) $\varphi$ is not a full map of $\mathbb{D}$ and there is $b \in \mathbb{D}$ with $\varphi(b) = b$;

(iii) $C_\varphi^n$ converges uniformly.

In that case, $C_\varphi^n$ converges to $P$, where $Pf = f(b)1_{\mathbb{D}}$ for all $f \in \mathcal{D}$. 

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Composition operators similar to an isometry


References


Thank You