Subnormality of operators of class Q

Jan Stochel

coauthors: S. Chavan, Z. J. Jabłoński, I. B. Jung

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The Cauchy dual operator $T'$ of a left-invertible operator $T \in B(H)$ is defined by

$$T' = T(T^* T)^{-1}.$$ 

If $T$ is left-invertible, then $T'$ is again left-invertible and

$$(T')' = T,$$

$$T^* T' = I \text{ and } T' T = I.$$ 

This notion has been introduced and studied by Shimorin in the context of the wandering subspace problem for Bergman-type operators (2001).
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This notion has been introduced and studied by Shimorin in the context of the wandering subspace problem for Bergman-type operators (2001).
Given \( m \geq 1 \), we say that an operator \( T \in B(\mathcal{H}) \) is an \( m \)-isometry if \( B_m(T) = 0 \), where

\[
B_m(T) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^* T^k.
\]

We say that \( T \) is:
- completely hyperexpansive if \( B_m(T) \leq 0 \) for all \( m \geq 1 \).
- 2-hyperexpansive if \( B_2(T) \leq 0 \).
- 2-hyperexpansive operator \( \leadsto \) Richter (1988)
- \( m \)-isometric operator \( \leadsto \) Agler (1990)
- completely hyperexpansive operator \( \leadsto \) Athavale (1996)
- a 2-isometry is \( m \)-isometric for every \( m \geq 2 \), and thus it is completely hyperexpansive,
  - a 2-hyperexpansive (e.g. 2-isometric) operator is left-invertible and its Cauchy dual \( T' \) is a contraction.
Complete hyperexpansivity; \( m \)-isometries

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An operator $T \in B(\mathcal{H})$ is said to be:

- **hyponormal** if $TT^* \leq T^*T$ (Halmos 1950),
- **subnormal** if there exist a Hilbert space $\mathcal{K}$ and a normal operator $N \in B(\mathcal{K})$, i.e. $N^*N = NN^*$, such that $\mathcal{H} \subseteq \mathcal{K}$ and $Th = Nh$ for all $h \in \mathcal{H}$ (Halmos 1950),
- **quasinormal** if $TTT^* = TT^*T$ (A. Brown 1953).

Quasinormal operators are subnormal and subnormal operators are hyponormal, but not reversely (if $\mathcal{H}$ is infinite dimensional).
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The Cauchy dual subnormality problem

- The map $T \mapsto T'$ sends
  - ♣ 2-hyperexpansive operators into hyponormal contractions (Shimorin 2002),
  - ♣ completely hyperexpansive unilateral weighted shifts into subnormal contractions (Athavale 1996).

- This leads to the Cauchy dual subnormality problem originally posed by Chavan (2007):
  - Is the Cauchy dual of a completely hyperexpansive operator a subnormal contraction?

- The Cauchy dual operator of a 2-hyperexpansive operator is power hyponormal contractions (Chavan 2013).

- The answer is NO even for 2-isometries (Anand, Chavan, Jablonski, JS 2017).
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The following question was addressed in (ACJS 2017):

- find subclasses of the class of 2-isometries for which the Cauchy dual subnormality problem has an affirmative solution.

It was proved in (ACJS 2017) that this is the case for:

- 2-isometries satisfying the kernel condition

\[ T^* T(\ker(T^*)) \subseteq \ker(T^*), \]

- the so-called quasi-Brownian isometries.

A recent generalization: in the class of quasi-Brownian isometries the map \( T \mapsto T' \) sends bijectively hyperexpansive operators onto subnormal contractions (Badea, Suciu 2018).

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Affirmative solutions

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• Quasi-Brownian isometries = \( \triangle_\gamma \)-regular 2-isometries (Majdak, Mbekhta, Suciu 2016) generalize Brownian isometries studied by Agler and Stankus (1995-1996).
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A block matrix representation

**Theorem**

If $T \in \mathcal{B}(\mathcal{H})$, then TFAE:

(i) $T$ is a quasi-Brownian isometry (resp., Brownian isometry),

(ii) $T$ has a block matrix form

$$
T = \begin{bmatrix}
V & E \\
0 & U
\end{bmatrix}
$$

with respect to an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $V \in \mathcal{B}(\mathcal{H}_1)$, $E \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $U \in \mathcal{B}(\mathcal{H}_2)$ are such that

V isometry, $V^*E = 0$, U isometry, $UE^*E = E^*EU$ \hspace{1cm} (2)

(resp., $V$ isometry, $V^*E = 0$, $U$ unitary, $UE^*E = E^*EU$), \hspace{1cm} (3)

(iii) $T$ is either an isometry or it has the block matrix form (1) with respect to a nontrivial orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $V \in \mathcal{B}(\mathcal{H}_1)$, $E \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $U \in \mathcal{B}(\mathcal{H}_2)$ satisfy (2) (resp. (3)) and $\ker E = \{0\}$. 

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Subnormality of operators of class Q
Aims of the talk

- This leads to a question why this phenomenon can happen.
- We will attempt to answer this by indicating and testing a certain class of operators closed for the operation of taking the Cauchy dual.

For this, we embed the class of quasi-Brownian isometries into an essentially larger class of operators having the $2 \times 2$ block matrix representation described by (1) and (2), not requiring that $U$ (the bottom right corner) is an isometry.

The entry $U$ can be replaced by a more general operator, namely by a normal, a quasinormal or a subnormal operator; $\mathcal{N}$, $\mathcal{Q}$ and $\mathcal{S}$ denote the respective classes of operators.

The most challenging problem is to characterize subnormality and complete hyperexpansivity within these classes. In my talk I will concentrate on the class $\mathcal{Q}$. 
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Jan Stochel [4ex]
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The class $\mathcal{Q}$

**Definition**

We say that an operator $T \in \mathcal{B}(\mathcal{H})$ is of class $\mathcal{Q}$ if $T$ has a block matrix form

$$
T = \begin{bmatrix}
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with respect to a nontrivial orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $V \in \mathcal{B}(\mathcal{H}_1)$, $E \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $Q \in \mathcal{B}(\mathcal{H}_2)$ satisfy the conditions

- $V$ isometry, $V^*E = 0$, $QE^*E = E^*EQ$,
- $Q$ is a quasinormal operator.

If this is the case, then we write $T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2}$. 
Proposition

Suppose \( T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2} \) is left invertible. Then \( \Omega := E^*E + Q^*Q \) is invertible in \( \mathcal{B}(\mathcal{H}) \), \( T' \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2} \) and

\[
T' = \begin{bmatrix} V & \tilde{E} \\ 0 & \tilde{Q} \end{bmatrix},
\]

where

\[
\tilde{E} = E\Omega^{-1} \quad \text{and} \quad \tilde{Q} = Q\Omega^{-1}.
\]
Suppose $T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2}$. Then

(i) $T^n = \begin{bmatrix} V^n & E^n \\ 0 & Q^n \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2}$ for any $n \in \mathbb{Z}_+$, where

$$E_n = \begin{cases} 0 & \text{if } n = 0, \\ \sum_{j=1}^{n} V^{j-1} EQ^{n-j} & \text{if } n \geq 1, \end{cases}$$

(ii) $T^*n T^n = \begin{bmatrix} I & 0 \\ 0 & \Omega_n \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2}$ for any $n \in \mathbb{Z}_+$, where

$$\Omega_n = \begin{cases} I & \text{if } n = 0, \\ E^* E \left( \sum_{j=0}^{n-1} (Q^* Q)^j \right) + (Q^* Q)^n & \text{if } n \geq 1. \end{cases}$$
Corollary

Suppose \( T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2} \). Then \( T \) is an isometry if and only if

\[
|Q|^2 + |E|^2 = I,
\]

(4)

or equivalently if and only if

\[
\sigma(|Q|, |E|) \subseteq \mathbb{T}_+,
\]

where \( \mathbb{T}_+ := \{(x_1, x_2) \in \mathbb{R}^2_+: x_1^2 + x_2^2 = 1\} \).

A pair \((T_1, T_2) \in \mathcal{B}(\mathcal{H})^2\) is said to be a spherical isometry if \( T_1^* T_1 + T_2^* T_2 = I \). Thus (4) means that the pair \((|Q|, |E|)\) is a spherical isometry.
Isometries

Corollary

Suppose $T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2}$. Then $T$ is an isometry if and only if

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\( \triangle_T \)-regularity

**Proposition**

Suppose \( T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2} \). Set \( \triangle_T = T^*T - I \) and \( \Omega = E^*E + Q^*Q \). Then the following conditions are equivalent:

(i) \( T \) is \( \triangle_T \)-regular, i.e., \( \triangle_T \geq 0 \) and \( \triangle_T T = \triangle_T^{1/2} T \triangle_T^{1/2} \) with \( \triangle_T = T^*T - I \),

(ii) \( \triangle_T \geq 0 \),

(iii) \( \Omega \geq I \),

(iv) \( \sigma(|Q|, |E|) \cap \mathbb{D}_+ = \emptyset \), where \( \mathbb{D}_+ = \{ (x_1, x_2) \in \mathbb{R}_+^2 : x_1^2 + x_2^2 < 1 \} \).
Theorem

Suppose \( T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2} \). Then the following conditions are equivalent:

(i) \( T \) is a quasi-Brownian isometry,

(ii) \( T \) is a 2-isometry,

(iii) \((|Q|^2 - I)(|Q|^2 + |E|^2 - I) = 0,\)

(iv) \( \sigma(|Q|, |E|) \subseteq \mathbb{T}_+ \cup (\{1\} \times \mathbb{R}_+) \),

(v) there exists an orthogonal decomposition \( \mathcal{H}_2 = \mathcal{H}_{2,1} \oplus \mathcal{H}_{2,2} \) such that \( \mathcal{H}_{2,1} \) and \( \mathcal{H}_{2,2} \) reduce \( Q \) and \( |E| \), \( Q|_{\mathcal{H}_{2,1}} \) is an isometry and \( \left( Q|_{\mathcal{H}_{2,2}}, |E|_{\mathcal{H}_{2,2}} \right) \) is a spherical isometry.
Proposition

Suppose $T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in Q_{\mathcal{H}_1, \mathcal{H}_2}$. Then the following conditions are equivalent:

(i) $T$ is a Brownian isometry, i.e., $T$ is a 2-isometry such that
\[ \triangle_T \triangle_T^* \triangle_T = 0, \]

(ii) $(|Q|^2 - I)(|Q|^2 + |E|^2 - I) = 0$

and

$(|Q^*|^2 - I)(|Q|^2 + |E|^2 - I)^2 = 0,$

(iii) there exists an orthogonal decomposition $\mathcal{H}_2 = \mathcal{H}_{2,1} \oplus \mathcal{H}_{2,2}$ such that $\mathcal{H}_{2,1}$ and $\mathcal{H}_{2,2}$ reduce $Q$ and $|E|$, $Q|_{\mathcal{H}_{2,1}}$ is an isometry,

\[ \left( Q|_{\mathcal{H}_{2,2}}, |E||_{\mathcal{H}_{2,2}} \right) \text{ is a spherical isometry and the spaces } \mathcal{H}_{2,1} \ominus Q(\mathcal{H}_{2,1}) \text{ and } |E|(\mathcal{H}_{2,1}) \text{ are orthogonal.} \]
Suppose $T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix} \in \mathcal{O}_{\mathcal{H}_1, \mathcal{H}_2}$. Then the following conditions are equivalent:

(i) $T$ is subnormal,

(ii) $\sigma(|Q|, |E|) \subseteq \bar{D}_+ \cup \left((1, \infty) \times \{0\}\right)$,

where $\sigma(|Q|, |E|)$ stands for the Taylor spectrum of $(|Q|, |E|)$ and

$\bar{D}_+ = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1^2 + x_2^2 < 1\}$. 
We say that a multi-sequence \( \{ \gamma_\alpha \}_{\alpha \in \mathbb{Z}^d_+} \subseteq \mathbb{R} \) is a Hamburger moment multi-sequence (or Hamburger moment sequence if \( d = 1 \)) if there exists a positive Borel measure \( \mu \) on \( \mathbb{R}^d \), called a representing measure of \( \{ \gamma_\alpha \}_{\alpha \in \mathbb{Z}^d_+} \), such that

\[
\gamma_\alpha = \int_{\mathbb{R}^d} x^\alpha \, d\mu(x), \quad \alpha \in \mathbb{Z}^d_+.
\] (5)

If such \( \mu \) is unique, then \( \{ \gamma_\alpha \}_{\alpha \in \mathbb{Z}^d_+} \) is said to be determinate. If (5) holds for some positive Borel measure \( \mu \) on \( \mathbb{R}^d \) supported in \( \mathbb{R}^d_+ \), then \( \{ \gamma_\alpha \}_{\alpha \in \mathbb{Z}^d_+} \) is called a Stieltjes moment multi-sequence (or Stieltjes moment sequence if \( d = 1 \)).
Lambert’s characterization of subnormality

Theorem (Lambert 1976)

An operator $T \in B(H)$ is subnormal if and only if for every $f \in H$, the sequence $\{\|T^n f\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence, i.e., there exists a positive Borel measure $\mu_f$ on $[0, \infty)$ such that

$$\|T^n f\|^2 = \int_{[0, \infty)} t^n d\mu_f(t), \quad n = 0, 1, 2, \ldots.$$
Lemma

Let $G : \mathcal{B}(X) \to \mathcal{B}(\mathcal{H})$ be a regular Borel spectral measure on a topological Hausdorff space $X$ such that $\text{supp} \ G$ is compact. Suppose that for every $n \in \mathbb{Z}_+$, $\varphi_n : X \to \mathbb{R}$ is a continuous function. Then TFAE:

(i) $\left\{ \int_X \varphi_n(x) \langle G(dx)f, f \rangle \right\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $f \in \mathcal{H}$,

(ii) $\left\{ \varphi_n(x) \right\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $x \in \text{supp} \ G$. 

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A characterization of subnormality

**Theorem**

Let $G : \mathcal{B}(X) \rightarrow \mathcal{B}(\mathcal{H})$ be a regular Borel spectral measure on a topological Hausdorff space $X$ such that $\text{supp } G$ is compact and let $T \in \mathcal{B}(\mathcal{H})$ be such that

$$T^* T^n = \int_X \varphi_n \, dG, \quad n \in \mathbb{Z}_+,$$

where $\varphi_n : X \rightarrow \mathbb{R}, n \in \mathbb{Z}_+$, are continuous functions. Suppose that for every $x \in X$, there exists a compactly supported complex Borel measure $\mu_x$ on $\mathbb{R}_+$ such that

$$\varphi_n(x) = \int_{\mathbb{R}_+} t^n \, d\mu_x(t), \quad n \in \mathbb{Z}_+, \ x \in X.$$

Then $T$ is subnormal if and only if for every $x \in \text{supp } G$, $\mu_x$ is a positive measure.
Let \( d \in \mathbb{N} \), \( \mu \) be a compactly supported complex Borel measure on \( \mathbb{R}^d \) and

\[
\gamma_{\alpha} = \int_{\mathbb{R}^d} x^\alpha d\mu(x), \quad \alpha \in \mathbb{Z}^d_+.
\]

Then the following conditions are equivalent:

(i) \( \{\gamma_{\alpha}\}_{\alpha \in \mathbb{R}^d} \) is a Hamburger moment multi-sequence,

(ii) \( \mu \) is a positive measure.

Moreover, if (i) holds, then \( \{\gamma_{\alpha}\}_{\alpha \in \mathbb{R}^d} \) is determinate.
Lemma

Suppose $d \geq 1$ and $\mu_1$ and $\mu_2$ are compactly supported complex Borel measures on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} x^\alpha d\mu_1(x) = \int_{\mathbb{R}^d} x^\alpha d\mu_2(x), \quad \alpha \in \mathbb{Z}_+^d.$$

Then $\mu_1 = \mu_2$. 

Jan Stochel [4ex] coauthors: S. Chavan, Z. J. J. Subnormality of operators of class Q
Any sequence \( \{ \gamma_n \}_{n=0}^{\infty} \subseteq \mathbb{R} \) has infinitely many representing complex measures. Indeed, by [Boas 1938, Durán], there is a complex Borel measure \( \rho \) on \( \mathbb{R} \) such that

\[
\gamma_n = \int_{\mathbb{R}} x^n d\rho(x), \quad n \in \mathbb{Z}_+.
\]

Let \( \{ s_n \}_{n=0}^{\infty} \) be an indeterminate Hamburger moment sequence with two distinct representing measures \( \mu_1 \) and \( \mu_2 \). Then \( \mu := \mu_1 - \mu_2 \) is a signed Borel measure on \( \mathbb{R} \) such that (Stieltjes)

\[
\int_{\mathbb{R}} x^n d\mu(x) = 0, \quad n \in \mathbb{Z}_+.
\]

As a consequence, we have

\[
\gamma_n = \int_{\mathbb{R}} x^n d(\rho + \vartheta \mu)(x), \quad n \in \mathbb{Z}_+, \quad \vartheta \in \mathbb{C}.
\]

Moreover, the mapping \( \mathbb{C} \ni \vartheta \mapsto \rho + \vartheta \mu \) is injective.
Lemma

Let for \( k = 1, 2, \) \( \{\gamma_k(n)\}_{n=0}^{\infty} \) be a Hamburger moment sequence having a compactly supported representing measure \( \mu_k \) and let \( p \in \mathbb{C}[x] \) be such that

\[
\gamma_1(n) = \gamma_2(n) + p(n), \quad n \in \mathbb{Z}_+.
\] (6)

Then \( p \) is a constant polynomial and \( \mu_1 = \mu_2 + p(0)\delta_1 \).

Corollary

Suppose \( p \in \mathbb{C}[x] \). Then the following conditions are equivalent:

(i) \( \{p(n)\}_{n=0}^{\infty} \) is a Hamburger moment sequence,

(ii) \( \{p(n)\}_{n=0}^{\infty} \) is a Stieltjes moment sequence,

(iii) \( p \) is a constant polynomial and \( p(0) \geq 0 \).
Lemma

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References


THANK YOU!